ECON 6090

Section Solutions

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1 August 29

Questions

1. Prove the following statements about preference relations:
   (a) If $\succsim$ is transitive, then $\succ$ is also transitive.
   (b) If $\succsim$ is transitive, then $\sim$ is also transitive.
   (c) If $\succsim$ is complete and transitive, then $\succsim$ is *negatively transitive*: if $x \succsim y$ then for any z, either $x \succsim z$ or $z \succsim y$ or both.
2. Let X be a finite set of alternatives and $\mathcal{B}$ the set of all nonempty subsets of X. Choice rule $C : \mathcal{B} \to \mathcal{B}$ satisfies *path independence* if for all $B, B' \in \mathcal{B}$, $C(B \cup B') = C(C(B) \cup C(B'))$.[1] Prove that any choice rule that is rationalized by a complete and transitive preference relation satisfies path independence.

Solutions

1. Short proofs are:
   (a) ***Proof.*** Take some $x, y, z \in X$ and assume that $x \succ y$ and $y \succ z$. Then we have that $x \succsim y$ and $y \succsim z$, so $x \succsim z$ by transitivity of $\succsim$. Towards a contradiction, argue that $z \succsim x$. Since $x \succsim y$ and $\succsim$ is transitive, then we would have that $z \succsim y$, which contradicts the assumption that $y \succ z \implies z \not\succsim y$. Thus, $z \not\succsim x$, so $x \succ z$. $\square$
   (b) ***Proof.*** This entire proof follows from the definition. Take some $x, y, z \in X$ and assume that $x \sim y$ and $y \sim z$. Then we have that (i) $x \succsim y$ and $y \succsim z$, so $x \succsim z$ by transitivity of $\succsim$, and (ii) $z \succsim y$ and $y \succsim x$, so $z \succsim x$ by transitivity of $\succsim$. Thus, $x \sim z$. $\square$

[1] This basically means that the choice is not affected if we first split the set into smaller sets, choose the best from each of those, and choose the best from the best of the smaller sets. Intuitively, this should be the same as choosing the best of the entire set.

(c) ***Proof.*** Assume that $\succsim$ is complete and transitive, and that there exist $x, y, z \in X$ such that $x \succsim y$. By completeness, it must be the case that either $x \succsim z$ or $z \succsim x$ (or both). If the first is true, the conclusion is immediate. If the second is true, then we have that $z \succsim x$ and $x \succsim y$, so by transitivity of $\succsim$ we have that $z \succsim y$, and conclusion follows. $\square$

2. ***Proof.*** Assume that we have a choice rule that is rationalized by a complete and transitive preference relation. We will show the equality by double set containment.

   ($\subseteq$): Take some $x \in C(B \cup B')$. Since $C(B \cup B') \subseteq B \cup B'$, we have that $x \in B \cup B'$. Without loss, assume that $x \in B$. We have that $x \succsim y \; \forall \, y \in B \cup B \implies x \succsim z \; \forall \, z \in B$. Therefore, $x \in C(B)$. Additionally, $\forall \, y \in C(B')$, $y \in B' \subseteq B \cup B'$, so $x \succsim y$. Since $x \in C(B) \cup C(B')$, and $x \succsim y \; \forall \, y \in C(B) \cup C(B')$, $x \in C(C(B) \cup C(B'))$.

   ($\supseteq$): Take some $x \in C(C(B) \cup C(B'))$. We have that for all $y \in C(B)$ and all $y' \in C(B')$, $x \succsim y$ and $x \succsim y'$. Then, for all $z \in B$ and $z' \in B'$, $y \succsim z$ and $y' \succsim z'$, so by transitivity $x \succsim z$ and $x \succsim z'$. Thus, $x \succsim z \; \forall \, z \in B \cup B'$, so $x \in C(B \cup B')$. $\square$

2 September 5

Questions

1. Are the following preference relations $\succsim$ rational?
   (a) Let $\succsim$ be defined on $\mathbb{R}$ by: $y \succsim x$ iff $y \geq x + \varepsilon, \varepsilon$ is a positive number.
   (b) Let $\succsim$ be defined on $\mathbb{R}$ by: $y \succsim x$ iff $y \geq x - \varepsilon, \varepsilon$ is a positive number.
   (c) $X = \{a, b, c\}.C^*(\{a, b\}, \succsim) = \{b\}.C^*(\{b, c\}, \succsim) = \{c\}.C^*(\{a, b, c\}, \succsim) = \{c\}$.
   (d) Agents 1 and 2 are facing the same choice set X. Agent 1 has a rational preference relation $\succsim_1$, consumer 2 's preference relation is given by $\succsim_2 := \succ_1$. Is consumer 2's preference rational?
   (e) Consider the lexicographic preference relation $\succsim$ on $\mathbb{R}^2_+ : (x_1, x_2) \succsim (y_1, y_2)$ if and only if $x_1 > y_1$ or $x_1 = y_1$ and $x_2 \geq y_2$. Is $\succsim$ a rational preference relation?
2. Consider the following definitions:

   Definition. A preference relation $\succsim$ on $\mathbb{R}^n_+$ is called *homothetic* if for all $x, y \in \mathbb{R}^n_+$ and all $\lambda > 0$, $x \succsim y$ if and only if $\lambda x \succsim \lambda y$.
   Definition. A function $u : \mathbb{R}^n_+ \to \mathbb{R}$ is called *homogeneous of degree one* if for all $x \in \mathbb{R}^n_+$ and all $\lambda > 0$, $u(\lambda \cdot x) = \lambda \cdot u(x)$.

   Show that a continuous strictly monotone preference relation $\succsim$ on $\mathbb{R}^n_+$ is homothetic if and only if it can be represented by a utility function which is homogeneous of degree one.

Solutions

1. The preference relations are:
   (a) Not rational. Taking $x = 1$, $y = 1$, $\varepsilon > 0$, $x \not\succsim y$ and $y \not\succsim x$. That's a completeness violation.
   (b) Not rational. Take $\varepsilon = \frac{1}{3}$, $x = -\frac{1}{4}$, $y = 0$, $z = \frac{1}{4}$. Then $x \succsim y$ and $y \succsim z$, but $x \not\succsim z$ since $-\frac{1}{4} < -\frac{1}{12}$.
   (c) Rational. We have that $c \succ b$ and $b \succ a$, and that $a \not\succ c$. This is not a transitivity violation, and it is vacuously complete (as we do not see any sets that contradict completeness).
   (d) Not rational. Set $\succsim_1 \equiv \geq$ on the real numbers, and set $x = 1$ and $y = 1$. Then $x \not\succ_1 y$ and $y \not\succ_1 x$, so $x \not\succsim_2 y$ and $y \not\succsim_2 x$. That's a completeness violation.

(e) Rational. Take some $x = (x_1, x_2)$ and $y = (y_1, y_2)$. We have that either $x_1 > y_1$, in which case $x \succsim y$, or $y_1 > x_1$, in which case $y \succsim x$, or $x_1 = y_1$. In the final case, either $x_2 \geq y_2$, in which case $x \succsim y$, $y_2 \geq x_2$, in which case $y \succsim x$, or both. Thus, the preference relation is complete. Take some $x, y, z \in \mathbb{R}^2_+$, and assume that $x \succsim y$ and $y \succsim z$. Either it is true that $x_1 > y_1$ or $y_1 > z_1$, in which case $x_1 > z_1$ and $x \succsim z$, or $x_1 = y_1 = z_1$. In that case, since $x \succsim y$ and $y \succsim z$, we have that $x_2 \geq y_2$ and $y_2 \geq z_2$, so by transitivity of $\geq$, we have that $x_2 \geq z_2$ and $x \succsim z$. Thus, the preference relation is transitive, and therefore rational.

2. ***Proof.*** ($\Leftarrow$): Fix some $x \succsim y$. We have that $u(x) \geq u(y)$, so $u(\lambda x) = \lambda u(x) \geq \lambda u(y) = u(\lambda y)$ so $\lambda x \succsim \lambda y$, and $x \succsim y \Longrightarrow \lambda x \succsim \lambda y$. The same proof works in the other direction

   ($\Rightarrow$): This proof will be constructive. Begin with the definition of the representative utility function for a continuous preference relation in the notes, where we assert that for any x, $\alpha e \sim x$ for some $\alpha \in \mathbb{R}$, and define $u(x) = \alpha$. Note that $x \sim y \Longleftrightarrow \lambda x \sim \lambda y$ from homotheticity. Thus, we must have that $\alpha e \sim x \Longleftrightarrow \alpha \lambda e \sim \lambda x$. Using the constructed utility function, we have that $u(\lambda x) = \lambda \alpha = \lambda u(x)$. □

3 September 12

Questions

1. (2009 Prelim 1) There are three goods with quantities denoted by $x = (x_1, x_2, x_3) \in \mathbb{R}^3_+$. A consumer's preferences can be represented by the utility function $u(x) = x_1 x_2^{1/2} x_3^{1/2}$. The prices of the goods are represented by $p \in \mathbb{R}^3_{++}$ and the consumer has wealth $w > 0$.

   (a) Write the consumer's decision problem as a constrained optimization problem.

   (b) Find the consumer's demand functions for the three goods.

   Now suppose that in addition to using money to purchase goods the consumer also has to provide coupons in order to make a purchase. The purchase of $y \geq 0$ units of any good requires y coupons. The consumer has an endowment of $c > 0$ coupons.

   (c) Write the consumer's new decision problem as a constrained optimization problem.

   (d) Is it possible that at a solution to the consumer's problem he has some left-over coupons? That is, can the coupon constraint ever be non-binding?

   (e) Suppose that $p = (1, 1, 1)$. Find the consumer's demands for the three goods.

Solutions

1. **2009 Prelim 1**

   (a) The consumer solves

   $$\max_{x \in \mathbb{R}^3_+} x_1 \cdot x_2^{1/2} \cdot x_3^{1/2} \text{ s.t. } p_1 \cdot x_1 + p_2 \cdot x_2 + p_3 \cdot x_3 \leq w$$

   (b) Note that utility is strictly increasing in each good, meaning that the constraint must bind. Also recall that a monotone transformation preserves orderings. For simplicity, we will log transform the utility function here. The consumer's problem becomes

   $$\max_{x \in \mathbb{R}^3_+} \ln x_1 + \frac{1}{2} \ln x_2 + \frac{1}{2} \ln x_3 \text{ s.t. } p_1 \cdot x_1 + p_2 \cdot x_2 + p_3 \cdot x_3 \leq w$$

   so we have the Lagrangian

   $$\mathcal{L} = \ln x_1 + \frac{1}{2} \ln x_2 + \frac{1}{2} \ln x_3 - \lambda(p_1 \cdot x_1 + p_2 \cdot x_2 + p_3 \cdot x_3 - w)$$

which admits first order conditions

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{1}{x_1} - \lambda p_1 = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{1}{2x_2} - \lambda p_2 = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_3} = \frac{1}{2x_3} - \lambda p_3 = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} \Longrightarrow p \cdot x = w$$

which, solving, get us:

$$x_1^\star = \frac{w}{2p_1} \qquad ; \qquad x_2^\star = \frac{w}{4p_2} \qquad ; \qquad x_3^\star = \frac{w}{4p_3}$$

(c) The consumer's new problem is

$$\max_{x \in \mathbb{R}^3_+} x_1 \cdot x_2^{1/2} \cdot x_3^{1/2}$$

subject to

$$p_1 \cdot x_1 + p_2 \cdot x_2 + p_3 \cdot x_3 \leq w \qquad ; \qquad x_1 + x_2 + x_3 \leq c$$

(d) Yes, if c is large enough that the optimal bundle from part (a) is affordable then that constraint will not bind.

(e) Note that the budget constraints and the coupon constraints are now parallel. If $w > c$, we have the problem from part (a). If $c > w$, then we have only the coupon constraint will bind. If they are equal, the problem is equivalent using either.

4 September 19

Questions

1. **June 2025 Microeconomics Q Exam (Part II)** Consider a consumer who has utility $u(x_1, \ldots, x_L, h)$ over the amounts of consumption $(x_1, \ldots, x_L)$ of goods $1, \ldots, L$, and hours worked h. The utility function $u(\cdot)$ is continuous and strictly increasing in each x_ℓ, and is weakly decreasing in h. Prices are given by $p = (p_1, \ldots, p_L)$. The wage rate is normalized to 1 throughout, so that the consumer can spend amount h to purchase consumption goods. Consumption and labor hours have to be non-negative, but assume that there is no upper bound on labor hours h.

   (a) Write down the "work-minimization problem": the problem of minimizing the hours of work subject to achieving a given level of utility and subject to the budget constraint.

   Call the value $h^\star$ of this problem the "minimal-work function". Does it have similar properties to the expenditure function? In particular:

   (b) Is the "minimal-work function" non-decreasing in p_i for $i \in \{1, \ldots, L\}$? (Hint: If $(x_1, \ldots, x_L, h)$ satisfies all constraints under higher prices, would it still satisfy all constraints under lower prices?).

   (c) Is the "minimal-work function" homogeneous of degree 1 in p? (Hint: if you doubled the prices of all goods, would the minimal-work-hours double? Try e.g. with $u(x_1, h) = x_1^2/h$.)

   (d) What would your answer to (b) and (c) be if you knew that the utility function is constant in work hours h?

   Prove your answers. For negative answers you can provide a counter-example. You do not need to re-prove known properties of the standard expenditure function.

Solutions

1. **June 2025 Microeconomics Q Exam (Part II)**

   (a) The work minimization problem takes some $\bar{u}$ as given, so the consumer solves

$$\begin{aligned} & \min_{h \in \mathbb{R}_+} h \\ \text{s.t.} \quad & u(x_1, \ldots, x_L, h) \geq \bar{u} \\ & p \cdot x \equiv \sum_{i=1}^{L} p_i \cdot x_i \leq h \end{aligned}$$

Since utility is strictly increasing in consumption, we can reduce this to:

$$\min_{x \in \mathbb{R}^L_+} p \cdot x \text{ s.t. } u(x_1, \ldots, x_L, p \cdot x) \geq \bar{u}$$

(b) Yes.

Proof. Note that a function being non-decreasing in an input is equivalent to it being non-increasing in the negative of that input. It therefore suffices to show that $h^\star$ is non-increasing in $-p_i$ for arbitrary p_i. Take some p and p' so that $p_i > p'_i$, and $p_j = p'_j$ for all $j \neq i$. We want to show that $h^\star(p, \bar{u}) \geq h^\star(p', \bar{u})$. Call $x^\star(p, \bar{u})$ the optimal output under p. Since utility is weakly decreasing in h, we have that $u(x^\star(p, \bar{u}), p' \cdot x^\star(p, \bar{u})) \geq \bar{u}$ since $p' \cdot x^\star(p, \bar{u}) \leq p \cdot x^\star(p, \bar{u})$. Thus, $x^\star(p, \bar{u})$ is in the feasible set under prices p'. From the properties of optimization problems, it must be the case that $h^\star(p', \bar{u}) \leq h^\star(p, \bar{u})$ since the minimal bundle under prices p is feasible under prices p'. Thus, $h^\star$ is non-increasing as p_i decreases, meaning that it is non-decreasing in p_i. $\square$

n.b. This is a really messy and confusing proof, there's got to be a better way to do this, but I can't think of one.

(c) No. For a counterexample, consider the given utility function $u(x, h) = \frac{x^2}{h}$, and consider $\bar{u} = 1$, with the price change $p = 1$ to $p' = 2$. (Note $x, p \in \mathbb{R}_+$). We have that

$$h^\star(p, \bar{u}) = \min_{x \in \mathbb{R}_+} 1 \cdot x \text{ s.t. } \frac{x^2}{1 \cdot x} \geq 1 = \min x \text{ s.t. } x \geq 1 = 1$$

$$h^\star(p', \bar{u}) = \min_{x \in \mathbb{R}_+} 2 \cdot x \text{ s.t. } \frac{x^2}{2 \cdot x} \geq 1 = \min 2x \text{ s.t. } x \geq 2 = 4$$

So since $h^\star(1, \bar{u}) = 1$ but $h^\star(2 \cdot 1, \bar{u}) = 4 > 2 = 2 \cdot h^\star(1, \bar{u})$, $h^\star$ is not homogeneous of degree 1.

(d) My answer to (b) would not change, in the sense that the same proof works as we do not differentiate between non-increasing in h and decreasing in h there. However, note that an increase in p_i would now lead to a strict decrease in x_i, meaning a strict decrease in $h^\star$, as long as the original demand for x_i was positive. My answer to (d) would now change. In the case that the utility function is constant in work hours h, the minimal work function would now be homogeneous of degree 1 in p.

Proof. Take some p and $\alpha > 0$, and fix $\bar{u}$. We have that

$$h^\star(p, \bar{u}) = \min_{x \in \mathbb{R}^L_+} p \cdot x \text{ s.t. } u(x, p \cdot x) \geq \bar{u}$$

and

$$h^\star(\alpha \cdot p, \bar{u}) = \min_{x \in \mathbb{R}^L_+} \alpha \cdot p \cdot x \text{ s.t. } u(x, \alpha \cdot p \cdot x) \geq \bar{u}$$

However, since $u(\cdot)$ is constant in h, we have that $u(x, p \cdot x) = u(x, \alpha \cdot p \cdot x) \; \forall \; \alpha$. Thus,

$$h^\star(\alpha \cdot p, \bar{u}) = \min_{x \in \mathbb{R}^L_+} \alpha \cdot p \cdot x \text{ s.t. } u(x, p \cdot x) \geq \bar{u} = \alpha \cdot \min_{x \in \mathbb{R}^L_+} p \cdot x \text{ s.t. } u(x, p \cdot x) \geq \bar{u}$$

So $h^\star(\alpha \cdot p, \bar{u}) = \alpha \cdot h^\star(p, \bar{u})$, so $h^\star$ is now homogeneous of degree 1. $\square$

5 September 26

Questions

1, 3, and 6 from the first problem set.

Solutions

See the problem set solutions.

6 October 3

Questions

1. In this question, we seek to investigate the relation between equivalent variations and compensating variations. Consider an economy with two goods: good 1 and good 2 respectively. The initial price vector is $p^0 = (p_1^0, p_2^0)$.

   (a) (adapted from 2002 Prelim 2) Suppose we only increase the price of good 1 and get a new price vector $p^1 = (p_1^1, p_2^0)$, where $p_1^1 > p_1^0$. Discuss in case where x_1 is normal good or inferior good and compare $CV(p^0, p^1, w)$ and $EV(p^0, p^1, w)$

   (b) Suppose a consumer's preference is quasi-linear in y with $u(x, y) = x^\alpha + y$, where $\alpha \in (0, 1), x \in \mathbb{R}_+, y \in (-\infty, +\infty)$. Now instead of only changing p_1, we allow both p_1 and p_2 changes, the new price vector is $p^1 = (p_1^1, p_2^1)$. Compare $CV(p^0, p^1, w)$ and $EV(p^0, p^1, w)$ in this case. How will your answer change from (a)?

2. (Adapted from MWG) Recall that the Gorman form is $v^i(p, w^i) = a^i(p) + b(p)w^i$. Verify that Gorman form implies a linear wealth expansion path, which means the Walrasian demand function is a linear function in wealth.

3. For each of the following utility functions, does a positive representative consumer exist?

   (a) $u^i(x, y) = x^{a^i} y^{1-a^i}$

   (b) $u^i(x_1, \ldots, x_N) = \sum_{n=1}^N \alpha_n \log x_n, \alpha_n > 0$

   (c) (2022 Prelim 2) $u^i(x, y) = ln(x) + y$, where wealth $w^i > 1$ and price $p_x > 0$ and $p_y = 1$.

Solutions

1. EV and CV:

   (a) Recall from above that

$$CV(p^0, p^1, w) = \int_{p_1^1}^{p_1^0} h_1(p_1, p_{-1}, \bar{u}^0)\, \partial p_1 \quad ; \quad EV(p^0, p^1, w) = \int_{p_1^1}^{p_1^0} h_1(p_1, p_{-1}, \bar{u}^1)\, \partial p_1$$

   Then we have that, since $h_1(p, \bar{u}) = x_1(p, e(p, \bar{u}))$:

$$\frac{\partial x_1(p, w)}{\partial w} \geq 0 \Longrightarrow \frac{\partial h_1(p, \bar{u})}{\partial \bar{u}} = \frac{\partial x_1}{\partial w} \cdot \frac{\partial e(p, \bar{u})}{\partial \bar{u}} \geq 0 \Longrightarrow CV \leq EV$$

   In the opposite direction, if good 1 is inferior, we will have $EV \leq CV$.

(b) In this case, internalizing the fact that the utility function is strictly increasing in both elements, we have that

$$e(p, \bar{u}) = \min_{x \geq 0} p_1 x + p_2(\bar{u} - x^{\alpha})$$

The FOC give us that

$$x^{\star}(p) = \left(\frac{\alpha p_2}{p_1}\right)^{\frac{1}{1-\alpha}}$$

Inputting this back in, we have that

$$e(p, \bar{u}) = p_2 \cdot \bar{u} - (1 - \alpha) p_2 \cdot [x^{\star}(p)]^{\alpha}$$

Thus, the expenditure function is linear in $\bar{u}$, so

$$CV = p_2^1(\bar{u}^1 - \bar{u}^0) \qquad ; \qquad EV = p_2^0(\bar{u}^1 - \bar{u}^0)$$

2. ***Proof.*** Suppose i has indirect utility $v^i(p, w^i) = a^i(p) + b(p) \cdot w^i$. By Roy's identity,

$$x^u(p, w^i) = -\frac{\nabla_p v^i(p, w^i)}{\partial v^i / \partial w} = -\frac{\nabla_p a^i(p)}{b(p)} - \frac{\nabla_p b(p)}{b(p)} \cdot w_i$$

So the Marshallian demand function is linear in wealth. □

3. Does a positive representative consumer exist?

(a) No. Individual demands are

$$x^i, y^i = \frac{a^i w^i}{p_x}, \frac{(1 - a^i) w^i}{p_y} \Longrightarrow v^i(p, w^i) = \left[(a^i)^{a^i}(1 - a^i)^{1-a^i}\right] p_x^{-a^i} p_y^{-(1-a^i)} \cdot w^i$$

So unless $a^i = a$ for all i, $b(\cdot)$ depends on i.

(b) Yes. Individual demands for each good are $x_i = \frac{\alpha_i}{\sum_j \alpha_j} \frac{1}{p_i} w_i$. Our indirect utility function is therefore

$$v^i(p, w^i) = \sum_n \alpha_n \ln w^i - \sum_n \alpha_n \ln p_n + \sum_n \alpha_n \ln \alpha_n$$

Which is separable over wealth, since the sum of all alphas is the same for all consumers.

(c) Yes. Individual demands are

$$x^i, y^i = \frac{1}{p_x}, w^i - 1 \Longrightarrow v^i(p, w^i) = \underbrace{-1 - \ln p_x}_{a^i(p)} + \underbrace{1}_{b(p)} \cdot w^i$$

7 October 10*

(note, really on October 8)

Questions are the entirety of PS3, solutions are posted on Canvas.

8 October 17

Questions are the midterm question, solutions are posted on Canvas. Either Section 08 or the Official Solutions.

9 October 24

Questions

1. **2025 June Q, Part I:** Consider the production possibilities set
$$Y = \left\{(q, -z) \in \mathbb{R}^2_+ \times \mathbb{R}^2_- : z_1^\alpha z_2^\beta \geq [q_1^\sigma + q_2^\sigma]^{\frac{1}{\sigma}}\right\}$$
where $\alpha = \beta = \frac{1}{3}$ and $\sigma > 0$.

   (a) Derive the conditional input demand function $z(w, q)$. (Note: Here we are considering a two-output technology, so $q \in \mathbb{R}^2_+$.)

   (b) Derive the cost function $C(w, q)$.

   (c) Now suppose $\sigma = \frac{3}{2}$. Derive the unconditional input demand function $x(p, w)$.

   (d) Give an expression for the derivative of the profit function with respect to w_1 and explain how you arrived at this expression. You should **not** accomplish this by first finding an expression for the profit function and then differentiating.

Solutions

(a) The conditional input demand function solves the cost minimization problem subject to a certain level of output, so
$$z^\star(w, q) \equiv \underset{z \in \mathbb{R}^2_+}{\operatorname{argmin}}\, w_1 \cdot z_1 + w_2 \cdot z_2 \text{ s.t. } z_1^\alpha z_2^\beta \geq [q_1^\sigma + q_2^\sigma]^{\frac{1}{\sigma}}$$
Recall that we can take logs of the condition and retain the same feasible set:
$$z^\star(w, q) \equiv \underset{z \in \mathbb{R}^2_+}{\operatorname{argmin}}\, w_1 \cdot z_1 + w_2 \cdot z_2 \text{ s.t. } \alpha \ln z_1 + \beta \ln z_2 \geq \frac{1}{\sigma} \ln [q_1^\sigma + q_2^\sigma]$$
Since this is convex, and the structure of the feasible set guarantees an interior solution (this could also be verified later), we solve this with a Lagrangian:
$$\mathcal{L} = w_1 \cdot z_1 + w_2 \cdot z_2 - \lambda \left(\alpha \ln z_1 + \beta \ln z_2 - \frac{1}{\sigma} \ln [q_1^\sigma + q_2^\sigma]\right)$$
which admits first order conditions
$$\frac{\partial \mathcal{L}}{\partial z_1} : w_1 - \frac{\lambda \alpha}{z_1} = 0 \Longrightarrow \lambda = \frac{z_1 w_1}{\alpha}$$
$$\frac{\partial \mathcal{L}}{\partial z_2} : w_2 - \frac{\lambda \beta}{z_2} = 0 \Longrightarrow \lambda = \frac{z_2 w_2}{\beta}$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} : \alpha \ln z_1 + \beta \ln z_2 = \frac{1}{\sigma} \ln [q_1^\sigma + q_2^\sigma]$$

So we have that

$$z_1 = \frac{\alpha}{\beta} \cdot \frac{w_2}{w_1} \cdot z_2 = \frac{w_2}{w_1} z_2 \implies \ln \frac{w_2}{w_1} z_2^2 = 3 \cdot \frac{1}{\sigma} \ln \left[q_1^\sigma + q_2^\sigma\right] \implies \ln \sqrt{\frac{w_2}{w_1}} z_2 = \frac{3}{2\sigma} \ln \left[q_1^\sigma + q_2^\sigma\right]$$

So

$$z_2^\star = \sqrt{\frac{w_1}{w_2}} \cdot \left[q_1^\sigma + q_2^\sigma\right]^{\frac{3}{2\sigma}}$$

and

$$z_1^\star = \frac{w_2}{w_1} \cdot z_2^\star = \sqrt{\frac{w_2}{w_1}} \left[q_1^\sigma + q_2^\sigma\right]^{\frac{3}{2\sigma}}$$

So formally,

$$z^\star(w, q) = \begin{bmatrix} \sqrt{\frac{w_2}{w_1}} \cdot \left[q_1^\sigma + q_2^\sigma\right]^{\frac{3}{2\sigma}} \\ \\ \sqrt{\frac{w_1}{w_2}} \cdot \left[q_1^\sigma + q_2^\sigma\right]^{\frac{3}{2\sigma}} \end{bmatrix}$$

(b) The cost function is the value function of the cost minimization problem subject to a certain level of output, so

$$C(w, q) \equiv \min_{z \in \mathbb{R}_+^2} w_1 \cdot z_1 + w_2 \cdot z_2 \text{ s.t. } z_1^\alpha z_2^\beta \geq \left[q_1^\sigma + q_2^\sigma\right]^{\frac{1}{\sigma}}$$

Because this is a value function and we've already found the optimal conditional inputs, we have that

$$C(w, q) = w_1 \cdot z_1^\star(w, q) + w_2 \cdot z_2^\star(w, q) = 2\sqrt{w_1 w_2} \cdot \left[q_1^\sigma + q_2^\sigma\right]^{\frac{3}{2\sigma}}$$

(c) The unconditional input demand function is the set of inputs that solves the full profit maximization problem, which implies the cost minimization problem. Note that with $\sigma = \frac{3}{2}$, our cost function becomes $C(w, q) = 2\sqrt{w_1 w_2} \cdot [q_1^\sigma + q_2^\sigma]$. The profit maximization problem is

$$\pi(p, w) = \max_{q \in \mathbb{R}_+^2} p_1 \cdot q_1 + p_2 \cdot q_2 - C(w, q) \equiv \max_{q \in \mathbb{R}_+^2} \underbrace{p_1 \cdot q_1 + p_2 \cdot q_2 - 2\sqrt{w_1 w_2} \cdot \left[q_1^\sigma + q_2^\sigma\right]}_{\mathcal{L}}$$

Again this is concave and has an interior solution, so we find the optimal outputs using first order conditions:

$$\frac{\partial \mathcal{L}}{\partial q_1} : p_1 - 2\sigma\sqrt{w_1 w_2} q_1^{\sigma - 1} = 0 \implies \sqrt{q_1} = \frac{p_1}{3\sqrt{w_1 w_2}} \implies q_1^\star(p, w) = \frac{p_1^2}{9 w_1 w_2}$$
$$\frac{\partial \mathcal{L}}{\partial q_2} : p_2 - 2\sigma\sqrt{w_1 w_2} q_2^{\sigma - 1} = 0 \implies \sqrt{q_2} = \frac{p_2}{3\sqrt{w_1 w_2}} \implies q_2^\star(p, w) = \frac{p_2^2}{9 w_1 w_2}$$

To find the unconditional input demand function, we substitute the optimal quantities into the conditional input demand functions, since $x(p,w) = z(w, q^\star(p,w))$. We get:

$$x_1^\star(p,w) = \sqrt{\frac{w_2}{w_1}}\left[\left(\frac{p_1^2}{9w_1w_2}\right)^{\frac{3}{2}} + \left(\frac{p_2^2}{9w_1w_2}\right)^{\frac{3}{2}}\right] = \frac{p_1^3+p_2^3}{27w_1^2w_2}$$
$$x_2^\star(p,w) = \sqrt{\frac{w_1}{w_2}}\left[\left(\frac{p_1^2}{9w_1w_2}\right)^{\frac{3}{2}} + \left(\frac{p_2^2}{9w_1w_2}\right)^{\frac{3}{2}}\right] = \frac{p_1^3+p_2^3}{27w_1w_2^2}$$

(d) The way we're meant to do this problem is to recall that from Shepherd's Lemma, $\frac{\partial \pi(p,w)}{\partial w_i} = -x_i^\star(p,w)$, so we have that

$$\frac{\partial \pi(p,w)}{\partial w_1} = -x_1^\star(p,w) = -\frac{p_1^3+p_2^3}{27w_1^2w_2}$$

The way I did it was to construct the profit function, take the derivative with respect to w_1, and then re-construct Shepherd's Lemma from there because I didn't perfectly remember it. This requires substituting the optimal outputs found in part (c) into the profit maximization problem, so:

$$\begin{aligned}
\pi(p,w) &= p_1 \cdot \frac{p_1^2}{9w_1w_2} + p_2 \cdot \frac{p_2^2}{9w_1w_2} - 2\sqrt{w_1w_2}\left[\left(\frac{p_1^2}{9w_1w_2}\right)^{\frac{3}{2}} + \left(\frac{p_2^2}{9w_1w_2}\right)^{\frac{3}{2}}\right] \\
&= \frac{p_1^3+p_2^3}{9w_1w_2} - 2\cdot\frac{p_1^3+p_2^3}{27w_1w_2} \\
&= \frac{p_1^3+p_2^3}{27w_1w_2}
\end{aligned}$$

So

$$\frac{\partial \pi(p,w)}{\partial w_1} = -\frac{p_1^3+p_2^3}{27w_1^2w_2} = -x_1^\star(p,w)$$

10 October 31

10.1 Questions

1. **2016 Prelim 2:** The set of prizes is $X = \{-1, 0, +1\}$ and a probability on these prizes is denoted by $p = (p_1, p_2, p_3)$. An individual strictly prefers a "small gamble", $p = (1/8, 3/4, 1/8)$, to certainty, $p = (0, 1, 0)$. However, the individual strictly prefers certainty to the "large gamble", $p = (1/2, 0, 1/2)$. Do this person's preferences have an objective expected utility representation? Explain.

2. **2014 June Q:** An individual has to decide how much of her wealth $w > 0$ to invest in a risky asset. This asset will have positive rate of return r with probability p, or a negative rate of return l with probability $1 - p$. So if the individual invests x dollars in the risk asset, the with probability p her wealth will be $w - x + (1 + r)x$ and with probability $1-p$ her wealth will be $w-x+(1+l)x$. Assume that the asset has a strictly positive expected rate of return $pr + (1 - p)l > 0$. Feasible investments in the risky asset are $x \geq 0$. Assume that this individual is an expected utility maximizer with Bernoulli payoff function $u(w)$ with $u' > 0$ and $u'' < 0$ for all non-negative wealths.

   (a) Show that the individual will invest a positive amount of wealth $x > 0$ in the risky asset.

   (b) It seems reasonable to suppose that as an individual's wealth increases he would invest more in the risk asset. Whether this is true or not depends on how the individual's risk aversion changes as their wealth changes. What is this relationship? This is, under what conditions on risk aversion does investment in the risky asset increase as wealth increases? [Hint: Absolute Risk Aversion of utility function $u(\cdot)$ at x is equal to $-u''(x)/u'(x)$]

10.2 Solutions

1. No. Assume there exists some Bernoulli expected utility function $u : X \to \mathbb{R}$. Then we must have that from the preferences,

$$\frac{1}{8}u(-1) + \frac{3}{4}u(0) + \frac{1}{8}u(1) > u(0) > \frac{1}{2}u(-1) + \frac{1}{2}u(1)$$

   The left inequality implies that $\frac{1}{2}u(-1)+\frac{1}{2}u(1) > u(0)$, directly contradicting the right inequality. Intuitively, this is a violation of the linearity of expectation.

2. **2014 June Q:**

   (a) The expected utility of bidding some x is $U(x;p) = p\cdot u(w+rx)+(1-p)\cdot u(w+lx)$, which is strictly concave because u is strictly concave. Therefore, first order conditions suffice to characterize an optimum. Also, we have that the partial with respect to x when $x = 0$ is:

$$\frac{\partial U(0;p)}{\partial x} = r \cdot p \cdot u'(w + rx) + l \cdot (1-p) \cdot u'(w + lx) = (pr + (1-p)l)u'(w) > 0$$

so the optimal investment is strictly greater than zero.

(b) Let $x^\star(w)$ be the optimal investment at some wealth w. Define the first order condition from part (a) as

$$F(w, x) = r \cdot p \cdot u'(w + rx) + l \cdot (1-p) \cdot u'(w + lx)$$

so we have that $F(w, x^\star(w)) = 0$. Conclusion follows from the Implicit Function Theorem. We have that

$$\frac{\partial x^\star(w)}{\partial w} = -\frac{F_x(w,x)}{F_w(w,x)} = -\frac{r^2 p u''(w + rx^\star) + l^2(1-p)u''(w + lx^\star)}{rpu''(w + rx^\star) + l(1-p)u''(w + lx^\star)}$$

Note that since $u''(\cdot) < 0$, the numerator is negative, so the sign depends purely on $F_w(w, x)$. Call the coefficient of absolute risk aversion $A(x)$, and note that $u''(x) = -u'(x)A(x)$. Then we have that

$$F_w(w,x) = rp \cdot (-u'(w + rx^\star)A(w + rx^\star)) + l(1-p) \cdot (-u'(w + lx^\star)A(w + lx^\star))$$

From the first order condition, we have that

$$rpu'(w + rx^\star) = -l(1-p)u'(w + lx^\star) := C > 0$$

so this becomes

$$F_w(w,x) = C(A(w + lx^\star) - A(w + rx^\star))$$

Thus,

$$F_w(w,x) > 0 \iff A(w + lx^\star) > A(w + rx^\star)$$

so we can say that the optimal investment in the risky asset is increasing in wealth only as long as the coefficient of absolute risk aversion is decreasing in wealth.