ECON 6090

Section 10

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Material

Definition. We have a (typically finite) set X of *outcomes (prizes)* and the set P of *lotteries*, or probability distributions over X. For each $p \in P$, we define the *support* of p as

$$\text{supp } p = \{x \in X : p(x) > 0\}$$

A preference relation $\succsim$ on P has an *expected utility representation* if there exists some $u : X \to \mathbb{R}$ such that

$$p \succsim q \iff \sum_{x \in \text{supp } p} u(x)p(x) \geq \sum_{y \in \text{supp } q} u(y)q(y)$$

People call u different things – I most often use *Bernoulli utility function*, but others call it a *payoff function*, especially in mechanism design contexts.

Example. *The St. Petersburg Paradox* A fair coin is tossed until tails. How much should you pay for a lottery ticket that pays 2^n dollars if tails appears first on the nth flip? The expected value is

$$EV = \frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 4 + \frac{1}{8} \cdot 8 + \cdots = 1 + 1 + 1 + \cdots = \infty$$

So, if you only care about expected value you should be willing to pay anything. What's happening here?

Assumption 1. *Assume that $\succsim$ is complete and transitive, that for all $0 < \alpha \leq 1$ and all $r \in P$, $p \succsim q \iff \alpha p + (1-\alpha)r \succsim \alpha q + (1-\alpha)r$* (independence)*, and that if $p \succ q \succ r$, there exists $\alpha, \beta \in (0,1)$ such that $\alpha p + (1-\alpha)r \succ q \succ \beta p + (1-\beta)r$* (the Archimedean property).

Theorem 1. *If $\succsim$ satisfies the above assumptions, then $\succsim$ has an expected utility representation $u : X \to \mathbb{R}$. Furthermore, if $v : X \to \mathbb{R}$ is another expected utility representation, then there exist constants $\alpha > 0$ and β such that $v(x) = \alpha \cdot u(x) + \beta$.*

Two Criticisms. First, the Archimedean property breaks if one outcome is infinitely preferred to another. This mathematically is odd, but not an unreasonable expectation in the real world. To fix this, we will often assume that X has no infinitely large or small elements.

Another criticism is independence. Are preferences linear in probabilities? The following is another famous paradox:

Example. *The Allais Paradox* Consider the following lotteries:

$$A = \begin{cases} \$1\text{M} & p = 1 \end{cases} \qquad B = \begin{cases} \$1\text{M} & p = 0.89 \\ \$5\text{M} & p = 0.10 \\ \$0 & p = 0.01 \end{cases}$$

$$C = \begin{cases} \$1\text{M} & p = 0.11 \\ \$0 & p = 0.89 \end{cases} \qquad D = \begin{cases} \$5\text{M} & p = 0.10 \\ \$0 & p = 0.90 \end{cases}$$

Most people prefer A to B, and prefer D to C. This violates the independence axiom.

Practice Questions

1. **2016 Prelim 2:** The set of prizes is $X = \{-1, 0, +1\}$ and a probability on these prizes is denoted by $p = (p_1, p_2, p_3)$. An individual strictly prefers a "small gamble", $p = (1/8, 3/4, 1/8)$, to certainty, $p = (0, 1, 0)$. However, the individual strictly prefers certainty to the "large gamble", $p = (1/2, 0, 1/2)$. Do this person's preferences have an objective expected utility representation? Explain.

2. **2014 June Q:** An individual has to decide how much of her wealth $w > 0$ to invest in a risky asset. This asset will have positive rate of return r with probability p, or a negative rate of return l with probability $1 - p$. So if the individual invests x dollars in the risk asset, the with probability p her wealth will be $w - x + (1 + r)x$ and with probability $1 - p$ her wealth will be $w - x + (1 + l)x$. Assume that the asset has a strictly positive expected rate of return $pr + (1 - p)l > 0$. Feasible investments in the risky asset are $x \geq 0$. Assume that this individual is an expected utility maximizer with Bernoulli payoff function $u(w)$ with $u' > 0$ and $u'' < 0$ for all non-negative wealths.

   (a) Show that the individual will invest a positive amount of wealth $x > 0$ in the risky asset.

   (b) It seems reasonable to suppose that as an individual's wealth increases he would invest more in the risk asset. Whether this is true or not depends on how the individual's risk aversion changes as their wealth changes. What is this relationship? This is, under what conditions on risk aversion does investment in the risky asset increase as wealth increases? [Hint: Absolute Risk Aversion of utility function $u(\cdot)$ at X is equal to $-u''(x)/u'(x)$]