ECON 6090

Section 8

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Fall 2025

Midterm Solutions

Problem 1

We will prove the claim, proceeding by double set containment.

Proof. ($\subseteq$) Take some $x \in C^\star(A \cup B, \succeq)$. We have that $x \succeq y\ \forall\ y \in A \cup B$. Additionally, since $C^\star(A \cup B, \succeq) \subseteq A \cup B$, we have that either $x \in A$, $x \in B$, or both. Without loss, let's say that $x \in A$. Then $x \succeq y\ \forall\ y \in A$ means that $x \in C^\star(A, \succeq)$. Furthermore, for all $z \in C^\star(B, \succeq)$, $z \in A \cup B$ so $x \succeq z$. This means that for all $y \in C^\star(A, \succeq) \cup C^\star(B, \succeq)$, $x \succeq y$. Thus, we must have that $x \in C^\star(C^\star(A, \succeq) \cup C^\star(B, \succeq), \succeq)$.

($\supseteq$) Take some $x \in C^\star(C^\star(A, \succeq) \cup C^\star(B, \succeq), \succeq)$. We have that, by the properties of choice correspondences $x \in C^\star(A, \succeq) \cup C^\star(B, \succeq)$. Without loss, let's say that $x \in C^\star(A, \succeq)$, so $x \in A \subseteq A \cup B$. It remains to show that $x \succeq y\ \forall\ y \in A \cup B$. Take some $y \in A$. Then $x \succeq y$ because $x \in C^\star(A, \succeq)$. Take some $z \in B$. Further, take some $y \in C^\star(B, \succeq)$ which is nonempty because $\succeq$ is rational. We have that, since $x \in C^\star(C^\star(A, \succeq) \cup C^\star(B, \succeq), \succeq)$, $x \succeq y$ and since $y \in C^\star(B, \succeq)$, $y \succeq z$. Thus, by transitivity $x \succeq z$, and for any $y \in A \cup B$, $x \succeq y$, so $x \in C^\star(A \cup B, \succeq)$. $\square$

Problem 2

1. We will find the value function and then use Roy's identity. First, to get the value function we use that $e(p, V(p, w)) = w$, so we have that

$$w = p_1 V(p, w) + g(\cdot) \Longrightarrow V(p, w) = \frac{w - g(p_2, \ldots, p_L)}{p_1}$$

Then, Roy's identity states that

$$x(p, w) = \frac{\nabla_p V(p, w)}{\partial V(p, w)/\partial w}$$

We will consider $x_1(p, w)$ and $x_i(p, w)$ for $i > 1$ separately. For the first good:

$$x_1(p, w) = \frac{g(p_2, \ldots, p_L) - w}{p_1}$$

For all other goods:

$$x_i(p, w) = g_i'(p_2, \ldots, p_L)$$

2. The income effect for the first good is

$$\frac{\partial x_1(p, w)}{\partial w} = \frac{1}{p_1}$$

The income effects for all other goods are zero:

$$\frac{\partial x_i(p, w)}{\partial w} = \frac{\partial}{\partial w}\Big[g_i'(p_2, \ldots, p_L)\Big] = 0$$

Thus, the total price effect for non-numeraire goods is

$$\frac{\partial x_i}{\partial p_j} = \frac{\partial h_i}{\partial p_j} + 0$$

The total price effect for the numeraire good is described by the Slutsky equation:

$$\frac{\partial x_1}{\partial p_j} = \frac{\partial h_1}{\partial p_j} + \frac{x_j}{p_1}$$

We do not need Hicksian theory to get a nice law of demand, as there are no income effects except for the numeraire good.

3. Recall that $e(p, V(p, w)) = w$. We thus have that

$$w_1 = p_1 \cdot V_1(p, w_1) + g_1(p_2, \ldots, p_L) \Longrightarrow V_1(p, w_1) = \frac{w_1}{p_1} - \frac{g_1(p_2, \ldots, p_L)}{p_1}$$
$$w_2 = p_1 \cdot V_2(p, w_2) + g_2(p_2, \ldots, p_L) \Longrightarrow V_2(p, w_2) = \frac{w_2}{p_1} - \frac{g_2(p_2, \ldots, p_L)}{p_1}$$

Since we can define $V_i(p, w_i) = a_i(p) + b(p) \cdot w_i$, where $b(p) = \frac{1}{p_1}$ is constant over all agents, we have that all indirect utility functions attain the Gorman form, which is a necessary and sufficient condition for a representative consumer to exist.

4. (not gonna do this here)

Problem 3

1. We have that the cost function solves the problem

$$C(w,q) = \min_{x,y\in\mathbb{R}^2_+} w_1 \cdot x + w_2 \cdot y \text{ s.t. } \min\{ax, by\} \geq q$$

Observe that, similarly to Leontief preferences, we must have that $ax = by$ at the optimum (otherwise, we could throw away some of the input we use more of and strictly pay less). Say that $y = \frac{ax}{b}$. Then the cost function is

$$C(w,q) = \min_{x,y\in\mathbb{R}^2_+} w_1 \cdot x + w_2 \cdot \frac{ax}{b} \text{ s.t. } ax \geq q$$

Since $a > 0$, we can say that the condition holds with equality, and $x = \frac{q}{a}$. Thus, we have that

$$C(w,q) = \min_{x,y\in\mathbb{R}^2_+} \frac{w_1 \cdot q}{a} + \frac{w_2 \cdot q}{b} = \frac{w_1 \cdot q}{a} + \frac{w_2 \cdot q}{b}$$

2. We have that $f(q)$ is homogeneous of degree $k < 1$, meaning that $f(\alpha z) = \alpha^k f(z)$. Our cost function is (by Problem Set 3) homogeneous of degree $\frac{1}{k} > 1$ in q. Take some q, q', and fix some $\alpha \in [0,1]$. We have that the cost function is convex if and only if

$$C(p, \alpha q + (1-\alpha)q') \leq \alpha C(p,q) + (1-\alpha)C(p,q')$$

Using the fact that the cost function is homogeneous of degree $1/k$, we can say that for any z, $C(p,z) = z^{1/k}C(p,1)$. We will use that fact to get that the above condition is equivalent to

$$(\alpha q + (1-\alpha)q')^{\frac{1}{k}} C(p,1) \leq \alpha \cdot q^{\frac{1}{k}} \cdot C(p,1) + (1-\alpha) \cdot (q')^{\frac{1}{k}} \cdot C(p,1)$$

which holds if and only if

$$(\alpha q + (1-\alpha)q')^{\frac{1}{k}} \leq \alpha \cdot q^{\frac{1}{k}} + (1-\alpha) \cdot (q')^{\frac{1}{k}}$$

Which holds if and only if the function $g(x) = x^{\frac{1}{k}}$ is convex for $x \geq 0$. Since $k < 1$, $1/k > 1$, so

$$g''(x) = \underbrace{\frac{1}{k}}_{>0} \cdot \underbrace{\left(\frac{1}{k} - 1\right)}_{>0} \cdot \underbrace{x^{\frac{1}{k}-2}}_{>0} > 0$$

so $g(x)$ is convex which implies that the cost function is convex in q.