ECON 6090

Section 7

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Problem Set 3 Solutions

1. Duality and Welfare with Quasi-Linear Preferences

   (a) We will apply Shephard's Lemma in the $i = 1$ case and in the $i \geq 2$ case. Recall that: $h(p, u) = \nabla_p e(p, u)$. We have that

   $$h_1(p, u) = u \qquad ; \qquad h_i(p, u) = g_i'(p_2, \dots, p_L)$$

   To find the Marshallian demand, we will apply the identities $w = e(p, V(p, w))$ and $x(p, w) = h(p, V(p, w))$. First, we get that the indirect utility function is defined by

   $$w = p_1 \cdot V(p, w) + g(p_2, \dots, p_L) \Longrightarrow V(p, w) = \frac{w - g(p_1, \dots, p_L)}{p_1}$$

   We then have that

   $$x_1(p, w) = V(p, w) = \frac{w - g(p_1, \dots, p_L)}{p_1} \qquad ; \qquad x_i(p, w) = g_i'(p_2, \dots, p_L)$$

   For non-numeraire goods, we have that $\frac{\partial x_i(p,w)}{\partial w} = 0$, which implies that the Marshallian and Hicksian demand curves are identical for these goods.

   (b) From above, we have that the indirect utility function is $V(p, w) = \frac{w - g(p_1, \dots, p_L)}{p_1}$. We use the duality relationship suggested in the hint. We normalize $p_1 = 1$ and substitute $w = p \cdot x = x_1 + \sum_{i=2}^{L} p_i x_i$ into the indirect utility function:

   $$V(1, p_{-1}, p \cdot x) = \left(x_1 + \sum_{i=2}^{L} p_i x_i \right) - g(p_2, \dots, p_L)$$

   Our goal is to find $u(x)$ by minimizing this expression with respect to the prices $(p_2, \dots, p_L)$. The expression can be split into two parts:

   $$u(x) = x_1 + \min_{p_2, \dots, p_L} \left\{ \sum_{i=2}^{L} p_i x_i - g(p_2, \dots, p_L) \right\}$$

The first term, x_1, is a constant with respect to the prices being minimized over. The second term is a well-defined minimization problem. Let's define the value of this second problem as a function ϕ:

$$\phi(x_2, \ldots, x_L) = \min_{p_2, \ldots, p_L} \left\{ \sum_{i=2}^{L} p_i x_i - g(p_2, \ldots, p_L) \right\}$$

This function ϕ takes the non-numeraire quantities as input and returns a real number. The first-order conditions for this minimization are $x_i - \frac{\partial g}{\partial p_i} = 0$ for $i = 2, \ldots, L$. Substituting this back, we find that the direct utility function must be of the form:

$$u(x) = x_1 + \phi(x_2, \ldots, x_L)$$

This is the definition of a quasi-linear utility function, which is additively separable in the numeraire good, x_1.

(c) The compensating variation, equivalent variation, and the change in consumer surplus are:

$$\text{CV} = \int_{p_2^1}^{p_2^0} h_2(p_1, p_2, p_{-1,2}, u^0)\, \partial p_2$$

$$\text{EV} = \int_{p_2^1}^{p_2^0} h_2(p_1, p_2, p_{-1,2}, u^1)\, \partial p_2$$

$$\Delta\text{CS} = \int_{p_2^1}^{p_2^0} x_2(p_1, p_2, p_{-1,2}, w)\, \partial p_2$$

We established earlier two relevant facts: that the Hicksian demand for non-numeraire goods does not depend on utility level, and that the Hicksian and Marshallian demands are identical for non-numeraire goods. These imply that the integrands are all the same, so conclusion follows.

2. Integral Form of Shephard's Lemma

(a) Verification is immediate.

(b) Let $\phi(\tau) = e(p(\tau), \bar{u})$. By the multivariable chain rule,

$$\frac{d\phi}{d\tau} = \nabla_p e(p(\tau), \bar{u}) \cdot \frac{dp(\tau)}{d\tau} = \nabla_p e(p(\tau), \bar{u}) \cdot (p' - p)$$

(c) We have that

$$\frac{d\phi}{d\tau} = h(p(\tau), \bar{u}) \cdot (p' - p).$$

(d) By the Fundamental Theorem of Calculus,

$$e(p', \bar{u}) - e(p, \bar{u}) = \phi(1) - \phi(0) = \int_0^1 \frac{d\phi}{d\tau}\, d\tau = \int_0^1 (p' - p) \cdot h(p(\tau), \bar{u})\, d\tau.$$

Substituting $p(\tau) = p + \tau(p' - p)$ yields the desired identity:

$$e(p', \bar{u}) - e(p, \bar{u}) = \int_0^1 (p' - p) \cdot h\big(p + \tau(p' - p), \bar{u}\big)\, d\tau.$$

3. Equivalence of WARP and Law of Compensated Demand

(a) Assume $x(p, w)$ satisfies WARP. We consider a compensated price change where the new wealth is defined as $w' = p' \cdot x^\star$. At the new budget (p', w'), the original bundle $x^\star$ is affordable by construction, since $p' \cdot x^\star = w'$. The consumer chooses x'. If $x' \neq x^\star$, WARP implies that at the original budget (p, w), the bundle x' must not have been affordable. This implication is expressed as the inequality $p \cdot x' > w$. By Walras's Law, we can state this as:

$$p \cdot x' > p \cdot x^\star$$

From the setup of a compensated price change, and using Walras's Law for the new choice, we also know:

$$p' \cdot x' = w' \quad \text{and} \quad p' \cdot x^\star = w' \implies p' \cdot x' = p' \cdot x^\star$$

Subtracting the inequality from the equality ($p' \cdot x' - p \cdot x' \leq p' \cdot x^\star - p \cdot x^\star$) yields:

$$(p' - p) \cdot x' \leq (p' - p) \cdot x^\star$$

$$(p' - p) \cdot (x' - x^\star) \leq 0$$

If $x' \neq x^\star$, the initial inequality from WARP is strict, which makes the final inequality strict. This proves the Law of Compensated Demand.

(b) We can show this direcctly, using the supplied facts that $p' \cdot x' = p' \cdot x = w'$ and $p \cdot x = w$. Start from the strict inequality for the Law of Compensated Demand (and clearly the weak version will follow):

$$\begin{aligned} (p' - p)(x' - x) &< 0 \\ (p' \cdot x' - p' \cdot x) - (p \cdot x' - p \cdot x) &< 0 \\ (w' - w') - (p \cdot x' - w) &< 0 \\ w &< p \cdot x' \end{aligned}$$

Which suffices to show that WARP holds.

4. The Gorman Form

(a) A quasi-linear consumer's problem can be solved in two stages. First, they allocate their budget between the numeraire good (x_1) and all other goods. For ***any*** amount of wealth w_{-1} spent on goods $2, \ldots, N$, they solve:

$$\max_{x_2,\ldots,x_N} \phi^i(x_2, \ldots, x_N) \quad \text{s.t.} \quad \sum_{j=2}^{N} p_j x_j \leq w_{-1}$$

Let the value of this sub-problem be an indirect utility function for the non-numeraire goods, $v^i_\phi(p_{-1}, w_{-1})$. The consumer's overall problem is then:

$$\max_{w_{-1}} \left\{ \frac{w - w_{-1}}{p_1} + v^i_\phi(p_{-1}, w_{-1}) \right\}$$

The first-order condition for the optimal expenditure w^*_{-1} is:

$$-\frac{1}{p_1} + \frac{\partial v^i_\phi(p_{-1}, w^*_{-1})}{\partial w_{-1}} = 0$$

This condition determines the optimal expenditure on non-numeraire goods, which depends only on their prices p_{-1}. Let's call the solution to this $w^*_{-1}(p_{-1})$. The total utility is then:

$$V^i(p, w^i) = \frac{w^i - w^*_{-1}(p_{-1})}{p_1} + v^i_\phi(p_{-1}, w^*_{-1}(p_{-1}))$$

We can rewrite this as:

$$V^i(p, w^i) = \underbrace{v^i_\phi(p_{-1}, w^*_{-1}(p_{-1})) - \frac{w^*_{-1}(p_{-1})}{p_1}}_{a^i(p)} + \underbrace{\left(\frac{1}{p_1}\right)}_{b(p)} w^i$$

This is exactly the Gorman form, $V^i = a^i(p) + b(p)w^i$. Crucially, the function $b(p) = 1/p_1$ is the same for all consumers, regardless of their specific utility function ϕ^i for the other goods.

(b) The Marshallian demands are $x_1 = \alpha_i w/p_1$ and $x_2 = (1-\alpha)w/p_2$. Substituting these into the utility function gives the indirect utility:

$$V_i(p_1, p_2, w) = \left(\frac{\alpha_i w}{p_1}\right)^{\alpha_i} \left(\frac{(1-\alpha_i)w}{p_2}\right)^{1-\alpha_i} = \left[\left(\frac{\alpha_i}{p_1}\right)^{\alpha_i} \left(\frac{1-\alpha_i}{p_2}\right)^{1-\alpha_i}\right] w$$

This is of the Gorman form $V_i(p, w) = a_i(p) + b_i(p)w$, where $a_i(p) = 0$ and $b_i(p) = {\alpha_i}^{\alpha_i}(1-\alpha_i)^{1-\alpha_i} p_1^{-\alpha_i} p_2^{-(1-\alpha_i)}$. Clearly, $b_i(p) = b(p)$, for some common $b(p)$ for all consumers, if and only if they all have $\alpha_i = \alpha$.

5. Properties of the Cost Function

(a) Increasing w by a factor of α is a positive monotonic transformation and therefore does not affect the optimal choice of z, but does increase $w \cdot z$ by that factor.

(b) Suppose $q' > q$. By free disposal, q can be produced from the same input vector used to produce q', so the minimizing input must be at least weakly lower.

(c) Homogeneity of degree k of f implies

$$f(z) = q \Longleftrightarrow \frac{1}{q} f(z) = 1 \Longleftrightarrow f\left(\frac{z}{q^{1/k}}\right) = 1$$

Therefore,

$$\begin{aligned}
C(w, q) &= \min_z w \cdot z \text{ st } f(z) = q \\
&= \min_z w \cdot z \text{ st } f\left(\frac{z}{q^{1/k}}\right) = 1 \\
&= q^{1/k} \min_z w \cdot \frac{z}{q^{1/k}} \text{ st } f\left(\frac{z}{q^{1/k}}\right) = 1 \\
&= q^{1/k} C(w, 1)
\end{aligned}$$