ECON 6090

Section 3

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1 Material

1.1 Utility Maximization Properties

Define the *utility maximization problem* for a consumer as

$$\max_{x \in \mathbb{R}^L_+} u(x) \text{ s.t. } p \cdot x \leq w$$

where we take the consumer's utility function $u(\cdot)$, the prices p, and the initial wealth w as given.

We define the *indirect utility function* as

$$V(p, w) = \max_{x \in \mathbb{R}^L_+} u(x) \text{ s.t. } p \cdot x \leq w$$

We define the *Marshallian demand correspondence* as

$$x(p, w) \in \operatorname*{argmax}_{x \in \mathbb{R}^L_+} u(x) \text{ s.t. } p \cdot x \leq w$$

Typical assumptions: $\succsim$ are locally non-satiated, u is continuous, $p \gg 0$, and $w > 0$.
Exercise. Assume that $u(x)$ has nice properties as necessary. Show that if $p'_i > p_i$ for all i, then $V(p', w) < V(p, w)$.

Under these assumptions, we have:
Proposition 1. *Let u be a continuous utility function representing $\succsim$ on $\mathbb{R}^L_+$.*

(i) If $p \in \mathbb{R}^L_{++}$ and $w \in \mathbb{R}_{++}$, then there exists an $x^\star \in \mathbb{R}^L_{++}$ that solves the consumer's problem

(ii) If $\lambda > 0$, then $x^\star$ also solves the consumer's problem for λp and λw (homogeneity of degree 0)

(iii) If in addition $\succsim$ is locally non-satiated, then Walras' Law holds, meaning that $p \cdot x^\star = w$

(iv) If in addition $\succsim$ is strictly convex (equiv. u strictly quasi-concave) then $x^\star$ is unique and the Walrasian demand function $x : \mathbb{R}^L_{++} \times \mathbb{R}_{++} \to \mathbb{R}^L_+$ is well-defined and continuous.

Proof.

(i) $B_{p,w}$ is nonempty and compact and u is continuous, so conclusion follows from the Extreme Value Theorem.

(ii) Observe that $p \cdot x \leq w \iff \lambda p \cdot x \leq \lambda w$, so the constraint set is the same in both problems.

(iii) Suppose not: $p \cdot x^\star < w$. Choose $\varepsilon > 0$ such that $p \cdot y < w$ for all $y \in B_\varepsilon(x^\star)$. By local non-satiation, there exists $y \in B_\varepsilon(x^\star)$ such that $y \succ x^\star$. This is a contradiction.

(iv) Suppose not: let $\hat{x}$ be a distinct solution. Fix $\alpha \in (0, 1)$. By strict convexity of preferences, $\alpha\hat{x} + (1 - \alpha)x^\star \succ x^\star$. By convexity of the budget set, $\alpha\hat{x} + (1 - \alpha)x^\star$ is affordable, contradicting that $x^\star$ is a global maximum. Continuity of x is annoying but proven elsewhere.

$\square$

Proposition 2. *The indirect utility function has the following properties: (i) continuous, (ii) nonincreasing in p_i, (iii) strictly increasing in w, (iv) quasiconvex, and (v) homogeneous of degree zero.*

Proof.

(i) In the case where the solution x is unique, $V = u \circ x$. We assumed continuity of u above, and continuity of x follows from the above proposition, as long as u is continuous. A full proof, when x is a correspondence, is omitted but follows from Berge's Theorem.

(ii) Fix i and suppose that $p'_i \geq p_i$. Then $B_{p',w} \subseteq B_{p,w}$, so $V(p', w) \leq V(p, w)$.

(iii) Suppose $w' > w$. Then $p \cdot x(p, w) < w'$, and by local non-satiation there exists $x' \succ x$ such that $p \cdot x' < w'$. Thus, $V(p, w') \geq u(x') > u(x(p, w)) = V(p, w)$.

(iv) Fix some $\alpha \in [0, 1]$ and some $(p, w), (p', w') \in \mathbb{R}^L_{++} \times \mathbb{R}_{++}$, and suppose that

$$x \in B(\alpha p + (1 - \alpha)p', \alpha w + (1 - \alpha)w')$$

Then we have that

$$\alpha(p \cdot x - w) + (1 - \alpha)(p' \cdot x - w') \leq 0 \implies x \in B_{p,w} \cup B_{p',w'}$$

Meaning that

$$B(\alpha p + (1 - \alpha)p', \alpha w + (1 - \alpha)w') \subseteq B_{p,w} \cup B_{p',w'}$$

Which implies that

$$V(\alpha p + (1 - \alpha)p', \alpha w + (1 - \alpha)w') \leq \max\{V(p, w), V(p', w')\}$$

So V is quasiconvex.

(v) This follows directly from x being homogeneous of degree 0.

$\square$

Proposition 3. *If u and x are continuously differentiable, then V is continuously differentiable and $\frac{\partial V}{\partial w} = \lambda$, where λ is the multiplier in $\mathcal{L}(\lambda, x) = u(x) - \lambda(p \cdot x - w)$.*

Proof. In class. $\square$

1.2 Expenditure Minimization Properties

The *expenditure minimization problem* is the optimization problem

$$\min_{x \in \mathbb{R}^L_+} p \cdot x \text{ s.t. } u(x) \geq \bar{u}$$

The *expenditure function* is defined by

$$e(p, \bar{u}) \coloneqq \min_{x \in \mathbb{R}^L_+} p \cdot x \text{ s.t. } u(x) \geq \bar{u}$$

The *Hicksian demand correspondence* gives solutions to the expenditure minimization problem:

$$h(p, \bar{u}) \coloneqq \operatorname*{argmin}_{x \in \mathbb{R}^L_+} p \cdot x \text{ s.t. } u(x) \geq \bar{u}$$

Proposition 4. *Assume that preferences are continuous. Then:*

(i) If $u(0) \leq \bar{u} \leq \sup_{x \in \mathbb{R}^L_+} u(x)$, where the right hand side is possibly infinite, then there exists $x^\star \in \mathbb{R}^L_+$ that solves the expenditure minimization problem.

(ii) If $\lambda > 0$, then this $x^\star$ also solves the consumer's problem for λp (homogeneity of degree 0).

(iii) If $x^\star$ solves the expenditure minimization problem, then $u(x^\star) = \bar{u}$.

(iv) If in addition, $\succsim$ is strictly convex then $x^\star$ is unique and the Hicksian demand function $h : \mathbb{R}^L_{++} \times \mathbb{R} \to \mathbb{R}^L_+$ is well-defined and continuous.

Proof.

(i) By the continuity of u and the Intermediate Value Theorem, there exists $x^0 \in \mathbb{R}^L_+$ such that $u(x^0) = \bar{u}$. We can then restrict the constraint set without changing the solution to $\{x \in \mathbb{R}^L_+ : u(x) \geq \bar{u} \text{ and } p \cdot x \leq p \cdot x^0\}$. This set is nonempty and compact, so conclusion follows from the Extreme Value Theorem.

(ii) This follows directly from the fact that $p \cdot x^\star \geq p \cdot x \iff \lambda p \cdot x^\star \geq \lambda p \cdot x$.

(iii) Suppose FSOC that $u(x^\star) > \bar{u}$. Then by continuity there exists $x \neq x^\star$ such that $x \leq x^\star$ and $\bar{u} \leq u(x) < u(x^\star)$. Since $p \in \mathbb{R}^L_{++}$, this implies that x is in the constraint set and attains a lower cost than $x^\star$, contradicting the fact that $x^\star$ is a global minimum.

(iv) Suppose FSOC that there exist $x_1^\star$ and $x_2^\star$ both (distinct) global optima, implying that $p \cdot x_1^\star = p \cdot x_2^\star$. By linearity, this means that taking some $\alpha \in (0,1)$, we have that $p \cdot (\alpha x_1^\star + (1-\alpha)x_2^\star) = p \cdot x_1^\star$, but by strict convexity we have that $u(\alpha x_1^\star + (1-\alpha)x_2^\star) > u(x_1^\star) \geq \bar{u}$, contradicting (iii). Continuity and existence follow from Berge's Theorem.

$\square$

Proposition 5. (Properties of e)

(i) *Continuous*

(ii) *Nondecreasing in* p_i *for* $i \in \{1, \ldots, L\}$

(iii) *Strictly increasing in* $\bar{u}$

(iv) *Homogeneous of degree 1 in* p

(v) *Concave in* p

Proof.

(i) Follows directly from Berge's Theorem

(ii) Let $p' \geq p$ and $x' \in H(p', \bar{u})$. Then $e(p', \bar{u}) = p' \cdot x' \geq p \cdot x' = e(p, \bar{u})$

(iii) Same as the proof of (iv) in Proposition 4 above.

(iv) Follows directly from H being homogeneous of degree 0

(v) Let $p'' \coloneqq \alpha p + (1-\alpha)p'$ for some $\alpha \in [0,1]$, $p, p' \in \mathbb{R}^L_{++}$, and $x'' \in H(p'', \bar{u})$. Then

$$e(p'', \bar{u}) = p'' \cdot x'' = \alpha p \cdot x'' + (1-\alpha)p' \cdot x'' \geq \alpha e(p, \bar{u}) + (1-\alpha)e(p', \bar{u})$$

$\square$

2 Practice Questions

1. (2009 Prelim 1) There are three goods with quantities denoted by $x = (x_1, x_2, x_3) \in \mathbb{R}^3_+$. A consumer's preferences can be represented by the utility function $u(x) = x_1 x_2^{1/2} x_3^{1/2}$. The prices of the goods are represented by $p \in \mathbb{R}^3_{++}$ and the consumer has wealth $w > 0$.

   (a) Write the consumer's decision problem as a constrained optimization problem.

   (b) Find the consumer's demand functions for the three goods.

   Now suppose that in addition to using money to purchase goods the consumer also has to provide coupons in order to make a purchase. The purchase of $y \geq 0$ units of any good requires y coupons. The consumer has an endowment of $c > 0$ coupons.

   (c) Write the consumer's new decision problem as a constrained optimization problem.

   (d) Is it possible that at a solution to the consumer's problem he has some left-over coupons? That is, can the coupon constraint ever be non-binding?

   (e) Suppose that $p = (1, 1, 1)$. Find the consumer's demands for the three goods.