ECON 6090

Section 2

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Fall 2025

1 Material

Choice Theory

Properties of preference relations We will use the following a lot in choice theory:

Definition. The preference relation $\succsim$ is *continuous* on X if the upper and lower contour sets are closed in X for all $x \in X$.

Or:

Definition. If X is a subset of a complete metric space (note the real numbers suffice), then $\succsim$ is *continuous* if and only if for all sequences $x_n \to x$ and $y_n \to y$, where $x_n \succsim y_n \; \forall \, n$, then $x \succsim y$.

Definition. The preference relation $\succsim$ is *monotonic* on $X \subseteq \mathbb{R}^n$ if $x \geq y \implies x \succsim y$. The preference relation $\succsim$ is *strictly monotonic* on $X \subseteq \mathbb{R}^n$ if $x \geq y$ and $x \neq y \implies x \succ y$.

Remark. What does this mean we can say about $(1, 2)$ and $(2, 1)$? More generally, what if two elements are not ordered?

and remember from last time:

Definition. The preference relation $\succsim$ is *rational* on X if it is both:

- *complete* on X, meaning that for all $x, y \in X$, either $x \succsim y$, $y \succsim x$, or both;
- *transitive* on X, meaning that for all $x, y, z \in X$, $x \succsim y$ and $y \succsim z \implies x \succsim z$

The main Choice Theory result for this week:

Theorem 1. (Debreu's Theorem) *If $\succsim$ is rational and continuous, then it is representable by a continuous real-valued utility function $u(\cdot)$.*

Proof. Presented in class, requires strict monotonicity. The full version is in Debreu (1954), but it's heavily topological and not particularly intuitive. $\square$

Two more definitions:

Definition. The preference relation $\succsim$ is *convex* on (convex) X if, for all $x, y, z \in X$ and all $\alpha \in [0, 1]$, $z \succsim x$ and $y \succsim x \implies \alpha \cdot z + (1 - \alpha) \cdot y \succsim x$. The preference relation $\succsim$ is *strictly convex* on X if, for any $\alpha \in (0, 1)$, $z \succsim x$ and $y \succsim x \implies \alpha \cdot z + (1 - \alpha) \cdot y \succ x$.

Definition. The function $u : \mathbb{R}^n \to \mathbb{R}$ is *quasiconcave* if for all $x, y \in \mathbb{R}^n$ and $\alpha \in [0,1]$,

$$u(\alpha \cdot x + (1-\alpha) \cdot y) \geq \min\{u(x), u(y)\}$$

Theorem 2. *$\succsim$ is convex if and only if $u(\cdot)$ representing $\succsim$ is quasiconcave.*

Proof. We're going to show a different proof based on an alternative definition of quasiconcavity:
Definition. A function $f : \mathbb{R}^L \to \mathbb{R}$ is *quasiconcave* if and only if the set

$$\{x \in \mathbb{R}^L : f(x) \geq r\}$$

is convex for all $r \in \mathbb{R}$.

($\Rightarrow$): Fix some $x, y \in \mathbb{R}^n$ and some $\alpha \in [0,1]$. WLOG, say that $y \succsim x$[1]. Then from reflexivity, $x \succsim x$, and since $y \succsim x$, from convexity of $\succsim$ we have that $\alpha \cdot x + (1-\alpha) \cdot y \succsim x$. Since $u(\cdot)$ represents $\succsim$, $u(y) \geq u(x)$ and $u(\alpha \cdot x + (1-\alpha) \cdot y) \geq u(x)$. Thus,

$$u(\alpha \cdot x + (1-\alpha) \cdot y) \geq \min\{u(x), u(y)\}$$

so $u(\cdot)$ is quasiconcave.

($\Leftarrow$): Fix some $x, y, z \in X$ and say that $z \succsim x$ and $y \succsim x$. Then $u(z) \geq u(x)$ and $u(y) \geq u(x)$. Setting $r = u(x)$, this means that $z, y \in \{a \in \mathbb{R}^L : u(a) \geq r\}$. Since that set is convex, for any $\alpha \in [0,1]$ it must be the case that $\alpha \cdot z + (1-\alpha) \cdot y \in \{a \in \mathbb{R}^L : u(a) \geq r\}$, which means that $u(\alpha \cdot z + (1-\alpha) \cdot y) \geq u(x)$, which implies that $\alpha \cdot z + (1-\alpha) \cdot y \succsim x$, so $\succsim$ is convex. $\square$

Some more properties of preference relations and utility functions:

If preferences are...	**then**	**utility representation(s) satisfy...**
monotonic on $X \subseteq \mathbb{R}^n$, meaning that $x \geq y \iff x \succsim y$	all	*monotonicity* of u: $x \geq y \iff u(x) \geq u(y)$
continuous on $X \subseteq \mathbb{R}^n$, meaning that $x_n \to x$, $y_n \to y$, $x_n \succsim y_n \; \forall \, n \implies x \succsim y$.	there exists	*continuity* of u, where $x_n \to x \implies u(x_n) \to u(x)$
local non-satiated on $X \subseteq \mathbb{R}^n$, meaning that $\forall \, x \in X$ and $\varepsilon > 0$, $\exists \, y \in B_\varepsilon(x)$ s.t. $y \succ x$	all	*no local maxima* of u, meaning that $\forall \, x \in X$ and $\varepsilon > 0$, $\exists \, y \in B_\varepsilon(X)$ s.t. $u(y) > u(x)$
convex on convex X, meaning that for $z \succsim x$, $y \succsim x$, and $\alpha \in [0,1]$, $\alpha z + (1-\alpha) y \succsim x$	all	*quasiconcavity* of u, meaning that for $\alpha \in [0,1]$, $u(\alpha x + (1-\alpha) y) \geq \min\{u(x), u(y)\}$
separable on $X \times Y$, meaning that $\forall$ $x, x' \in X$ and $y, y' \in Y$, $(x, y) \succsim (x', y) \iff (x, y') \succsim (x', y')$	there exists	*separability* of u, meaning that $\exists \, v, U$ such that $u(x, y) = U(v(x), y)$ and U is increasing in the first argument
homothetic on $X \subseteq \mathbb{R}^n$, meaning that for $\lambda > 0$, $x \succsim y \iff \lambda x \succsim \lambda y$	there exists	*homogeneity of degree 1* of u, meaning that $\forall \, \lambda > 0$, $u(\lambda x) = \lambda u(x)$.

[1] We could just rename them if the opposite was true – recall that we have rationality here!

2 Practice Questions

Q1

Are the following preference relations $\succsim$ rational?

(a) Let $\succsim$ be defined on $\mathbb{R}$ by: $y \succsim x$ iff $y \geq x + \varepsilon, \varepsilon$ is a positive number.

(b) Let $\succsim$ be defined on $\mathbb{R}$ by: $y \succsim x$ iff $y \geq x - \varepsilon, \varepsilon$ is a positive number.

(c) $X = \{a, b, c\}. C^*(\{a, b\}, \succsim) = \{b\}. C^*(\{b, c\}, \succsim) = \{c\}. C^*(\{a, b, c\}, \succsim) = \{c\}.$

(d) Agents 1 and 2 are facing the same choice set X. Agent 1 has a rational preference relation $\succsim_1$, consumer 2 's preference relation is given by $\succsim_2 := \succ_1$. Is consumer 2 's preference rational?

(e) Consider the lexicographic preference relation $\succsim$ on $\mathbb{R}_+^2 : (x_1, x_2) \succsim (y_1, y_2)$ if and only if $x_1 > y_1$ or $x_1 = y_1$ and $x_2 \geq y_2$. Is $\succsim$ a rational preference relation?

Q2

Definition. A preference relation $\succsim$ on $\mathbb{R}_+^n$ is called *homothetic* if for all $x, y \in \mathbb{R}_+^n$ and all $\lambda > 0$, $x \succsim y$ if and only if $\lambda x \succsim \lambda y$.
Definition. A function $u : \mathbb{R}_+^n \rightarrow \mathbb{R}$ is called *homogeneous of degree one* if for all $x \in \mathbb{R}_+^n$ and all $\lambda > 0$, $u(\lambda \cdot x) = \lambda \cdot u(x)$.

Show that a continuous strictly monotone preference relation $\succsim$ on $\mathbb{R}_+^n$ is homothetic if and only if it can be represented by a utility function which is homogeneous of degree one.