ECON 6090

Section 1

Gabe Sekeres

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1 Material

Preference Relations

Definition. A *preference relation* is a binary relation over a set of alternatives X, denoted by $\succsim \subseteq X^2$, where for $x, y \in X$, $x \succsim y$ is interpreted as *'x is at least as good as y'*, or *'x is weakly preferred to y'*.

We say that $x \succ y$ if $x \succsim y$ and $y \not\succsim x$ (*'x is strictly preferred to y'*), and we say that $x \sim y$ if $x \succsim y$ and $y \succsim x$ (*'x is indifferent to y'*).

Definition. The preference relation $\succsim$ is *complete* on X if for all $x, y \in X$, either $x \succsim y$, $y \succsim x$, or both.

Definition. The preference relation $\succsim$ is *transitive* on X if for all $x, y, z \in X$,

$$x \succsim y \text{ and } y \succsim z \Longrightarrow x \succsim z$$

Definition. For any subset $B \subseteq X$, define the *choice correspondence* as $C^\star : B \rightrightarrows B$, where

$$C^\star(B, \succsim) \coloneqq \{x \in B \mid x \succsim y \;\forall\, y \in B\}$$

Remark. $C^\star(B, \succsim) \subseteq B$.

We can get the following results, built only from these definitions:

Definition. A preference relation is called *rational* if it is complete and transitive.

Proposition 1. *For any nonempty $B \subset X$, $\succsim$ being rational $\Longrightarrow C^\star(B, \succsim) \neq \emptyset$.*

Proof. By induction, in the notes. $\square$

Revealed Choice

Definition. The *power set* $\mathcal{P}(X)$ is the set of all (nonempty) subsets of X.

Definition. A *choice correspondence* $C : \mathcal{P}(X) \to \mathcal{P}(X)$ is a correspondence with the property that, for some $\mathcal{D} \subseteq \mathcal{P}(X)$, for any $A \in \mathcal{D}$, $C(A) \subseteq A$ and $C(A) \neq \emptyset$. We define a *choice structure* as $\mathscr{S} = (\mathcal{D}, C)$.

Axiom 1. (WARP) *The choice structure $\mathscr{S}$ satisfies the Weak Axiom of Revealed Preference if and only if*

$$\forall\ A, B \subseteq \mathcal{D};\ x, y \in A \cap B;\ x \in C(A);\ y \in C(B) \Longrightarrow x \in C(B)$$

Alternatively:
Axiom 2. (WARP 2) *The choice structure $\mathscr{S}$ satisfies the Weak Axiom of Revealed Preference if*

$$\exists\ A \in \mathcal{D};\ x, y \in A \text{ s.t. } x \in C(A), y \notin C(A) \Longrightarrow\ \forall\ B \in \mathcal{D} \text{ s.t. } x, y \in B, y \notin C(B)$$

Axiom 3. (Sen's α, IIA) *The choice structure $\mathscr{S}$ satisfies Independence of Irrelevant Alternatives if*

$$\forall\ A, B \in \mathcal{D};\ A \subseteq B;\ x \in A;\ x \in C(B) \Longrightarrow x \in C(A)$$

Axiom 4. (Sen's β, EC) *The choice structure $\mathscr{S}$ satisfies Expansion Consistency if*

$$\forall\ A, B \in \mathcal{D};\ A \subseteq B;\ x, y \in C(A);\ x \in C(B) \Longrightarrow y \in C(B)$$

Proposition 2. *If $\forall\ A, B \in \mathcal{D}, A \cap B \in \mathcal{D}$, then WARP $\Longleftrightarrow$ Sen's α and β*

Proof. ($\Longleftarrow$) in the notes, ($\Longrightarrow$): We will take α and β in turn:

1. WARP $\Longrightarrow \alpha$: Say that we have some $A, B \subseteq \mathcal{D}$ where $A \subseteq B$, $x \in A$, and $x \in C(B)$. Towards a contradiction, assume that $x \notin C(A)$. Then there exists some $y \in C(A)$ where $y \neq x$. Since $A \cap B = A$, we have that $x, y \in A \cap B$, $x \in C(B)$, and $y \in C(A)$, but $x \notin C(A)$. This is a contradiction of WARP, so it must be the case that the choice structure satisfies Sen's α.

2. WARP $\Longrightarrow \beta$: Say that we have some $A, B \subseteq \mathcal{D}$ where $A \subseteq B$, $x, y \in C(A)$, and $x \in C(B)$. Since $A \cap B = A$, we hve that $x, y \in A \cap B$, $y \in C(A)$, and $x \in C(B)$. Then by WARP, we must have that $y \in C(B)$, so the choice structure satisfies Sen's β.

$\square$

Definition. We say that x is *revealed preferred* to y, according to the choice structure $\mathscr{S}$, denoted as $x \succsim_{\mathscr{S}} y$, if there exists some $A \in \mathcal{D}$ such that $x, y \in A$ and $x \in C(A)$.
Proposition 3. *The following are equivalent as long as $\mathcal{D} = \mathcal{P}(X)$:*

1. *$\mathscr{S}$ satisfies WARP*

2. *$\mathscr{S}$ satisfies Sen's α and β*

3. *$\succsim_{\mathscr{S}}$ is rational and $C(B) = C^{\star}(B. \succsim_{\mathscr{S}})\ \forall\ B \in \mathcal{D}$*

Proof. WARP $\Longrightarrow$ rational in the notes, the rest left as an exercise. $\square$

2 Practice Question

1. Prove the following statements about preference relations:
   (a) If $\succsim$ is transitive, then $\succ$ is also transitive.
   (b) If $\succsim$ is transitive, then $\sim$ is also transitive.
   (c) If $\succsim$ is complete and transitive, then $\succsim$ is *negatively transitive*: if $x \succsim y$ then for any z, either $x \succsim z$ or $z \succsim y$ or both.
2. Let X be a finite set of alternatives and $\mathcal{B}$ the set of all nonempty subsets of X. Choice rule $C : \mathcal{B} \to \mathcal{B}$ satisfies *path independence* if for all $B, B' \in \mathcal{B}$, $C(B \cup B') = C(C(B) \cup C(B'))$.[1] Prove that any choice rule that is rationalized by a complete and transitive preference relation satisfies path independence.

[1] This basically means that the choice is not affected if we first split the set into smaller sets, choose the best from each of those, and choose the best from the best of the smaller sets. Intuitively, this should be the same as choosing the best of the entire set.