ECON 6140

Macroeconomics Notes

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1.1 Introduction

Administration Homeworks are due at the beginning of the lecture, and no late homework will be accepted. However, only $N - 1$ will be evaluated. Please feel free to work in groups, but turn in your own write-up. The midterm (like last semester, the final for this portion of the course) will be on Tuesday, March 11. Julieta will look for a room so we can take a full two hours. This course will occasionally refer to the notes by Dirk Krueger, as well as the same textbooks as last semester, Recursive Methods in Macroeconomic Dynamics (SLP) and Recursive Macroeconomic Theory (LS).

This first week will be somewhat review, but the pace will pick up quickly. You need to do the homeworks. Homeworks will be heavy, but that's because the way you study is by doing them. Remember: at this point in your career, the grades are completely useless. They function as a progress marker, but in and of themselves they don't matter at all. They don't make you a good economist.

Motivation Why are people in the US today much richer than they were in 1800? Why are Germany and France much richer than Argentina and Kenya? Does growth generate inequality? What is the role of frictions in hindering growth?

Macroeconomists aim at answering these questions building *quantitative* models – models that can be contested with empirical facts. "The weight of evidence for an extraordinary claim must be proportioned to its strangeness" – Laplace, 1812. We will think about abstractions, yes, but *useful* (read: empirically testable) abstractions.

The main questions in this course are:

1. Why are some countries richer than others?
2. Why do some countries grow faster than others?
3. What is the effect of wealth and consumption inequality for cross-country differences in income per capita?
4. What is the role of financial frictions?
5. What is the role of education for long-run growth?
6. How do firms innovate, and what's their impact on economic growth?
7. Why do some firms operate older technologies while better ones are available?

Broadly, we will consider (in order):

1. One-sector growth model (Ramsey-Cass-Koopmans)
   (a) Computation
   (b) Extensions to multiple sectors
2. Competitive equilibrium
   (a) Heterogeneity and aggregation
   (b) The income fluctuations problem, incomplete markets
3. Overlapping generations
   (a) Recursive representation

(b) Dynamic inefficiencies

4. Long-run growth

   (a) Human capital

   (b) AK model

   (c) Endogenous growth: externalities and innovation

Macroeconomics, more than anything, is a course on the *long-term*. It's not necessarily about monetary policy, or trade, or anything specific, but more the study of behavior over a long time horizon. It's the only field able to study topics like climate change and demographic change for that reason.

1.2 Growth Model

"Who does what, when" – Sargent, on models. Consider: Time, Preferences, Technology.
Model. *One-sector Growth Model (redux)*

- Time: Discrete, infinite horizon
- Preferences: representative dynasty preferences

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

  where $\beta \in (0,1)$ and u strictly concave, increasing, and differentiable

- Technology:

$$c_t + x_t + g_t \leq f(k_t)$$
$$k_{t+1} \leq x_t + (1-\delta)k_t$$

  where x_t is investment, g_t is government spending, and $f(k_t)$ is output. $f(k_t)$ is concave, strictly increasing, and $f(0) = 0$. We also have the following two conditions: $\lim_{k \to 0} f'(k_t) > \frac{1}{\beta} - (1-\delta)$, and $\lim_{k \to \infty} f'(k_t) < \frac{1}{\beta} - (1-\delta)$

Question. Why is this called a one-sector growth model? It looks like there are two sectors!

Because the marginal rate of transformation between consumption and next-period capital (investment) is one-to-one.

Assumption 1.1. *All quantities must be non-negative.*

We have the following maximization problem, given a sequence of $\{g_t\}$ and a k_0:

$$\max_{\{c_t, x_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \text{ subject to}$$

$$c_t + x_t + g_t \leq f(k_t)$$
$$k_{t+1} \leq x_t + (1-\delta)k_t$$
$$(c_t, x_t, k_{t+1}) \geq (0,0,0)$$

Definition. An *allocation* is a set of sequences $\{c_t, x_t, k_{t+1}\}$. An allocation is *feasible* if it satisfies

$$c_t + x_t + g_t \leq f(k_t)$$

$$k_{t+1} \leq x_t + (1-\delta)k_t$$

$$(c_t, x_t, k_{t+1}) \geq (0, 0, 0)$$

How will we solve this? We consider it as a finite horizon problem, and then derive (and apply) a transversality condition. We will solve the finite problem by an application of the KKT conditions, where the Lagrangian is

$$\mathcal{L}(c, x, k, \lambda, \theta, \gamma) = \sum_{t=0}^{T} \beta^t \{u(c_t) + \lambda_t[f(k_t) - (c_t + x_t + g_t)] + \theta_t[x_t + (1-\delta)k_t - k_{t+1}] + \ldots\}$$

where ... are the non-negativity conditions with γ multipliers (trivially will hold).

The first-order necessary conditions are:

$$\begin{aligned}
c_t &: u'(c_t) - \lambda_t + \gamma_{1t} = 0 && t = 0, 1, \ldots, T \\
x_t &: -\lambda_t + \theta_t + \gamma_{2t} = 0 && t = 0, 1, \ldots, T \\
k_{t+1} &: -\theta_t + \gamma_{3t} + \beta\lambda_{t+1} f'(k_{t+1}) + \beta(1-\delta)\theta_{t+1} = 0 && t = 0, 1, \ldots, T \\
k_{T+1} &: -\theta_T + \gamma_{3T} = 0
\end{aligned}$$

as well as complimentary slackness conditions. The region we think about the finite horizon is that this last period is different – you're going to die, so there is no payoff for accumulating capital. We will use the complimentary slackness condition for time $t = T$ to construct the transversality condition. We need the agent to not want to accrue capital forever, for some payoff long in the future. The complimentary slackness condition is that $\beta^T k_{T+1} \gamma_{3T} = 0$. We can convert this to $\beta^T \theta_T k_{T+1}$, and the transversality condition will hold as long as $\lim_{T \to \infty} \beta^T \theta_T k_{T+1} = 0$.
Remark. Most importantly!! Never repeat this approach outside of this course.

Our first order necessary conditions are now:

$$\begin{aligned}
c_t &: u'(c_t) = \lambda_t \\
x_t &: \theta_t = \lambda_t \\
k_{t+1} &: \beta\lambda_{t+1}[f'(k_{t+1} + (1-\delta)] = \lambda_t \\
\text{TC} &: \lim_{T \to \infty} \beta^T \theta_T k_{T+1} = 0 \\
f &: c_t + x_t + g_t \leq f(k_t) \\
k_t &: k_{t+1} \leq x_t + (1-\delta)k_t
\end{aligned}$$

Rearranging, we get

$$\begin{aligned}
k_{t+1} &: \beta u'(c_{t+1})[f'(k_{t+1}) + (1-\delta)] = u'(c_t) \\
\text{TC} &: \lim_{T \to \infty} \beta^T \theta_T k_{T+1} = 0
\end{aligned}$$

From Micro 101, we can recognize that

$$\begin{aligned}
\frac{u'(c_t)}{\beta u'(c_{t+1})} &= [f'(k_{t+1}) + (1-\delta)] \\
MRS &= MRT
\end{aligned}$$

Now suppose that $g_t = g \; \forall \; t$. We have the following:
Definition. A *steady state* is an allocation such that, for all t, $c_t = c$, $x_t = x$, and $k_{t+1} = k$. At the steady

state, our first order necessary conditions become

$$\begin{aligned} c_t &: u'(c^\star) = \lambda^\star \\ k_{t+1} &: \beta[f'(k^\star) + (1-\delta)] = 1 \\ f &: c^\star + x^\star + g = f(k^\star) \\ k_t &: \delta k^\star = x^\star \end{aligned}$$

Simplifying, we have that

$$\beta[f'(k^\star) + (1-\delta)] = 1 \qquad \text{(Euler)}$$
$$c^\star + \delta k^\star + g = f(k^\star) \qquad \text{(Feasibility)}$$

Essentially, the level of capital is fully determined by the production technology. To know consumption, we just need the feasible consumption rate.
Question. Does this steady state necessarily exist? If so, is it unique?
Solution. Define $L(k) = \beta[f'(k^\star)+(1-\delta)]$. We know from the assumptions (way above) that $\lim_{k\to 0} L(k) > 1$ and $\lim_{k\to\infty} L(k) < 1$. So what else do we need? Just continuity! Which we have from our assumption that this was a constant returns to scale production function in capital and labor, so decreasing in labor, so f' is decreasing! Conclusion follows from the Intermediate Value Theorem – we have both existence and uniqueness.

Are we done yet? We need to check feasibility! Is $c^\star > 0$? Suppose that $g = 0$. We need to check $f'(k) < \delta$ at the point $\hat{k}$ such that $f(\hat{k}) = \delta\hat{k}$, which implies that $\hat{c} = 0$. However, for all $k < \hat{k}$, $f(k) > \delta k$. It remains to show that $k^\star < \hat{k}$, which follows directly because:

$$f'(k^\star) = \frac{1}{\beta} - (1-\delta) > \delta$$

for $\beta \in (0, 1)$.

1.3 Dynamics in the One-Sector Growth Model

Suppose now that we have a technology level z so that output is $zf(k)$, which is analogous to total factor productivity. Our first order conditions are now

$$\begin{aligned} c_t &: u'(c_t) = \lambda_t \\ k_{t+1} &: \beta\lambda_{t+1}[zf'(k_{t+1}) + (1-\delta)] = \lambda_t \\ \text{TC} &: \lim_{t\to\infty} \beta^T \lambda_T k_{T+1} = 0 \\ f &: c_t + x_t + g_t \leq zf(k_t) \\ k_t &: k_{t+1} \leq x_t + (1-\delta)k_t \end{aligned}$$

Remark. One could describe the dynamics in terms of consumption, which is what most books do, or in terms of shadow value, which is what we will do.

We define $u'(c(\lambda)) \equiv \lambda$ and will assume $g_t = g$ sufficiently 'small'[1] for now. Our key conditions are, as before

$$\beta\lambda_{t+1}[zf'(k_{t+1}) + (1-\delta)] = \lambda_t \qquad \text{(Euler)}$$
$$c_t + k_{t+1} - (1-\delta)k_t + g_t) \leq zf(k_t) \qquad \text{(Feasibility)}$$

[1]Basically, that we will not run into the non-negativity constraint on consumption.

We can solve this using a phase diagram, which we saw last semester. We can't explicitly solve an *infinite* system of non-linear equations explicitly, but we can do it using programs – specifically, for this QuantEcon has a good codebase.

In this case, we will use finite difference methods – specifically, the shooting algorithm. We will approximate k and c (or λ, in our case) with N discrete points in the time dimension. Denote the distance between grid points as Δt. Let (k_n, c_n) be a point on the grid, and use the equations that characterize the optimum with k_0 given:

$$\begin{aligned} c_{n+1} &= \beta c_n [z f'(f(k_n) - (1-\delta)k_n - c_n) + (1-\delta)] \\ k_{n+1} &= f(k_n) - (1-\delta)k_n - c_n \end{aligned}$$

Remark. Why is this linear in c? We are explicitly assuming log utility here, that's how we convert it. Any CARA utility would have an exponent, but we assume log for simplicity.

Explicitly, we have:
Algorithm. *Shooting Method*

1. Guess c_0
2. Obtain (c_n, k_n) for $n = 1, \ldots, N$ by running the equations above forward
3. If the sequence converges to $(c^\star, k^\star)$ then you have the correct saddle path. If not, update c_0 and go back to 1

This will converge because the solution is unique – there is explicitly only one steady state, and since the problem is convex there's a unique path to that steady state.

Now take the sequential planner's problem, which is

$$\max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t$$

$$c_t, k_{t+1} \geq 0 \text{ and } k_0 > 0 \text{ given}$$

We have the value function

$$V(k_0) = \max_{\{(c_t, k_{t+1}) \in \Gamma(k_t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

where

$$\Gamma(k_t) = \{(c_t, k_{t+1}) : c_t \geq 0, k_{t+1} \geq 0, c_t + k_{t+1} \leq f(k_t) + (1-\delta)k_t\}$$

Our recursive formulation is

$$V(k_0) = \max_{(c_0, k_1) \in \Gamma(k_0)} u(c_0) + \beta V(k_1)$$

In general,

$$V(k) = \max_{(c, k') \in \Gamma(k)} u(c) + \beta V(k')$$

where k is the state variable (or set of state variables). As long as u is differentiable, the solution to this problem satisfies

$$k' = f(k) + (1-\delta)k - c \qquad \text{and} \qquad \frac{\partial u(c)}{\partial c} = \beta \frac{\partial V(k')}{\partial k}$$

From the envelope condition, we can recover the Euler equation:

$$\frac{\partial V(k')}{\partial k} = \frac{\partial u(c')}{\partial c}\left[z\frac{\partial f(k')}{\partial k} + (1-\delta)\right]$$

As we saw last semester, the solution to this problem is a policy function $g : K \to K$ such that $k' = g(k)$ and a value function $V^\star : K \to \mathbb{R}_+$. We will approximate g and $V^\star$ numerically, with methods that differ depending on the characteristics of the problem. In general, our algorithm will be either value function iteration, or finding policy functions from the Euler equations. Both of these *should* give us the same solutions.

Numerical Solutions. We have a set of non-linear equations to be solved over the state space K (these are our Euler equations):

$$\frac{\partial u(c)}{\partial c} = \beta\frac{\partial u(c')}{\partial c}\left[z\frac{\partial f(k')}{\partial k} + (1-\delta)\right]$$
$$\implies \frac{\partial u(f(k)+(1-\delta)k-k')}{\partial c} = \beta u'(f(k')+(1-\delta)k'-k'')\left[\frac{\partial f(k')}{\partial k} + (1-\delta)\right]$$

We will discretize K into N nodes, and solve the system of equations: (and then make sure $k = k^\star$ belongs to the set K)

$$\frac{\partial u(f(k)+(1-\delta)k-g(k))}{\partial c} = \beta u'(f(g(k))+(1-\delta)g(k)-g(g(k)))\left[\frac{\partial f(g(k))}{\partial k} + (1-\delta)\right]$$

Algorithm. *Value Function Iteration*

1. Discretize the state space K
2. Make a guess for the value function $V_0(k)$
3. Use the optimization algorithm to solve for $k'(k)$
4. Compute[2]
$$V_1(k) = u(f(k)+k(1-\delta)-k'(k)) + \beta V_0(k'(k))$$
5. Check the distance between $V_0(k)$ and $V_1(k)$. If larger than the tolerance, update $V_0(k) = V_1(k)$ and go back to 3. Otherwise, stop!

Remark. With full depreciation ($\delta = 1$) and log utility, you can (and we did!) do this by hand!
Example. Suppose we want to know the answer to questions of the following form:

1. What is the role of the capital output ratio...
2. What is the role of productivity...

for differences in output per capita across countries?

We will choose functional forms for technology and preferences where parameters have clear economic interpretations, *i.e.*

$$u(c_t) = \frac{c_t^{1-\sigma}-1}{1-\sigma} \qquad \text{and} \qquad f(k) = zk_t^\alpha$$

We will set $z = 1$ in the US for now, and have four free parameters: σ, δ, β, and α. The literature suggests two approaches: estimation and calibration.

[2] There are various methods to approximate $V_0(k')$ for 'off-grid' points.

Question. Why not always estimate? The model is an abstraction in which we've deliberately abstracted away from some features. Standard formal statistical procedures for estimation use criteria that may not make economic sense. This was a big discussion in the 80s and 90s.

Alternative: choose the aspects of the data that your model was designed to capture.

The key idea of calibration is that choosing parameters boils down to choosing moments in the data to match. You can think of calibration as an exactly identified generalized method of moments (GMM) estimation.

This model is designed to explain the capital accumulation process. The key statistics that we care about are k/y, x/y, and r. Think of the modern US as fluctuations around the steady state (*i.e.* take averages to calculate the steady state). If we take one period as one year, we get that $k/y \approx 2.5$, $x/y \approx 0.2$, and $r \approx 0.04$.
Remark. Note that we have three moments, but four parameters! That's because σ does not affect the steady state, so it cannot be identified from the steady state. Estimates in the data have that $\sigma \in [1, 2.5]$. We will use $\sigma = 1$, corresponds to log utility.

Our steady state conditions give us that

$$\frac{x}{y} = \delta \frac{k}{y} \Longrightarrow \delta \approx 0.08 \qquad ; \qquad r = \alpha \frac{y}{k} \Longrightarrow \alpha \approx 0.1 \qquad ; \qquad \beta[r + (1-\delta)] = 1 \Longrightarrow \beta \approx 1$$

How much of the observed differences in income per capita are accounted for by differences in z, or by differences in k/y? Recall that $Y_t = zK_t^{\alpha} H_t^{1-\alpha}$. Taking logs, we get

$$\ln(Y_t) = \underbrace{\alpha}_{\beta_k} \ln(K_t) + \underbrace{(1-\alpha)}_{\beta_H} \ln(H_t) + \underbrace{\ln(A_t)}_{\varepsilon}$$

Remark. We could estimate this as a regression directly... but orthogonality between TFP and the covariates is extremely unlikely. This condition will not be satisfied. Julieta gave a full lecture on this, if you are interested.

An important take-away from the one sector growth model: differences in TFP induce differences in K, but $\frac{K}{Y}$ is independent from TFP in steady state! We have from the Euler equation that

$$1 = \text{discount} \left[\underbrace{\alpha \frac{Y}{K}}_{MPK} + (1 - \text{depreciation}) \right]$$

So output per worker is

$$\frac{Y}{L} = Z \left(\frac{K}{Y} \right)^{\frac{\alpha}{1-\alpha}} \frac{H}{L} \qquad \text{where } Z = A^{\frac{1}{1-\alpha}}$$

in logs:

$$\ln\left(\frac{Y}{L}\right) = \ln(Z) + \frac{\alpha}{1-\alpha} \ln\left(\frac{K}{Y}\right) + \ln\left(\frac{H}{L}\right)$$

If you want to play with these data to see the relationships yourself, Julieta has a Github repo with some broad growth data and code.
Remark. In accounting exercises, we very often get that TFP explains most of the difference in output per capita. That's really unsatisfying – it's basically a residual. However, macroeconomists are constantly looking under the hood of TFP to find the actual paths for that difference.

1.4 Equilibria

1.4.1 Competitive Equilibria

Question. How many markets are open?

In the most simple case, all markets are open, but this is unwieldy and not so realistic – recall that agents live forever in many models. Instead, we will look at a (recursive) market structure.

Model. *Competitive Equilibrium* Firms rent inputs in spot markets, and specifically do not own capital. They solve

$$\max_{\{k_t, n_t\}} c_t + p_{k_t} x_t - r_t k_t - w_t n_t$$

subject to

$$c_t + x_t \leq F(k_t, n_t) \qquad t = 0, 1, \ldots$$

and a bunch of non-negativity constraints.

Assumption 1.2. *F is strictly increasing in each argument, concave, twice continuously differentiable, and homogeneous of degree 1.*

This means we have constant returns to scale for production, meaning that the size of the firm does not matter. In an interior solution, $p_{k_t} = 1$, and $r_t = F_k(k_t, n_t)$ and $w_t = F_n(k_t, n_t)$. Let $f(k) = F(k, 1)$. Equilibrium factor prices satisfy $r_t = f'(k_t)$ and $w_t = f(k_t) - k_t f'(k_t)$. Firms choose the level of employment.

Households solve the problem

$$\max_{\{c_t, x_t, k_{t+1}, b_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$\begin{aligned} c_t + p_{k_t} x_t + b_{t+1} &\leq w_t + r_t k_t + R_t b_t && t = 0, 1, \ldots \\ k_{t+1} &\leq x_t + k_t(1 - \delta) && t = 0, 1, \ldots \\ 0 &= \lim_{T \to \infty} \beta^T u(c_T) b_{T+1} \\ 0 &\leq c_t, x_t, k_{t+1} && t = 0, 1, \ldots \end{aligned}$$

Why do we need a no-ponzi-scheme condition on b_t? We don't want agents carrying debt forward forever, in order to maximize consumption.

Remark. This is *not* a transversality condition – the transversality condition is an optimality condition, this is a restriction on the problem structure itself. As we can see from real life – Ponzi, Madoff, etc. – this can happen, and would be optimal for maximizing consumption. We could alternatively impose $b_t \geq \underline{b}$ for some $\underline{b} \in (-\infty, 0)$.

The Lagrangian of this problem is:

$$\begin{aligned} \mathcal{L} = \sum_{t=0}^{\infty} \beta^t \Big\{ u(c_t) + \\ &+ \lambda_t \left[w_t + r_t k_t + R_t b_t - c_t - p_{k_t} x_t - b_{t+1} \right] \\ &+ \theta_t \left[(1 - \delta) k_t + x_t - k_{t+1} \right] \\ &+ \gamma_{1t} c_t + \gamma_{2t} x_t + \gamma_{3t} k_{t+1} \Big\} \end{aligned}$$

The first order conditions are similar to before, but have some new conditions:

$$\begin{aligned}
c_t &: u'(c_t) = \lambda_t \\
x_t &: \lambda_t = \theta_t \\
k_{t+1} &: \theta_t = \beta[\theta_{t+1}(1-\delta) + \lambda_{t+1} r_{t+1} \\
b_{t+1} &: \lambda_t = \beta R_{t+1} \lambda_{t+1} \\
TVC_k &: \lim_{T\to\infty} \beta^T \lambda_T k_{T+1} = 0 \\
TVC_b &: \lim_{T\to\infty} \beta^T \lambda_T b_{T+1} = 0
\end{aligned}$$

We end up with one main Euler equation:

$$u'(c_t) = \beta u'(c_{t+1})[1 - \delta + r_{t+1}]$$

We can also use the FOC on the choice of capital to obtain

$$u'(c_t) = \beta u'(c_{t+1})[1 - \delta + f'(k_{t+1})]$$

The Euler equation for bonds is

$$u'(c_t) = \beta u'(c_{t+1}) R_{t+1}$$

Note that one of these two (capital and bonds) Euler equations implies the other!

Definition. A *recursive competitive equilibrium (RCE)* is a colelction of price sequences $\{\{w_t^\star, r_t^\star, p_{k_t}^\star, R_t^\star\}$, an allocation $\{c_t^\star, x_t^\star, k_{t+1}^\star\}$, and a sequence of bond holdings $\{b_{t+1}\}$ such that

1. Given prices, the allocation and sequence of bonds is utility maximizing.
2. Given prices, the allocation is profit maximizing
3. Markets clear
4. $b_0^\star = b_0 = 0$, $k_0^\star = k_0 > 0$ is given.

Proposition 1.1. *Any budget feasible allocation must satisfy*

$$\sum_{t=0}^{\infty} q_t c_t = \sum_{t=0}^{\infty} q_t w_t + q_0[1 - \delta + r_0]k_0 + q_0 R_0 b_0$$

where $q_t/q_0 = \prod_{j=1}^{t} R_j^{-1}$ *for* $t \geq 1$.

Remark. q_t/q_0 is the price of consumption at time t, in terms of consumption at time 0. These are the prices in an Arrow-Debreu economy with time zero trading.

Proof. Our budget constraint at time $t = 0$ is

$$b_1 = w_0 + r_0 k_0 - c_0 - x_0$$

while at time $t = 1$ it is

$$c_1 + x_1 + b_2 = w_1 + r_1 k_1 + R_1 b_1$$

We can combine these, and get

$$c_0 + x_0 + R_1^{-1}(c_1 + x_1) = w_0 + r_0 k_0 + R_1^{-1}(w_1 + r_1 k_1) + R_0 b_0 - R_1^{-1} b_2$$

Repeating, we get that for any finite T,

$$\sum_{t=0}^{T} q_t(c_t + x_t) = \sum_{t=0}^{T} q_t(w_t + r_t k_t) + q_0 R_0 b_0 - q_T b_{T+1}$$

No Ponzi implies that we can take T to ∞, and use the Euler equation for bonds to get

$$\sum_{t=0}^{\infty} q_t(c_t + x_t) = \sum_{t=0}^{\infty} q_t(w_t + r_t k_t) + q_0 R_0 b_0$$

We want to show that

$$\sum_{t=0}^{\infty} q_t(r_t k_t - x_t) = \sum_{t=0}^{\infty} q_t(r_t k_t - k_{t+1} + (1-\delta)k_t) = q_0[1 - \delta + r_0]k_0$$

We can rearrange the terms of the second sum to read

$$\lim_{T\to\infty} q_0[1-\delta+r_0]k_0 + k_1[-q_0 + q_1(1-\delta+r_1)] + \cdots + k_T[-q_{T-1} + q_T(1-\delta+r_T)]$$

And conclusion follows from the Transversality Condition □

Proposition 1.2. *In an equilibrium, $b_t^\star = 0$ for all t*

Proof. From the household budget constraint,

$$w_t^\star + r_t^\star k_t^\star + R_t^\star b_t^\star = c_t^\star + x_t^\star + b_{t+1}^\star$$

and using the equilibrium conditions for w_t, r_t we have that

$$f(k_t^\star) - k_t^\star f'(k_t^\star) + R_t^\star b_t^\star = c_t^\star + x_t^\star + b_{t+1}^\star$$

And conclusion follows from noting that market clearing implies that $f(k_t^\star) = c_t^\star + x_t^\star$, so the budget constraint for the private sector becomes

$$R_t^\star + b_t^\star = b_{t+1}^\star$$

and conclusion follows from $b_0^\star = 0$. □

1.4.2 Taxes

Question. How do deficits affect the economy?

Theorem 1.1. Ricardian Equivalence

1. *Budget policy* does not matter – *for any given sequence $\{g_t\}$ all non-distortionary tax structures that raise the appropriate level of revenue are associated to the same real allocation*
2. *Timing of the tax collection is irrelevant, in the sense that prices and allocations are independent of the timing. This result is often described as the Ricardian proposition*

Proof. Let $g = \{g_t\}$ be a sequence of expenditure. Define $\phi = \{\tau_t\}$ the sequence of lump-sum taxes to finance g and any initial debt. Consider an alternative sequence of taxes $\hat{\phi}$ such that $\hat{\tau}_0 < \tau_0$. Then

$$\sum_{t=0}^{\infty} q_t c_t = \sum_{t=0}^{\infty} q_t(w_t - \tau_t) + q_0[1-\delta+r_0]k_0 + q_0 R_0 b_0$$

and assume that $R_t b_t + g_t = \tau_t + b_{t+1}$. This implies that

$$\sum_{t=0}^{\infty} q_t(\tau_t - g_t) = q_0 R_0 b_0 - q_T b_{T+1}$$

Impose a No-Ponzi condition, so $\lim_{T\to\infty} q_T b_{T+1} = 0$. Then $\sum_{t=0}^{\infty} q_t \tau_t = \sum_{t=0}^{\infty} q_t \hat{\tau}_t$, and

$$\sum_{t=0}^{\infty} q_t c_t = \sum_{t=0}^{\infty} q_t(w_t - g_t) + q_0[1 - \delta + r_0]k_0$$

Guess that prices did not change after the tax change. Given that the budget constraint of the household is the same, it chose the same investment and consumption sequences, and firms chose the same allocations to maximize profits as before. Market clearing requires that

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t - g_t$$

which is the same since g_t did not change! □

Question. Why does the higher deficit not generate shifts in the interest rate?

For $\hat{\tau}_0 < \tau_0$,

$$b_1 + \tau_0 = g_0 + R_0 b_0 = \hat{b}_1 + \hat{\tau}_0 \implies \hat{b}_1 - b_1 = \hat{\tau}_0 - \tau_0$$

So debt goes up! However, households understand that lower taxes today will be compensated with higher taxes tomorrow. Households increase saving, at exactly the same rate as the government will increase taxes in the future. Since everyone is perfectly forward-looking, they do not change their habits at all.

Model. *Distortionary Taxes* We have a representative household, which solves

$$\max_{\{c_t, n_t, x_t, k_{t+1}, b_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - n_t)$$

subject to, for $t = 0, 1, 2, \ldots$

$$\begin{aligned}
(1+\tau_t^c)c_t + (1+\tau_t^x)x_t + b_{t+1} &\leq (1+\tau_t^n)w_t n_t + r_t k_t - \tau_t^k(r_t - \delta_t)k_t + (1 + (1-\tau_t^b r_t^b) b_t \\
k_{t+1} &\leq x_t + k_t(1-\delta) \\
0 &= \lim_{T\to\infty} \beta^T u'(c_T) b_{T+1} \\
0 &\leq c_t, x_t, k_{t+1}
\end{aligned}$$

The representative firm's problem is

$$\max_{k_t, n_t} F(k_t, n_t) - r_t k_t - w_t n_t$$

The first order conditions for the household are

$$\begin{aligned}
u_c'(c_t, 1-n_t) &= \lambda_t(1+\tau_t^c) && (c_t) \\
(1+\tau_t^x)\lambda_t &= \beta\lambda_{t+1}\left[(1-\delta_k)(1+\tau_{t+1}^x) + (1-\tau_{t+1}^k)r_{t+1} + \tau_{t+1}^k \delta_k\right] && (k_{t+1}) \\
\lambda^t &= \beta\lambda_{t+1}\left[1 + (1-\tau_{t+1}^b)r_{t+1}^b\right] && (b_{t+1}) \\
u_n'(c_t, 1-n_t) &= \lambda_t(1-\tau_t^w)w_t && (n_t) \\
0 &= \lim_{T\to\infty} \beta^T \lambda_T k_{T+1} && \text{(TVC)} \\
0 &= \lim_{T\to\infty} \beta^t \lambda_T b_{T+1} && \text{(No Ponzi)}
\end{aligned}$$

And the first order conditions for the firm are

$$F_k(k_t, n_t) = r_t \qquad ; \qquad F_n(k_t, n_t) = w_t$$

So in steady state, we have

$$\begin{aligned} \rho + \tau^x(\rho + \delta_k) &= (1 - \tau^k)(F_k(k, n) - \delta_k) && (k) \\ 1 &= \beta[1 + \rho] && (b) \\ u_n'(c, 1 - n) &= u_c'(c, 1 - n)\frac{1 - \tau^w}{1 + \tau^c} F_n(k, n) && (n) \\ F(k, n) &= c + g + \delta_k k && \text{(Feasibility)} \end{aligned}$$

where $\rho \equiv (1 - \tau^b) r^b$. Thus, we have that taxes on bonds just change r^b. Similarly, consumption and income taxes are equivalent, and investment and capital taxes are equivalent. This leads to two extremely famous results:

Theorem 1.2. First Welfare Theorem *If $[(w^\star, r^\star, p_k^\star, R^\star), (c^\star, x^\star, k^\star), b^\star]$ is an interior competitive equilibrium, then $(c^\star, x^\star, k^\star)$ solves the planner's problem.*

Proof. Since $b^\star = 0$ in any equilibrium, the household budget constraint is

$$f(k^\star) = c^\star + x^\star \qquad ; \qquad (k^\star)' = (1 - \delta)k^\star + x^\star$$

where $(k^\star)'$ is a vector of capital stock with first element $k_1^\star$. Also, the optimality conditions for the firm and the household together imply that

$$\begin{aligned} u'(c_t) &= \beta u'(c_{t+1})[(1 - \delta) + f'(k_{t+1})] \\ 0 &= \lim_{T \to \infty} \beta^T u'(c_T) k_{T+1} \end{aligned}$$

□

Theorem 1.3. Second Welfare Theorem *If $(c^\star, x^\star, k^\star)$ is an interior solution to the planner's problem, then there exist prices $(w^\star, r^\star, p_k^\star, R^\star)$ and a sequence of bond holdings $b^\star$ such that $[(w^\star, r^\star, p_k^\star, R^\star), (c^\star, x^\star, k^\star), b^\star]$ is a competitive equilibrium of an economy with a representative agent who has initial wealth $b_0 = 0$ and initial capital holdings $k_0 > 0$ equal to the initial endowment of capital in the economy.*

Proof. Let $b^\star = 0$. Since the solution to the planner's problem is feasible, we need to define prices and ensure that the optimality conditions (including transversality conditions) are met. Define

$$\begin{aligned} w^\star &= f(k^\star) - f'(k^\star)k^\star \\ r^\star &= f'(k^\star) \\ R^\star &= (1 - \delta) + f'(k^\star) \end{aligned}$$

At these prices, the optimality conditions of the firms and households are satisfied. Finally, we check the transversality conditions. The condition on bonds is trivially met since $b^\star = 0$. If we set $\lambda_t = u'(c_t)$, the transversality condition of the consumer problem in capital is the same as the one the planner faces. This will be a competitive equilibrium. □

1.5 Overlapping Generations Model

There are three models you should know to read macroeconomic papers: growth models (like we've been doing), overlapping generations models, and partial equilibrium models. This will be a primer on OLG, which is the second major workhorse of modern macroeconomics. Why do we use this model? Well, individuals don't actually live forever. In what sense is a model where people live for finite time equivalent to infinite lifecycles model? We want models (that macroeconomists call altruistic bequest motive models) where agents have interesting life-cycles: born, education, labor income, plan for retirement, partner, have children, retire, and die. This integration of micro data into macro models is very modern.

Model. *Overlapping Generations (OLG)* Time is discrete ($t = 1, 2, \dots$) and the economy lasts forever. There is a single non-storable consumption good in each period. A new generation is born each period, indexed by the year born. People live for two periods and then die.[3] What happens to the population?

Generation t is endowed with the consumption good in periods 1 and 2 of life, (e_t^t, e_{t+1}^t). Generation t's consumption in those periods is (c_t^t, c_{t+1}^t). At each point in time t, there are two generations alive: an old generation with endowment and consumption (e_t^{t-1}, c_t^{t-1}) and a young generation with endowment and consumption (e_t^t, c_t^t). At time 0, there is one old generation (e_1^0, c_1^0). We also have exponential population growth where $L_0 = 1$ and $L_t = (1+n)^t L_0$.

We assume that the only endowment is labor (time) – one unit supplied inelastically when young in return for w_t (*i.e.* you retire in year 2). We have constant returns to supply (CRS) technology for production $Y_t = F(K_t, L_t)$ and competitive factor markets. We will assume that $\delta = 1$, define $k \equiv \frac{K}{L}$, $f(k) \equiv F(k, 1)$, and the gross return on savings and wage rates

$$1 + r_t = R_t = f'(k_t) \qquad \text{and} \qquad w_t = f(k_t) - k_t f'(k_t)$$

Each generation solves the problem

$$\max_{c_t^t, c_{t+1}^t, s_t} u(c_t^t) + \beta u(c_{t+1}^t)$$

subject to

$$c_t^t + s_t \leq w_t \qquad \text{and} \qquad c_{t+1}^t \leq R_{t+1} s_t$$

So old agents rent their savings to firms as capital, and as long as u is strictly increasing with Inada, the constraints hold with equality. We need no non-negativity constraints – why?

The Euler equation is

$$u'(c_t^t) = \beta R_{t+1} u'(c_{t+1}^t)$$

and since the individual problem is concave, this suffices. We obtain a savings function $s : \mathbb{R}^2 \to \mathbb{R}$ with $s_t = s(w_t, R_{t+1})$, where s is increasing in w and increasing or decreasing in R. Aggregate savings are $S_t = L_t s_t$, and with full depreciation capital stock is $K_{t+1} = L_t s(w_t, R_{t+1})$.

Definition. An *OLG competitive equilibrium* is a sequence of aggregate capital stocks, individual consumption, and factor prices $\{K_t, (c_t^t, c_{t+1}^t), R_t, w_t\}_{t=0}^{\infty}$ such that the factor price sequence satisfies the first two conditions, individual consumption decisions are given by the Euler equation, and the savings function, and the aggregate capital stocks follow the law of motion.

Steady state is defined as normal, holding $k \equiv \frac{K}{L}$ constant, and the equilibrium characterization requires normalizing by the size of the population $L_{t+1} = (1+n)L_t$. We have that

$$k_{t+1} = \frac{K_{t+1}}{L_{t+1}} = \frac{L_t}{L_{t+1}} s(w_t, R_{t+1})$$

and combining with the above, we get

$$k_{t+1} = \frac{s(f(k_t) - k_t f'(k_t), f'(k_{t+1}))}{n+1}$$

Steady state is a fixed point of this expression. Note that since s can take any form, in principle multiple steady states (as well as complicated dynamics) are possible. If we assume CRRA utility and Cobb-Douglas

[3] Alternatively, death can be stochastic. See Blanchard 1985.

production:

$$U_t = \frac{(c_t^t)^{1-\sigma} - 1}{1-\sigma} + \beta \frac{(c_{t+1}^t)^{1-\sigma} - 1}{1-\sigma} \qquad ; \qquad f(k) = k^\alpha$$

for $\theta > 0$, $\alpha, \beta \in (0,1)$. The key outcome is the Euler equation

$$\frac{c_{t+1}^t}{c_t^t} = (\beta R_{t+1})^{\frac{1}{\theta}}$$

Rewritten in terms of savings rate, we have

$$s_t^\theta \beta R_{t+1}^{1-\theta} = (w_t - s_t)^{-\theta} \Longrightarrow s_t = \frac{w_t}{\phi_{t+1}} \quad \text{where } \phi_{t+1} \equiv \left[1 + \beta^{-\frac{1}{\theta}} R_{t+1}^{-\frac{1-\theta}{\theta}}\right] > 1 \Longrightarrow s_t < w_t$$

We have comparative statics for savings with respect to wages

$$s_w = \frac{\partial s}{\partial w} = \frac{1}{\phi_{t+1}} \in (0,1)$$

and with respect to capital

$$s_R = \frac{\partial s}{\partial R} = \frac{1-\theta}{\theta} (\beta R_{t+1})^{-\frac{1}{\theta}} \frac{s_t}{\phi_{t+1}}$$

where the sign depends on θ, the intertemporal elasticity of substitution. By combining the above, we can define a steady state implicitly by

$$k^\star = \frac{f(k^\star) - k^\star f'(k^\star)}{(1+n)\left[1 + \beta^{-\frac{1}{\theta}} f'(k^\star)^{-\frac{1-\theta}{\theta}}\right]}$$

We could also solve for the interest rate $R^\star = \alpha(k^\star)^{\alpha-1}$, and get

$$(1+n)\left[1 + \beta^{-\frac{1}{\theta}} (R^\star)^{-\frac{1-\theta}{\theta}}\right] = \frac{1-\alpha}{\alpha} R^\star$$

and finally, dynamics are given by the difference equation

$$k_{t+1} = \frac{(1-\alpha) k_t^\alpha}{(1+n)\left[1 + \beta^{-\frac{1}{\theta}} (\alpha k_{t+1}^{\alpha-1})^{-\frac{1-\theta}{\theta}}\right]}$$

Proposition 1.3. *In the OLG model with generations that live for two periods, Cobb-Douglas technology, and CRRA preferences, there exists a unique and stable steady state for all $k(0) > 0$.*
Remark. In this specific well-behaved case, equilibrium dynamics ≈ the Solow Model. Even with CRRA and Cobb-Douglas the model can get quite messy – the canonical model assumes log preferences.

The social planner solves

$$\sum_{t=0}^\infty \beta_s^t U_t \equiv \sum_{t=0}^\infty \beta_s^t (u(c_t^t) + \beta u(c_{t+1}^t))$$

subject to

$$F(K_t, L_t) = K_{t+1} + c_t^t L_t + c_t^{t-1} L_{t-1}$$

where β_s is the planner's discount factor across generations. Dividing by L_t, we get

$$f(k_t) = (1+n) k_{t+1} + c_t^t + \frac{c_t^{t-1}}{1+n}$$

We have the Lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta_s^t \left[(u(c_t^t) + \beta u(c_{t+1}^t)) + \lambda_t \left(f(k_t) - (1+n)k_{t+1} + c_t^t + \frac{c_t^{t-1}}{1+n} \right) \right]$$

The Euler equation comes from the first order necessary conditions:

$$u'(c_t^t) = \beta f'(k_{t+1}) u'(c_{t+1}^t)$$

Since $f'(k_{t+1}) = R_{t+1}$, the intertemporal consumption decisions are identical to the household, meaning that there is no distortion in consumption allocation over time when shifting to the social planner's problem. However, there may be differences across generations – the social planner may weight them differently. The steady state gives us

$$f'(k^\star) - (1+n)k^\star = \overbrace{c_1^\star + \frac{c_2^\star}{1+n}}^{c^\star}$$

where c_1 and c_2 are consumption when young and old respectively. We maximize overall consumption when

$$\frac{\partial c^\star}{\partial k^\star} = f'(k^\star) - (1+n) = 0 \Longrightarrow \exists\, k^{\text{gold}} \text{ s.t. } f'(k^{\text{gold}}) = (1+n)$$

so k^{gold} is the steady state capital that maximizes consumption (*golden rule capital*). Importantly, since f is concave, if $k^\star > k^{\text{gold}}$, then $f'(k^\star) < 1+n \Longrightarrow \frac{\partial c^\star}{\partial k^\star} < 0$, so lower savings would increase consumption for everyone. This is depicted in Figure 1.

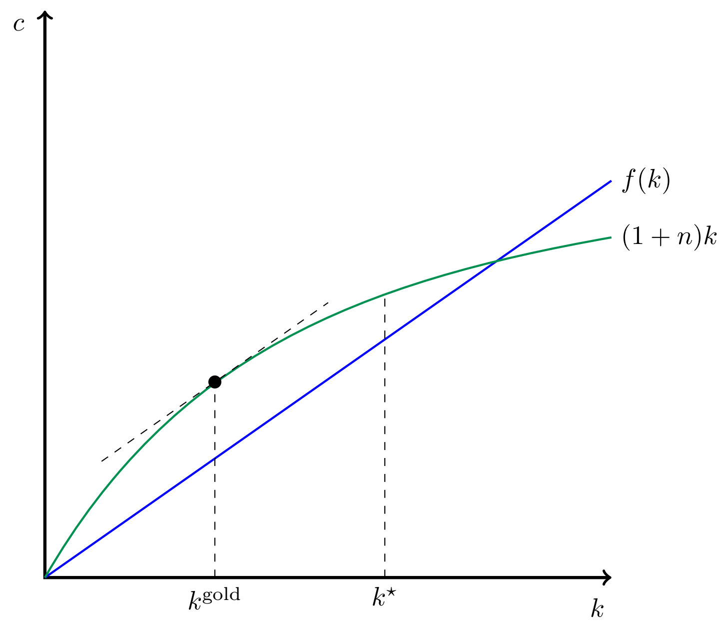

Figure 1: Golden Rule Capital

Definition. The economy is *dynamically inefficient* if it involves overaccumulation (*i.e.* if $k^\star > k^{\text{gold}}$). An alternative to this condition is $R^\star < 1+n \Longleftrightarrow r^\star < n$. Transversality in a standard one-sector growth model requires $r > g+n$, but we do not impose transversality in the OLG model, since agents live for two periods and solve finite problems.

Remark. Individuals born at time t face prices determined by the stock of capital chosen by the previous generation. Pecuniary externality: actions of the previous generation affect those further on. These typically do not matter for welfare (because they are second order), but these affect an infinite stream of newborn agents. These pecuniary externalities can be exploited, and we will see this in the application.

Proposition 1.4. *In the baseline OLD, the compaetitive equilibrium is not necessarily Pareto optimal. Whenever $r^\star < n$ the economy is dynamically inefficient. Hence, it is possible to reduce the capital stock in the steady state and increase consumption for all generations.*

Proof. Consider changing next period's capital stock so that $\Delta k < 0$, and then move towards the steady state. This will lead to lower savings in the first period, so $\Delta c_T = (1+n)\Delta k > 0$, meaning that everyone is happier. Further, since $k^\star > k^{\text{gold}}$, for small Δk,

$$\Delta c_t = -(f'(k^\star - \Delta k) - (1+n))\Delta k \Longrightarrow f'(k^\star - \Delta k) - (1+n) < 0 \Longrightarrow \Delta c_t > 0 \; \forall \; t > T$$

Thus, we've improved consumption in all future periods, showing a Pareto improvement. □

Remark. We can think about two types of systems to deal with dynamic inefficiencies:

1. Fully funded (social security): Young make contributions to the social security system, which are paid back when they are old
2. Unfunded (pay-as-you-go): Transfers go directly from the young to the current old

Pay-as-you-go discourages savings, so it may lead to a Pareto improvement.

Example. *Fully-Funded Social Security* The household's problem is

$$\max_{c_t^t, c_{t+1}^t, s_t, d_t} u(c_t^t) + \beta u(c_{t+1}^t)$$

subject to

$$c_t^t + s_t + d_t \leq w_t \qquad \text{and} \qquad c_{t+1}^t \leq R_{t+1}(s_t + d_t)$$

The government raises d_t from the young, invests it in the capital stock, and pays it back when they are old. Market clearing for capital requires $s_t + d_t = (1+n)k_{t+1}$, but the household does not necessarily choose $s_t > 0$. If s_t is unconstrained, then given a (feasible) sequence $\{d_t\}_{t=0}^{\infty}$, the set of competitive equilibria without social security is the set of competitive equilibria with social security if $s_t > 0$. If you impose $s_t \geq 0$ (no borrowing), then a (feasible) sequence $\{d_t\}_{t=0}^{\infty}$ is a competitive equilibrium if the equilibrium savings is such that $s_t > 0$ for all t. This implies that there cannot be a Pareto improvement if we impose $s_t \geq 0$.

Example. *Unfunded Social Security* The household's problem is

$$\max_{c_t^t, c_{t+1}^t, s_t} u(c_t^t) + \beta u(c_{t+1}^t)$$

subject to

$$c_t^t + s_t + d_t \leq w_t \qquad \text{and} \qquad c_{t+1}^t \leq R_{t+1}s_t + (1+n)d_{t+1}$$

The government raises d_t from the young and distributed $(1+n)d_t$ to the old. The rate of return on social security is $1+n$ rather than R_{t+1}, and only s_t goes to capital accumulation.

Remark. Unfunded social security reduces capital accumulation! What effect will that have on growth and welfare? If the economy is dynamically inefficient, this may be good... however most poorer countries have capital accumulation that is too low rather than too high. However, social security transfers resources from future generations to the initial old generation. With no dynamic inefficiency, this will make some future generation worse off!

1.6 Heterogeneity

We will talk about heterogeneity and consumption distribution, Gorman aggregation, and variance of consumption. We will typically take income as exogenous, and consider differences in consumption as the objects of interest with respect to the heterogeneity.

Piketty talked in capital about how capital growth, when wages are stagnant, can lead to increases in inequality. Specifically, recent automation is a large concern here. This is a recent question – in older history, growth was associated with sharp declines in inequality, rather than increases. Weirdly, inequality in the 20th century seemed to decline until the 1960s, when inequality in the U.S. increased by massive amounts.
Question. Is the one-sector growth model consistent with some degree of heterogeneity across households?

Under some conditions, we can show that heterogeneity in initial wealth, effective labor (*i.e.* human capital), and (limited) differences in utility do not affect equilibrium. Why? Because the 'averages' remain the same across the economy, which is identical to the representative agent economy. The basic ideas go back to Gorman on aggregation.
Model. *Heterogeneity* We have N households, each characterized by a vector (θ_i, a_i, e_i) where a_i are assets and e_i endowments of labor, and individual utility

$$u_i(c) = \frac{(c+\theta_i)^\eta}{1-\eta}$$

where $\eta > 0$ and θ_i can be either positive or negative. The household problem is

$$\max_{\{c_{i,t}, a_{i,t+1}\}} \sum_{t=0}^{\infty} \beta^t u_i(c_{i,t})$$

subject to

$$c_{i,t} + a_{i,t+1} \leq w_t e_i + R_t a_{i,t} \; \forall \, t = 0, 1, \dots \qquad \text{and} \qquad \lim_{T\to\infty} \beta^T u_i'(c_{i,T}) a_{i,T+1} = u$$

with $a_{i,0}$ given. We also have a present value budget constraint, where

$$\sum_{t=0}^{\infty} q_t c_{i,t} \leq \sum_{t=0}^{\infty} q_t w_t e_i + q_0 a_{i,0} \text{ where } q_t \equiv \prod_{j=1}^{t} R_j^{-1}$$

The representative firm solves the standard problem

$$\max_{c_t, x_t} c_t + p_{kt} x_t - w_t e_t - r_t k_t$$

subject to

$$c_t + x_t = F(k_t, e_t) \qquad \text{and} \qquad k_{t+1} = (1-\delta)k_t + x_t$$

Let N be the number of households in the economy. For any variable z_{it}, let $z_t = N^{-1} \sum_{i \leq N} z_{i,t}$, meaning the population average. We have first and second moments (including covariance) as usual, and we define $\theta \equiv N^{-1} \sum_{i=1}^{N} \theta_i$, $a_t \equiv N^{-1} \sum_{i=1}^{N} a_{it}$, and $e \equiv N^{-1} \sum_{i=1}^{N} e_i$.
Definition. A competitive equilibrium is a collection of price sequences $\{q_t, w_t, r_t, R_t\}$, an allocation $\{x_t, k_t, \{c_{it}\}\}$ and a sequence of asset holdings $\{a_{it}\}$ such that

1. Given prices, the allocation and sequence of assets maximizes utility
2. Given the equilibrium prices, the allocation maximizes profits

3. The allocation is feasible: market clearing and aggregate law of motion for capital

4. $a_0 = k_0 > 0$ given

Proposition 1.5. *Average quantities corresponding to a competitive equilibrium also solve the planner's problem:*

$$\max_{\{c_t, x_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \frac{(c_t+\theta)^{1-\eta}}{1-\eta}$$

subject to $c_t + x_t \le F(k_t, e_t)$ *and* $k_{t+1} = (1-\delta)k_t + x_t$.

Proof. Assume an interior solution. Thef irst order condition for the planner's problem is just

$$(c_t+\theta)^{-\eta} = \beta(c_{t+1}+\theta)^{-\eta}[1-\delta+F_k(k_{t+1}, e_{t+1})]$$

The Euler equation for family i is

$$(c_{it}+\theta_i)^{-\eta} = \beta(c_{it+1}+\theta_i)^{-\eta}R_{t+1}$$

but $R_{t+1} = [1-\delta+F_k(k_{t+1}, e_{t+1})]$, which implies that

$$(c_{it}+\theta_i) = \beta^{-1/\eta}(c_{it+1}+\theta_i)[1-\delta+F_k(k_{t+1}, e_{t+1})]^{-1/\eta}$$

Averaging each side, we get

$$N^{-1}\sum_{i=1}^{N}(c_{it}+\theta_i) = N^{-1}\sum_{i=1}^{N}\beta^{-1/\eta}(c_{it+1}+\theta_i)[1-\delta+F_k(k_{t+1}, e_{t+1})]^{-1/\eta}$$

□

Question. What are the implications in this model for consumption mobility and cross-sectional dispersion of consumption?

If we write the FOC of the household with the present-value budget constraint, we get

$$c_{it}+\theta_i)^{-\eta} = \lambda_i \frac{q_t}{\beta^t} \Longrightarrow c_{it}+\theta_i) = \left(\lambda_i \frac{q_t}{\beta^t}\right)^{-1/\eta}$$

We can use this expression in the budget constraint to solve for λ_i:

$$\lambda_i^{-1/\eta}\sum_{t=0}^{\infty} q_t \left(\frac{q_t}{\beta^t}\right)^{-1/\eta} = e_i \sum_{t=0}^{\infty} q_t w_t + \theta_i \sum_{t=0}^{\infty} q_t + q_0 a_{i0}$$

For any sequence z_t, let $v(z,q) = \sum_{t=0}^{\infty} q_t z_t$. Then we have

$$\lambda_i^{-1/\eta} v\left(\left(\frac{q_t}{\beta^t}\right)^{-1/\eta}, q\right) = e_i v(w,q) + \theta_i v(1,q) + q_0 a_{i0}$$

Let $\Phi_t = \frac{q_t}{\beta_t}$ and $\Phi_0 = 1$. Then we have

$$\lambda_i^{-1/\eta} = \frac{e_i v(w,q) + \theta_i v(1,q) + a_{i0}}{v(\Phi_t^{-1/\eta}, q)}$$

Using this multiplier in the FOC of the household, we get

$$c_{it} = \frac{e_i v(w,q) + \theta_i v(1,q) + a_{i0}}{v(\Phi_t^{-1/\eta}, q)} \Phi_t^{-1/\eta} - \theta_i$$

Note that consumption for each household is a linear function of e_i, θ_i, a_{i0}. Let M_i be such that

$$M_i \equiv m_e e_i + m_\theta \theta_i + m_a a_{i0}$$

where $m_e = \frac{v(w,q)}{v(\Phi_t^{-1/\eta},q)}$, $m_\theta = \frac{v(1,q)}{v(\Phi_t^{-1/\eta},q)}$, and $m_a = \frac{1}{v(\Phi_t^{-1/\eta},q)}$. Also, let $M = m_e e + m_\theta \theta + m_a a_0$. Then we have that

$$c_{it} = M_i \Phi_t^{-1/\eta} - \theta_i \qquad \text{and} \qquad c_t = M \Phi_t^{-1/\eta} - \theta$$

Remark. Key features here: all individuals face the same prices, aggregate demand functions are independent of the distribution of the personal characteristics, and a necessary and sufficient condition is that the Engel curves be affine.

Household i's relative consumption is $c_{it}^R = \frac{c_{it}}{c_t}$.

Proposition 1.6. *The long-run distribution of consumption is non-degenerate (*i.e. *it is not true that as $t \to \infty$, $c_{it}^R = 1$).*

Proof. We know that $c_t \to c^\star$, and if $k_0 < k^\star$ then c_t is monotonically increasing. Thus, $\Phi_t^{-1/\eta} \to (\Phi^\star)^{-1/\eta} > 0$ for some $\Phi^\star$, and

$$c_i^{R^\star} = \frac{M_i (\Phi^\star)^{-1/\eta} - \theta_i}{M (\Phi^\star)^{-1/\eta} - \theta} \neq 1$$

□

Question. When do we observe $c_i^{R^\star} < 1$? Well,

$$c_i^{R^\star} < 1 \iff M_i (\Phi^\star)^{-1/\eta} - \theta_i < M (\Phi^\star)^{-1/\eta} - \theta$$

which holds if

$$(\Phi^\star)^{-1/\eta} \left[m_e (e_i - e) + m_a (a_{i0} - a_0) \right] < (\theta_i - \theta) \left(1 - (\Phi^\star)^{-1/\eta} m_\theta \right)$$

Remark. Some comparative statics: higher e_i or a_{i0} leads to higher $c_i^{R^\star}$; lower θ_i leads to lower $c_i^{R^\star}$. We also have the (painful) second moments:

$$\sigma^2(c_i^{R^\star}) = \frac{1}{\left[M(\Phi^\star)^{-1/\eta} - \theta_i \right]^2} \Bigg\{ (\Phi^\star)^{-2/\eta} \left[m_e^2 \sigma^2(e) + m_a^2 \sigma^2(a_0) \right] + \left((\Phi^\star)^{-1/\eta} m_\theta - 1 \right)^2 \sigma^2(\theta)$$
$$+ 2(\Phi^\star)^{-1/\eta} \left[m_e m_a \sigma(e, a_0) + (m_\theta - 1)(m_e \sigma(e,\theta) + m_a \sigma(a_0, \theta)) \right] \Bigg\}$$

Notice that $\sigma^2(c_i^\star) = \sigma^2(c_i^{R^\star}) \left[M(\Phi^\star)^{-1/\eta} - \theta_i \right]^2$, so even if we have two economies A and B with $\sigma_A^2(e) = \sigma_B^2(e)$ and $\sigma_A^2(a_0) = \sigma_B^2(a_0)$, we will have differences in consumption variance. A positive covariance between any two of the elements that make up a household's type increases long-run variance in consumption.

Question. What do you think of the statement 'Controlling for initial wealth inequality, all countries display the same amount of consumption inequality'?

Model. *Bewley Economies* We have that agents are *ex ante* identical, where the only source of heterogeneity is income risk. Agents cannot insure against these shocks due to incomplete markets. We have discrete time and no aggregate uncertainty, so the aggregate endowment $\bar{e}$ is constant. We have a measure 1 of infinitely-lived households, with preferences

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_{it})$$

We have random endowments $e_{it} \in E$ where $|E| < \infty$, and transition matrix $\pi(e_{it+1}, e_{it})$ with stationary distribution $\Pi(e)$, where $\Pi(e) =$ the measure of households with endowment e. The households solve

$$\max_{c_{it}, a_{it+1}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_{it})$$

subject to a budget constraint and a borrowing constraint:

$$c_{it} + a_{it+1} = e_{it} + (1+r)a_{it} \qquad \text{and} \qquad a_{it+1} \geq -b, \text{ where } b \in \mathbb{R}_+$$

In recursive form, we have that

$$v(a, e) = \max_c u(c) + \beta \sum_{e'} \pi(e', e) v(a', e')$$

subject to

$$c + a' = e + (1+r)a \qquad \text{and} \qquad a' \geq -b$$

Remark. We need to be careful here! The borrowing constraint may bind here.

This Bellman equation becomes

$$v(a, e) = \max_{a'} u(e + (1+r)a - a') + \beta \sum_{e'} \pi(e' \mid e) v(a', e') + \mu(a' + b)$$

Sufficient conditions for optimality are

$$\frac{\partial u(c)}{\partial c} = \beta \sum_{e'} \pi(e' \mid e) \frac{\partial v(a', e')}{\partial a'} + \mu \qquad (a')$$

$$\frac{\partial v(a, e)}{\partial a} = \frac{\partial u(c)}{\partial c}(1+r) \qquad (a)$$

$$\mu(a' + b) = 0 \qquad \text{(CS)}$$

The Euler equation is

$$\frac{\partial u(c)}{\partial c} \geq \beta \sum_{e'} \pi(e' \mid e) \frac{\partial u(c')}{\partial c'}(1+r)$$

with equality whenever $a' > -b$ (*i.e.* when $\mu = 0$). Solutions to the household problem are functions $v(a, e)$, $a'(a, e)$ that satisfy the sufficient conditions, for a given r.

Assume shocks are i.i.d. Then consumption and savings decisions depend on the current value of income $x \equiv e + (1+r)a$, and savings are increasing in the current value of income $\partial a'/\partial x > 0$. If x is sufficiently high, choose $a' > -b$ and the Euler equation will hold with equality. If not, choose $a' = -b$ and let the Euler equation be violated. Essentially, current consumption is lower than optimal whenever the constraint binds.

The aggregate state is the joint distribution of assets and endowments $\Phi(a, e)$. In a stationary recursive competitive equilibrium, aggregate quantities and prices are constant over time.

Definition. A *stationary equilibrium* is an allocation and prices such that

1. Households maximize utility per $v(a, e)$, $a'(a, e)$

2. Markets clear

3. $\Phi(a, e)$ is time invariant

For market clearing, we need aggregate consumption:

$$C = \int_a \int_e c(a, e)\Phi(da, de) = \int_0^1 e\Pi(de) = e$$

and aggregate assets:

$$\int_a \int_e a'(a, e)\Phi(da, de) = 0$$

(since the households borrow from each other). We define a transition function $Q((a, e), (A, E))$ which denotes the probability (or mass of households) in state (a, e) that transition to state $(a', e') \in (A, E)$ tomorrow – so

$$Q((a, e), (A, E)) = \begin{cases} \sum_{e' \in E} \pi(e' \mid e) & \text{if } a'(a, e) \in A \\ 0 & \text{otherwise} \end{cases}$$

Note that a' is determined today. Finally, we have the law of motion for Φ

$$\Phi'(A, E) = \int_e \int_e Q((a, e), (A, E))\Phi(da, de)$$

Algorithm. To solve for the stationary equilibrium:

1. Given the interest rate, solve for the policy function of the household.
2. Given the policy function, iterate over the law of motion of the aggregate state until $\Phi' = \Phi$.
3. Using the stationary distribution, check market clearing.
4. If aggregate asset positions are positive, lower the interest rate and go back to 1.
5. If aggregate asset positions are negative, raise the interest rate and go back to 1.
6. Iterate until market clearing is satisfied.

Example. Huggett (1993) used six periods per year, a time discount $\beta = 0.66^{1/6} = 0.993$ per period, CRRA utility, a two-state Markov chain with $y_H = 1$, $y_L = 0.1$, and transition probabilities $\pi(H \mid H) = 0.925$, $\pi(L \mid L) = 0.5$, solved on a grid of borrowing constraints $\bar{a}$. The asset policy was basically linear, with optimal for the high types being almost exactly the 45° line, and the low types close to a linear downward shift from them. The consumption policy looked as usual, high types always above low types, with the difference being large when asset levels are low and small when asset levels are high. Finally, excess demand is clearly a decreasing, continuous function of the bond price, and so is an increasing and continuous function of the interest rate.

He additionally assumed complete risk-sharing so $c_{it} = C = Y$, bond price (interest rate) $q \equiv \frac{1}{1+r} = \beta$, and asset dynamics $a_{it+1} = (1 + r)(a_{it} + e_{it} - Y)$. He found with low risk-aversion, the risk-free rates were

$\bar{a}$	r	q
-2	-7.1%	1.0124
-4	2.3%	0.9962
-6	3.4%	0.9944
-8	4.0%	0.9935

So a tighter borrowing constraint (higher $\bar{a}$) led to higher demands for savings ($\uparrow q, \downarrow r$), and looser borrowing constraint (lower $\bar{a}$) led to lower demands for savings ($\downarrow q, \uparrow r$), with convergence to complete market equilibrium. With high risk-aversion, the risk-free rates were

$\bar{a}$	r	q
-2	-23%	1.0448
-4	-2.6%	1.0045
-6	1.8%	0.9970
-8	3.7%	0.9940

Higher risk aversion α leads to higher demand for savings, meaning lower r for all $\bar{a}$.
Example. *Numerical Tools for Incomplete Markets* We want to solve the problem

$$V(a,\varepsilon) = \max_{a',\tilde{c}\geq 0} u(\tilde{c}) + \beta\mathbb{E}\left[V(a',\varepsilon')\right]$$

subject to

$$\begin{aligned}\tilde{c} + a' &= w\varepsilon + (1+r)a \\ a' &\geq \underline{a} \\ \varepsilon' &= \rho\varepsilon + \epsilon \\ \epsilon &\sim \text{i.i.d}\end{aligned}$$

where ϵ are errors from the autoregressive process. We will solve this using the *Collocation method*, where we say that $V(a,\varepsilon) \approx B(a,\varepsilon)c$ where B are basis functions and c are *collocation coefficients*. Our objective is to find $\hat{f}$ that minimizes

$$\|f-\hat{f}\|_\infty \equiv \sup_{x\in D} |f(x)-\hat{f}(x)|$$

where a standard function interpolation would have that $\hat{f}(x,c)$ is a linear combination of polynomials with coefficients c, where we choose c to minimize $|f-\hat{f}|$ at a finite number of nodes. The key is our choice of polynomial and nodes.
Algorithm. *Collocation Method*

1. Choose basis functions $b_1(x),\ldots,b_n(x)$
   Remark. $b_n(x) = x^n$ typically a bad idea. Often better are Chebyshev orthogonal polynomials or Splines k-th order polynomials spliced together (popular choice for linear models).

2. Choose nodes $x = x_1,\ldots,x_n$
   Remark. Equidistant typically are bad. Chebyshev nodes, the roots of the n-th degree Chebyshev polynomial, are optimal.

The Chebyshev polynomial has a certain number of waves, depending on the basis. Equidistant nodes will lead to massive tail errors, as n increases. However, using Chebyshev nodes the approximation error decreases as n increases, and we can get a really good approximation of most smooth functions.

Splines basis functions are basically indicator functions, each has a peak at a certain point among the domain. A nice feature of these is that they are shape-preserving, and can be combined to span basically any function you want. Splines can be non-linear, but linear splines actually tend to do a good job.
Algorithm. *Collocation Method Computation*

1. Choose order n of polynomial

2. Choose $m \geq n$ nodes

3. Evaluate at nodes $B(x)$

4. Find c using $F = Bc$, then compute $c = B\setminus F$ (where F is also evaluated on x).

The nodes $B(x)$ are defined as

$$B(x) = \begin{bmatrix} b_1(x_1) & b_2(x_1) & \cdots & b_n(x_1) \\ b_1(x_2) & b_2(x_2) & \cdots & b_n(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ b_1(x_m) & b_2(x_m) & \cdots & b_n(x_m) \end{bmatrix}$$

If we are approximating a multidimensional function $f(x, y)$, we generate x and y nodes, choose basis functions b_i^x, b_i^y, and approximate

$$F = \sum_{i=1}^{n_x} \sum_{i=1}^{n_y} b_i^x(x) b_i^y(y) c_{ij}$$

Finally, we find c with $c = B \setminus F$, where $B = B_m \otimes B_{m-1} \otimes \cdots \otimes B_1$.

Back to the incomplete markets model, we will discretize the shock process and solve it using the collocation method, where $V(a, \varepsilon) \approx B(a, \varepsilon)c$. Commonly, people use Tauchen or Rouwenhorst to do this discretization. A common assumption is that $\epsilon \sim \mathcal{N}(0, \sigma^2)$. In broad strokes, we will:

1. Choose the parameters and grid
2. Define the space, meaning B and B^E
   Remark. Note that B^E can be computed using B and c:

$$\mathbb{E}[V(a, \varepsilon)] = B(a, \varepsilon)c_E = \sum_{\omega_i} \omega_i V(a, \rho\varepsilon + \epsilon_i) = \sum_{\omega_i} \omega_i B(a, \rho\varepsilon + \epsilon_i)c \Longrightarrow c_E = \underbrace{B(a, \varepsilon)^{-1} \sum_{\omega_i} \omega_i B(a, \rho\varepsilon + \epsilon_i)c}_{B^E}$$

3. Then we guess c_0 and solve the following linear equation until convergence:

$$B(a, \varepsilon)c_{j+1} = \max_{a' \geq 0} u(w\varepsilon + (1 + r)a - a') + (a', \varepsilon)B^E c_j$$

Aggregate assets are $A = \sum_{\varepsilon} a \cdot \phi(a, \varepsilon)$, and we will assume $A = K$ in equilibrium, and find r such that markets clear. To aggregate we need to compute the *Transition Probability Matrix (TPM)* P using the policy functions. We have a complete ergodic distribution $\phi(a, \varepsilon)$ such that P is a fixed point: $\phi(a, \varepsilon)P = \phi(a, \varepsilon)$. We can solve it by computing the TPM and fining the eigenvectors.
Remark. A quick detour on Markov processes. A stochastic process is a sequence of random vectors, indexed by time. A stochastic process has the *Markov property* if for all t and $k \geq 1$,

$$\mathbb{P}(x_{t+1} \mid x_t, \ldots, x_{t-k}) = \mathbb{P}(x_{t+1} \mid x_t)$$

Assuming the Markov property, we can characterize a process by a *Markov chain*, where π_0 is the initial distribution, P is the transition matrix where

$$P_{ij} = \mathbb{P}(x_{t+1} = e_j \mid x_t = e_i), \qquad \sum_j P_{ij} = 1$$

and e_i is a vector of zeros except for 1 in entry i. The probability of being in state j in two periods is:

$$P_{ij}^2 = \mathbb{P}(x_{t+2} = e_j \mid x_t = e_i) = \sum_{h=1}^{n} \mathbb{P}(x_{t+2} = e_j \mid x_{t+1} = e_h)\mathbb{P}(x_{t+1} = e_h \mid x_{t+1} = e_i)$$

Therefore, unconditional probabilities are determined by

$$\pi_1' = \mathbb{P}(x_1) = \pi_0' P \qquad \text{and} \qquad \pi_2' = \mathbb{P}(x_2) = \pi_0' P^2$$

A *stationary distribution satisfies* $\pi' = \pi' P \implies \pi'(I - P) = 0 \implies (I - P')\pi = 0$. Does this distribution necessarily exist? Yes! P has at least one eigenvalue and at least one eigenvector π, which we can normalize such that $\sum_i \pi_i = 1$. Multiplicity is possible, however. For a given π_0, it might be the case that

$$\lim_{t\to\infty} \pi_t = \pi_\infty$$

If this equation holds and $\pi_\infty \perp\!\!\!\perp \pi_0$, then the process is asymptotically stationary with a unique distribution π_∞, called the *asymptotic* (or *invariant*) distribution of P.
Remark. Some examples that are interesting to work through:[4]

1. Discrete and continuous choice problems
2. Heterogeneous firm models
3. Panel simulations:
   (a) We can also simulate many households over time
   (b) For example: Select a very long period of time and N agents, draw ε many times, use the policy function to compute the consumption and assets of each agent, for initial conditions to remain irrelevant, remove the first periods of the simulations, and the simulation can either be discrete or continuous.
   (c) We can study micro behavior of consumption and income jointly, checking if the distribution converges to to the one we compute by inverting the TPM, etc
   (d) Further notes on how to do this in the slides.

1.7 Continuous Time Growth

Some facts about continuous time growth (assuming that production technology is Cobb-Douglas, where $Y_t = A_t M_t K_t^\alpha H_t^{1-\alpha}$ for H_t human capital, M_t is a measure of uncertainty, and A_t is the level of technology):

1. Mathematically, we have that

$$\frac{Y_t}{L_t} = Z_t \frac{H_t}{L_t}\left(\frac{K_t}{Y_t}\right)^{\frac{\alpha}{1-\alpha}} \qquad \text{and} \qquad Z_t = (A_t M_t)^{\frac{1}{1-\alpha}}$$

   So all long-term growth in the Ramsay-Cass-Koopmans model is from either A or population L, which is exogenous!

2. From the one-sector growth model, we have that (i) $\frac{Y_t}{L_t}$ and $\frac{K_t}{L_t}$ grow over time at roughly constant equal rates (implication: $\frac{K_t}{Y_t}$ constant over time), (ii) that $\frac{I_t}{Y_t}$ is constant, and (iii) that $\frac{R_t K_t}{Y_t}$ and $\frac{W_t L_t}{Y_t}$ constant.
3. These predictions are consistent with the Kaldor facts!
4. They also are consistent with *some* catch-up dynamics, but transitions are too fast.

Model. *Ramsay-Cass-Koopmans (RCK)* We have a representative, infinitely-lived family (or dynasty), with population growth $n > 0$, labor force $L(t) = \exp(nt)$ where $L(0) = 1$ by assumption, constant returns

[4] CompEcon is very flexible and has some very handy computational tools!

technology $Y(t) = F(K(t), A(t)L(t))$, and capital dynamics:

$$\dot{K}(t) + \delta K(t) = F(K(t), A(t)L(t)) - C(t)$$

define $c_t = \frac{C(t)}{L(t)}$, $\zeta(t) = \frac{C(t)}{A(t)L(t)}$, and $\kappa(t) = \frac{K(t)}{A(t)L(t)}$ (this is in *efficiency units*). We have that

$$\frac{\dot{K}(t)}{A(t)L(t)} + \frac{\delta K(t)}{A(t)L(t)} = F\left(\frac{K(t)}{A(t)L(t)}, 1\right) - \frac{C(t)}{A(t)L(t)}$$

which simplifies to

$$\dot{\kappa}(t) = F(\kappa(t)) - \zeta(t) - (n + g + \delta)\kappa(t)$$

and thus

$$\dot{\kappa}(t) = \dot{\left(\frac{K(t)}{A(t)L(t)}\right)} = \frac{\dot{K}(t)}{A(t)L(t)} - n\kappa(t) - g\kappa(t)$$

Note the intuition here: we have that $\frac{1}{A(t)}\frac{\partial A(t)}{\partial t} = g$, and that $\frac{1}{L(t)}\frac{\partial L(t)}{\partial t} = n$. We can show this by using Ekaterina's trick of converting to logarithmic forms, taking derivatives, and showing that those partials are equivalent to the (more complicated) fractions.

Problem: F is decreasing in capital, meaning no endogenous long-run growth!

Solution: Constant returns to capital! See below.

Model. *The AK Model.* The simplest version of this assumes that $Y(t) = A(t)K(t)$, capital per capital is $\frac{K(t)}{L(t)}$, and feasibility is

$$\dot{k}(t) = Ak(t) - c(t) - \delta k(t)$$

or, with population growth,

$$\dot{k}(t) = Ak(t) - c(t) - (n + \delta)k(t)$$

The family values *per capita* consumption as

$$u(c) = \int_0^\infty \exp(-\rho t)U(c(t))dt$$

where ρ is the time discount factor, and $U(c(t))$ is a CRRA function, so $U(c) = \frac{c^{1-\sigma}}{1-\sigma}$. We can show that an allocation (k, c) is Pareto optimal if and only if it solve the social planner's problem, which is

$$\max_{k,c\geq 0} \int_0^\infty \exp(-\rho t)U(c(t))dt$$

subject to

$$\dot{k}(t) = Ak(t) - c(t) - (n + \delta)k(t) \qquad \text{with} \qquad k(0) = k_0$$

We can solve this problem using Pontryagin's Maximum Principle, with control variable c, state variable k, and co-state variable λ. The present-value Hamiltonian is

$$\mathcal{H}(t, k, c, \lambda) = \exp(-\rho t)U(c(t)) + \lambda(t)\left[f(k(t)) - c(t) - (n + \delta)k(t)\right]$$

which admits sufficient conditions

$$\begin{aligned} 0 &= \exp(-\rho t)U'(c(t)) - \lambda(t) && (c) \\ \dot{\lambda}(t) &= -\left[f'(k(t)) - (n + \delta)\right]\lambda(t) && (k) \\ 0 &= \lim_{t\to\infty} \lambda(t)k(t) && (TVC) \end{aligned}$$

plus the constraint

$$\dot{k}(t) = Ak(t) - c(t) - (n+\delta)k(t)$$

We can get rid of the co-state variable by differentiating

$$\dot{\lambda}(t) = \exp(-\rho t)U''(c(t))\dot{c}(t) - \rho\exp(-\rho t)U'(c(t)) \Longrightarrow \frac{\dot{\lambda}(t)}{\lambda(t)} = \frac{U''(c(t))}{U'(c(t))}\dot{c}(t) - \rho$$

where we replace

$$-\left[f(k(t)) - (n+\delta+\rho)\right]c(t) = \frac{U''(c(t))}{U'(c(t))}c(t)\dot{c}(t)$$

and because U is CRRA,

$$\sigma\dot{c}(t) = \left[A - (n+\delta+\rho)\right]c(t)$$

Definition. A *balanced growth path (BGP)* is an allocation such that consumption, capital, and output grow at a constant (possibly different) growth rate

$$\frac{\dot{x}(t)}{x(t)} = \gamma_x > 0 \text{ for any variable } x$$

In this particular problem, the consumption dynamics are independent of capital:[5]

$$c(t) = \exp\left(\frac{1}{\sigma}\left[A - (n+\delta+\rho)\right]t\right)c(0)$$

So consumption grows at a constant rate from the beginning. Capital dynamics are

$$\frac{\dot{k}(t)}{k(t)} = A - \frac{c(t)}{k(t)} - (n+\delta)$$

Along the BGP, consumption and capital grow at the same rate, so the economy is on the BGP from time $t=0$. We need an additional condition so that utility does not blow up, so U remains well-defined. Observe that:

$$\int_0^\infty \exp(-\rho t)U(c(t))dt = c(0)^{\frac{1-\sigma}{\sigma}}\int_0^\infty \exp(-\rho t)\frac{\exp\left(\frac{1-\sigma}{\sigma}\left[A-(n+\delta+\rho)\right]t\right)dt}{1-\sigma}$$

So we need that

$$\frac{1-\sigma}{\sigma}\left[A - (n+\delta) - \frac{\rho}{1-\sigma}\right] < 0$$

Remark. The main implication is that no country can ever catch up. However, we see some of that in the data – how do we reconcile that? What would happen if we added labor? With increasing returns to scale, we don't have a well-defined equilibrium in the standard AK model. Romer (1986) argued for increasing returns to scale but the agents in the economy behave as if constant returns to scale.
Question. Why do we consider discounting rather than no discount factor and maximize mean consumption in each period rather than have discounting?
Answer. We don't exactly have a Permanent Income Hypothesis here, as this is a production economy. We essentially, in most cases, have a single optimal path, and it's always worth it to get on the optimal path as quickly as possible. See the Turnpike Theorem.

[5] Note not true in the standard RCK model, remember the sequential economy.

Model. *RCK with Productivity Growth* Recall that the optimal allocation satisfies

$$\begin{aligned}\dot{\zeta}(t) &= \frac{1}{\sigma}\left[f'(\kappa(t)) - (n + \sigma g + \delta + \rho)\right]\zeta(t)\\ \dot{\kappa}(t) &= f(\kappa(t)) - \zeta(t) - (n+g+\delta)\kappa(t)\\ 0 &= \lim_{t\to\infty} \exp(-\hat{\rho}t)U'(\zeta(t))\kappa(t)\end{aligned}$$

where g is exogenous growth of technology, and $\kappa(t) = \frac{K(t)}{A(t)L(t)}$, $\zeta(t) = \frac{C(t)}{A(t)L(t)}$. We have steady state analysis $x^\star = (\zeta^\star, \kappa^\star)$, and saddle paths. From the theory of linear approximation, we know that when the path is "nice" (this condition is quite important! think of it as a convex problem that is everywhere differentiable), the behavior around the steady state is well-approximated by the behavior of a linear system around the steady state. We use the first-order Taylor approximation

$$f(x) = f(x^\star) + \nabla f(x^\star)\cdot(x - x^\star)$$

where $\nabla f(x^\star)$ is the $\mathbb{R}^n$ gradient to f at $x^\star$. In our case, $x^\star = (\zeta^\star, \kappa^\star)$ and we have g_1 and g_2 that solve

$$\begin{aligned}g_1(\zeta(t),\kappa(t)) &= \dot{\zeta}(t) = \frac{1}{\sigma}\left[f'(\kappa(t)) - (n+g+\delta+\rho)\right]\zeta(t)\\ g_2(\zeta(t),\kappa(t)) &= \dot{\kappa}(t) = f(\kappa(t)) - \zeta(t) - (n+g+\delta)\kappa(t)\end{aligned}$$

where $g_1(\zeta^\star,\kappa^\star) = g_2(\zeta^\star,\kappa^\star) = 0$. From our linear approximation, we have

$$\begin{bmatrix}\dot{\zeta}(t)\\ \dot{\kappa}(t)\end{bmatrix} \approx \begin{bmatrix}-\frac{1}{\sigma}[f'(\kappa(t)) - (n+g+\delta+\rho)] & -\frac{1}{\sigma}f''(\kappa(t))\zeta(t)\\ -1 & f'(\kappa(t)) - (n+g+\delta)\end{bmatrix}_{(\zeta^\star,\kappa^\star)}\begin{bmatrix}\zeta(t)-\zeta^\star\\ \kappa(t)-\kappa^\star\end{bmatrix}$$

which becomes

$$\begin{bmatrix}\dot{\zeta}(t)\\ \dot{\kappa}(t)\end{bmatrix} \approx \begin{bmatrix}0 & -\frac{1}{\sigma}f''(\kappa^\star)\zeta^\star\\ -1 & \rho\end{bmatrix}_{(\zeta^\star,\kappa^\star)}\begin{bmatrix}\zeta(t)-\zeta^\star\\ \kappa(t)-\kappa^\star\end{bmatrix}$$

This is a two-dimensional difference equation that can be solved analytically. We look at the eigenvalues of $\nabla f(\zeta^\star,\kappa^\star)$, λ, which satisfy

$$0 = \det(\nabla f(\zeta^\star,\kappa^\star) - \lambda I) = -\lambda(\rho-\lambda) + \frac{1}{\sigma}f''(\kappa^\star)\zeta^\star$$

This quadratic equation has two roots, one negative and one positive. Let l be the number of negative eigenvalues, and let m be the number of state variables of the problem. Stability comes directly from these: if $l = m$, we are *saddle-path stable*, we have a unique optimal trajectory (the negative eigenvalue governs the speed of convergence), if $l < m$, unstable, no convergence to the steady state, and if $l > m$, indeterminacy, multiple optimal trajectories.

The speed of convergence is

$$\kappa(t) - \kappa^\star \approx e^{-|\lambda_1|t}(\kappa(0) - \kappa^\star)$$

with half-life

$$\kappa(t_{1/2}) - \kappa^\star \approx \frac{1}{2}(\kappa(0)-\kappa^\star) \text{ hence } t_{1/2} = \frac{\ln(2)}{|\lambda_1|}$$

1.8 Endogenous Growth Theory

1.8.1 Externalities

The AK model generates long-term growth. Typically, we think about this as either spillovers or learning by doing. How do we endogenize growth?

Remark. First, what worked in the AK model? Constant returns in a reproducible factor (capital), and we left out the non-reproducible factor (labor). The dilemma is that we left out the non-reproducible factor. Paul Romer had an elegant solution.

Model. *Endogenous Growth Model* (from Romer (1986)) The economy is the same as the AK model setup but we have a continuum of firms that produce output with technology

$$y_i(t) = F(k_i(t), l_i(t)K(t))$$

where firms are indexed with $i \in [0,1]$, k_i, l_i are capital and labor at firm i, $K(t) = \int k_i(t)di$ is aggregate capital, and $F(\cdot)$ has constant returns to scale with respect to k_i and l_i. This gives us an externality, where firms do not understand that requesting more capital will increase the aggregate capital, they just take that as given.[6] The firm's problem is

$$\max_{k_i, l_i} F(k_i(t), l_i(t)K(t)) - w(t)l_i(t) - r(t)k_i(t)$$

where $K(t)$ is *exogenous to the firm*. Note that there are increasing returns to scale overall:

$$F(\theta k_i(t), \theta l_i(t) \underbrace{\theta K(t)}_{=\int \theta k_i(t)di}) = F(\theta k_i(t), \theta^2 l_i(t)K(t)) > \theta F(k_i(t), l_i(t)K(t))$$

for any $\theta > 1$. The competitive equilibrium exists in this economy because from the firm's perspective there are constant returns to scale. Will it be Pareto Optimal? Not in general! This is straightforward to see, in a world with externalities.

As before, the households are a size L of identical people, with no population growth.

Definition. A *competitive equilibrium* is a set of allocations $\{\hat{c}(t), \hat{a}(t)\}$ for the representative household, a set of allocations $\{\hat{k}_i(t), \hat{l}_i(t)\}$ for each firm i, a stream of aggregate capital stock $\{\hat{K}(t)\}$, and a stream of prices $\{\hat{w}(t), \hat{r}(t)\}$ such that:

1. Given prices, $\{\hat{c}(t), \hat{a}(t)\}$ solves
$$\max_{\{c(t), a(t)\}} \int \exp(-\rho t)\frac{c(t)^{1-\sigma}}{1-\sigma}dt$$
subject to
$$c(t) + \dot{a}(t) + a(t) = w(t) + (r(t) - \delta)a(t)$$
$$a(0) = k(0) \text{ given}$$
$$\lim_{t\to\infty} a(t)\exp\left(-\int_0^t (r(\tau) - \delta)d\tau\right) \geq 0$$

2. Given $\{\hat{w}(t), \hat{r}(t)\}$ and $\{\hat{K}(t)\}$, the path $\{\hat{k}_i(t), \hat{l}_i(t)\}$ maximizes firm profits in each period for each firm

[6] This is often called the *Big K / small k model*.

3. Feasibility: for all t, markets clear:

$$L\hat{c}(t) + \dot{\hat{K}}(t) + \hat{K}(t)\delta = \int_0^1 F(\hat{k}_i(t), \hat{l}_i(t)\hat{K}(t))di$$

$$\int_0^1 \hat{l}_i(t)di = L \qquad \text{and} \qquad \int_0^1 \hat{k}_i(t)di = L\hat{a}(t)$$

4. Rational Expectations:

$$\int_0^1 \hat{k}_i(t)di = \hat{K}(t)$$

The planner's problem is

$$\max_{c(t),K(t)} \int \exp(-\rho t)\frac{c(t)^{1-\sigma}}{1-\sigma}dt$$

subject to

$$Lc(t) + \dot{K}(t) - K(t)\delta = F(K(t), LK(t)) \text{ with } K(0) = Lk(0)$$

Note that the planner can deal in terms of total capital and total labor, because the firms have identical production functions, so when they choose capital and labor so that the marginal products of capital and labor respectively equal the wage and rental price, they all choose the same levels, so they can be aggregated. The planner can choose first the aggregates and next how to split them across the agents. Optimality requires that

$$\gamma_C^{SP}(t) = \frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma}\Big[F_1(K(t), LK(t)) + F_2(K(t), LK(t))L - (\delta + \rho)\Big]$$

Since F is homogeneous of degree 1 in capital, F' is homogeneous of degree 0, so we have that

$$F_1(K(t), LK(t)) + F_2(K(t), LK(t))L = F_1(1, L) + F_2(1, L)L$$

Thus, consumption growth is

$$\gamma_C^{SP}(t) = \frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma}\Big[F_1(1, L) + F_2(1, L)L - (\delta + \rho)\Big]$$

which is constant in time! From the aggregate resource constraint:

$$L\frac{c(t)}{K(t)} + \frac{\dot{K}(t)}{K(t)} + \delta = F(1, L)$$

so in the balanced growth path, $\gamma_C^{SP}(t) = \gamma_k^{SP}(t) = \gamma_K^{SP}(t)$. From the first order conditions of the household and the firm, we have that

$$\gamma_C^{CE}(t) = \frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma}\Big[r(t) - (\delta + \rho)\Big] \qquad \text{and} \qquad r(t) = F_1(k_i(t), l_i(t)K(t))$$

Since all firms are identical and choose the same allocations, $k_i(t) = k(t) = \int k_i(t)di = K(t)$ and $l_i(t) = L$, so $r(t) = F_1(1, L)$, so

$$\gamma_C^{CE}(t) = \frac{1}{\sigma}\Big[F_1(1, L) - (\delta + \rho)\Big] < \gamma_C^{SP}(t)$$

since $F_2(1, L) > 0$. So in the balanced growth path, $\gamma_C^{CE} = \gamma_k^{CE} = \gamma_K^{CE}$. How do we bring γ_K^{CE} to γ_K^{SP}? With a subsidy per unit of capital invested, equal to $F_2(1, L)L$. That will make the cost of capital to the firms $r(t) - F_2(1, L)L$.

Remark. A prediction of this model is that larger economies grow more, where

$$\gamma_C^{CE}(t) = \frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma}\left[F_1(1,L) - (\delta+\rho)\right] < \gamma_C^{SP}(t)$$

since $\frac{\partial F(1,L)}{\partial L} > 0$. If L is defined as the labor force of a country (or population), the data across post-World War II countries show no evidence of this fact. Technically, this is because we have constant returns in K and increasing returns in K and L. We can avoid this by assuming that productivity depends on the ratio $\frac{K}{L}$ rather than the aggregate. An example of this is the following:

Model. *Externalities in Human Capital* (from Lucas, 1988) We have households with measure 1 and human capital h_i for $i \in (0,1)$, with $h_i(0) = h_0$ and $k_i(0) = k_0$. The household chooses time to work $1 - s_i(t)$ and time to accumulate human capital $s_i(t)$, with budget constraint(s)

$$\begin{aligned} c_i(t) + \dot{a}_i(t) &= (r(t) - \delta)a_i(t) + (1 - s_i(t))h_i(t)w(t) \\ \dot{h}_i(t) &= \theta h_i(t)s_i(t) - \delta h_i(t) \end{aligned}$$

and production technology

$$Y(t) = AK(t)^{\alpha}L(t)^{1-\alpha}H(t)^{\beta}$$

Firms choose capital and labor and they take $H(t)$ as given. Since the total supply of labor is $(1-s(t))H(t)$, this becomes

$$AK(t)^{\alpha}H(t)^{1-\alpha}H(t)^{\beta}(1-s(t))^{1-\alpha} = AK(t)^{\alpha}H(t)^{\beta+(1-\alpha)}(1-s(t))^{1-\alpha}$$

The planner's problem is

$$\max \int_0^{\infty} \exp(-\rho t)\frac{c(t)^{1-\sigma}}{1-\sigma}dt$$

subject to

$$\begin{aligned} C(t) + \dot{K}(t) + \delta K(t) &= AK(t)^{\alpha}H(t)^{\beta+(1-\alpha)}(1-s(t))^{1-\alpha} \\ \dot{H}(t) &= \theta H(t)s(t) - \delta H(t) \end{aligned}$$

with $H(0)$ and $S(0)$ given, $s(t) \in [0,1]$, and we have used the fact that

$$\int (1 - s_i(t))h_i(t)di = L(t) \quad ; \quad \int a_i(t)di = K(t) \quad ; \quad \int c_i(t)di = C(t)$$

Firms solve the simple

$$\max_{L(t),K(t)} Y(t) - w(t)L(t) - r(t)K(t)$$

where

$$r(t) = \alpha\frac{Y(t)}{K(t)} \qquad \text{and} \qquad w(t) = (1-\alpha)\frac{Y(t)}{(1-s)H(t)}$$

Households solve the Hamiltonian

$$\begin{aligned} \mathcal{H}(\cdot) = \exp(-\rho t)\frac{c_i(t)^{1-\sigma}}{1-\sigma} &+ \lambda(t)\left[(r(t)-\delta)a_i(t) + (1-s_i(t))h_i(t)w(t) - c_i(t)\right] \\ &+ \mu(t)\left[\theta h_i(t)s_i(t) - \delta h_i(t)\right] \end{aligned}$$

Again, we have that $\gamma_C^{SP} > \gamma_C^{CE}$.

1.8.2 Innovation

In the economies we've studied so far, we've been on the balanced growth path from the beginning. We can think of transition dynamics by relaxing that, and think of growth through variety innovation.
Model. *Transition Dynamics* (from Jones & Manuelli, 1990) We have production technology

$$Y = F(K, L) = AK + BK^{\alpha}L^{1-\alpha}$$

so that $y = Ak + Bk^{\alpha}$ and $\lim_{t\to\infty} f'(k) = A$. We have growth rates

$$\frac{\dot{k}}{k} = \frac{f(k)}{k} - \frac{c}{k} - (n+\delta) \qquad \text{and} \qquad \frac{\dot{c}}{c} = \frac{1}{\theta}\left(A + B\alpha k^{\alpha-1} - (\rho+\delta)\right)$$

where θ is the elasticity of substitution in consumption. In the balanced growth path, we have growth rates

$$\gamma^{\star} = \frac{1}{\theta}\left(A - (\rho+\delta)\right)$$

The problem is that we have no steady state. We can rewrite the variables in stationary terms (*i.e.* *detrend* them). Which variables do we detrend with respect to? Here, capital. Generally it depends on the problem. We have

$$z = \frac{f(k)}{k} \qquad \text{and} \qquad \chi = \frac{c}{k}$$

By doing a bunch of (annoying) algebra, we can rewrite the dynamic system as

$$\dot{z} = -(1-\alpha)(z-A)(z-\chi-n-\delta)$$
$$\dot{\chi} = \chi\left((\chi-\varphi) - \frac{\theta-\alpha}{\theta}(z-A)\right)$$

where $\varphi \equiv (A-\delta)\frac{\theta-1}{\theta} + \frac{\rho}{\theta} - n$. We can now draw a phase diagram in (χ, z) space!

Writing Recursively. (*Hamilton-Jacobi-Bellman Equations*) We have

$$V(k_0) = \max_{c(t)} \int_0^{\infty} \exp(-\rho t) U(c(t)) dt$$

subject to

$$\dot{k}(t) = F(k(t)) - \delta k(t) - c(t)$$

for $t \geq 0$, with $k(0) = k_0$ given. Our state x is $k(t)$, and the control u is $c(t)$. Let $h(x,u) = U(u)$ and $g(x,u) = F(x) - \delta x - u$. The value function of the generic optimal control problem solves the Hamilton-Jacobi-Bellman equation

$$\rho V(x) = \max_u h(x,u) + V'(x) g(x,u)$$

With multiple states, $V'(x)$ is a vector of dimension m. This implies that

$$\rho V(k) = \max_c U(c) + V'(k)\left[F(k) - \delta k - c\right] \Longleftrightarrow U'(c) = V'(k)$$

We define a discount factor $\beta(\Delta) = \exp(-\rho\Delta)$, where Δ is the length of a period. We multiply all of our flows by Δ, and keep the stocks the same. The Bellman equation is

$$V(k_t) = \max_{c_t} \Delta U(c_t) + \exp(-\rho\Delta) V(k_{t+\Delta})$$

subject to

$$k_{t+\Delta} = \Delta[F(k_t) - \delta k_t - c_t] + k_t$$

For small Δ, $\exp(-\rho\Delta) \sim (1-\rho\Delta)$, so

$$\rho\Delta V(k_t) = \max_{c_t} \Delta U(c_t) + (1-\rho\Delta)(V(k_{t+\Delta}) - V(k_t))$$

and so

$$\rho V(k_t) = \max_{c_t} U(c_t) + (1-\rho\Delta)\left(\frac{V(k_{t+\Delta}) - V(k_t)}{k_{t+\Delta} - k_t}\frac{k_{t+\Delta} - k_t}{\Delta}\right)$$

and taking the limit as $\Delta \to 0$, we get

$$\rho V(k_t) = \max_{c_t} U(c_t) + V'(k_t)\dot{k}_t$$

Recall that we have had the two general forms:

$$\underbrace{\mathcal{H}(x,u,\lambda) \equiv h(x,u) + \lambda g(x,u)}_{\text{Hamiltonian}} \quad ; \quad \underbrace{\rho V(x) = \max_u h(x,u) + V'(x)g(x,u)}_{\text{Bellman}}$$

so we can directly see that $\lambda(t) = V'(x(t))$, the co-state variable is the shadow value of the future. Thus:

$$\rho V(x) = \max_{u \in U} \mathcal{H}(x,u,V'(x))$$

we get the reason for the 'Hamilton' in the name of this subsection!

Model. *Endogenous Technological Change* (from Romer, 1990) We can think about expanding input varieties, where a greater variety of inputs increases the 'division of labor' which raises the productivity of final goods firms. We have competitive markets for final goods, monopolistic competition for intermediate goods, and competitive markets for R&D. We have final goods, produced by:

$$Y(t) = L(t)^{1-\alpha}\left(\int_0^{A(t)} x_i(t)^{1-\mu} di\right)^{\frac{\alpha}{1-\mu}}$$

where $1/\mu$ is the elasticity of substitution. We are interested in $\mu \in (0,1)$, meaning that there is *some* substitution. We also have intermediate goods, produced by

$$x_i(t) = a l_i(t)$$

and R&D, with constant returns to scale:

$$\dot{A}(t) = bX(t)$$

where $X(t)$ are final goods devoted to R&D. We normalize the population to 1, and the planner's problem is

$$\max \int_0^\infty \exp(-\rho t)\frac{c(t)^{1-\sigma}}{1-\sigma} dt$$

subject to

$$c(t) + X(t) = L(t)^{1-\alpha} \left(\int_0^{A(t)} x_i(t)^{1-\mu} di \right)^{\frac{\alpha}{1-\mu}}$$
$$x_i(t) = al_i(t)$$
$$\dot{A}(t) = bX(t)$$
$$L(t) + \int_0^{A(t)} l_i(t) di = 1$$

Given that there is imperfect substitution, the optimal planner strategy is

$$x_i(t) = x(t) \qquad \text{and} \qquad l_i(t) = l(t)$$

Hence,

$$c(t) + X(t) = L(t)^{1-\alpha} \left(A(t)(al(t))^{1-\mu} \right)^{\frac{\alpha}{1-\mu}}$$

Feasibility requires that $l(t) = \frac{1-L(t)}{A(t)}$, and equilibrium aggregate output is

$$Y(t) = a^\alpha L(t)^{1-\alpha}(1-L(t))^\alpha A(t)^{\frac{\alpha\mu}{1-\mu}}$$

Output maximization implies that $L(t) = 1-\alpha$. The planner's problem is

$$\max \int_0^\infty e^{-\rho t} \frac{c(t)^{1-\sigma}}{1-\sigma} dt \qquad \text{s.t. } c(t) + \frac{\dot{A}(t)}{b} = CA(t)^{\frac{\alpha\mu}{1-\mu}}$$

where $C = a^\alpha(1-\alpha)^{1-\alpha}\alpha^\alpha$. If $\frac{\alpha\mu}{1-\mu} \in (0,1)$, we have Ramsey-Cass-Koopmans. Romer requires that $\alpha\mu = 1-\mu$, so we have an AK model where

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma}[bC - \rho]$$

so we are on the balanced growth path from the beginning.

Alternatively, we could solve the decentralized economy. When a new idea is introduced, the R&D sector charges $k(t)$, where since the sector is competitive and has CRS technology, so $\pi_{R\&D} = 0$. We will assume that $k(t)$ is the present value of the profits of the monopolistic firm, and let $p_i(t)$ be the price of intermediate goods, $p(t)$ be the price of final goods, and $w(t)$ the cost of labor. The final goods firms maximize profits, so we attain the first order conditions

$$w(t) = (1-\alpha)p(t)L(t)^{-\alpha} \left(\int_0^{A(t)} x_i(t)^{1-\mu} di \right)^{\frac{\alpha}{1-\mu}} \qquad (L(t))$$
$$p_i(t) = \alpha p_t L(t)^{1-\alpha} \left(\int_0^{A(t)} x_i(t)^{1-\mu} di \right)^{\frac{\alpha}{1-\mu}-1} x_i(t)^{-\mu} \qquad (x_i(t))$$

which simplify to

$$w(t) = (1-\alpha)p(t)\frac{Y(t)}{L(t)} \qquad \text{and} \qquad x_i(t)^\mu = \alpha \frac{p(t)}{p_i(t)} \frac{Y(t)}{\int_0^{A(t)} x_i(t)^{1-\mu} di}$$

Since markets are competitive, $\pi = 0$, and we can use the zero profit condition to get that

$$x_i^\star(t) = \alpha^{\frac{1}{\mu}} \left(\frac{p(t)}{p_i(t)}\right)^{\frac{1}{\mu}} Y(t)^{\frac{\mu+\alpha-1}{\alpha\mu}} L(t)^{\frac{(1-\mu)(1-\alpha)}{\alpha\mu}}$$

The intermediate goods producer, who has already paid the fixed cost, solves

$$\max_{p_i(t)} p_i(t)x_i(t) - w(t)\frac{x_i(t)}{a}$$

Because of monopolistic competition, x_i is a function of $p_i(t)$. Optimality yields that $p_i(t) = \frac{w(t)}{a(1-\mu)}$, which is a constant markup over marginal cost.

In equilibrium, all firms are identical so $p_i(t) = \hat{p}(t)$. We could show that $L^{CE}(t) = \frac{1-\alpha}{1-\alpha\mu} > 1 - \alpha$ so there is more labor in the final goods sector than in the planner's allocation. Intuitively, $x_i(t)$ is relatively expensive, so firms switch to labor rather than innovation – there is less investment in ideas. The growth rates are

$$\frac{\dot{c}^{CE}(t)}{c^{CE}(t)} = \frac{1}{\sigma}\left(b\alpha^\alpha \alpha\mu \frac{1-\alpha}{1-\alpha\mu} - \rho\right) < \frac{\dot{c}(t)}{c(t)}$$

Remark. Variety models deal with horizontal innovation, but typically innovations both (i) improve the quality of the good, and (ii) lower the cost of production. Endogenous growth, in the Schumpeterian economies involves price competition, the replacement of old vintages, and business stealing effects (entrants).
Model. *Endogenous Vertical Innovation* (from Aghion & Howitt, 1992; Howitt & Aghion, 1998, and Grossman & Helpman, 1991) We have a representative household with CRRA preferences, constant population, and inelastic labor supply L. The resource constraint is that $C(t) + X(t) + Z(t) = Y(t)$ for R&D $Z(t)$, investment $X(t)$, and consumption $C(t)$. The final good is produced:

$$Y(t) = \frac{1}{1-\beta} L(t)^\beta \left(\int_0^1 q(v,t)x(v,t \mid q)^{1-\beta} dv\right)$$

where $x(v, t \mid q)$ is the quantity of machines of vintage v of quality $q(v, t)$ at time t. The source of growth here is *quality improvements*. We have a *quality ladder* for each machine type, where

$$q(v,t) = \lambda^{n(v,t)} q(v,0)$$

where $\lambda > 1$ and $n(v, t)$ is the number of innovations so far. At any point in time, only one quality of any machine v is used – *i.e.* creative destruction, where the invention of a higher-quality machine 'destroys' (replaces) an older machine. Once a machine of quality $q(v, t)$ is invented, any quantity can be produced at cost $\psi q(v, t)$.
Remark. Innovation is driven by entrants. This is sometimes called *Arrow's Replacement Effect*, where incumbents have little incentive to innovate as that would destroy their own profits.

Innovation requires investment, where $Z(v, t)$ units of the final good are used for research in line v with quality $q(v, t)$. The rate of innovation is $z(v, t \mid q) = \eta \frac{Z(v,t)}{q(v,t)}$, and we have (i) free entry into research, and (ii) perpetual patents for innovators.
Definition. An *allocation* in this economy is a time path for (i) consumption levels, aggregate spending on machines, and aggregate R&D spending $\{C(t), X(t), Z(t)\}_{t=0}^\infty$; (ii) machine qualities $\{q(v, t)\}_{t=0}^\infty$ for $v \in [0, 1]$; (iii) prices and quantities of each machine and the net present value of profits from each machine: $\{p^x(v, t \mid q), x(v, t \mid q), V(v, t \mid q)\}_{t=0}^\infty$ for $v \in [0, 1]$; and (iv) interest rates and wages $\{r(t), w(t)\}_{t=0}^\infty$.

The final good producer solves

$$\max_{L,x(v)} \frac{1}{1-\beta}L(t)^\beta \left(\int_0^1 q(v,t)x(v,t \mid q)^{1-\beta}dv\right) - w(t)L(t) - \int_0^1 p^x(v,t)x(v,t \mid q)dv$$

The optimal machine demand is

$$x(v,t \mid q) = \left(\frac{q(v,t)}{p^x(v,t \mid q)}\right)^{\frac{1}{\beta}} L$$

There are typically two regimes: (i) drastic innovation, where firms charge monopoly prices, and (ii) limit prices. For now, we assume (i), attained when λ is sufficiently large, such that $\lambda \geq \left(\frac{1}{1-\beta}\right)^{(1-\beta)/\beta}$. We normalize $\psi = 1-\beta$, and we have that

$$\begin{aligned}\pi(v,t) &= \max_x p^x(v,t \mid q)x(v,t \mid q) - \psi q(v,t)x(v,t \mid q) \\ &= \max_x q(v,t)L^\beta x(v,t \mid q)^{1-\beta} - \psi q(v,t)x(v,t \mid q)\end{aligned}$$

then for a profit maximizing monopoly, we have that

$$x(v,t \mid q) = L \quad ; \quad p^x(v,t \mid q) = q(v,t) \quad ; \quad \pi(v,t) = \beta q(v,t)L$$

Total output will be

$$Y(t) = \frac{1}{1-\beta}Q(t)L \qquad \text{where} \quad Q(t) = \int_0^1 q(v,t)dv$$

and aggregate spending in machines (and the implied equilibrium wage rate) is therefore

$$\int_0^1 p^x(v,t \mid q)x(v,t \mid q)dv = Q(t)L \qquad \text{which implies that} \qquad w(t) = \frac{\beta}{1-\beta}Q(t)$$

The monopolist with vintage v and quality $q(v,t)$ has a value function defined by

$$r(t)V(v,t \mid q) - \dot{V}(v,t \mid q) = \pi(v,t \mid q) - z(v,t \mid q)V(v,t \mid q)$$

where $z(v,t \mid q)$ is the arrival rate of innovations to vintage v. The creative destruction implies that when an innovation occurs, the monopolist loses its monopoly and is replaced by a higher quality producer; and from there the innovation has zero value. This implies that $z(v,t \mid q)$ is the rate of replacement of incumbents in vintage v. Since we have free entry, entrants have

$$\eta V(v,t \mid q) \leq \frac{q(v,t)}{\lambda} \qquad \text{with equality if } z(v,t \mid q) > 0$$

The consumer's maximization problem admits the Euler equation

$$\frac{\dot{C}(t)}{C(t)} = \frac{1}{\theta}(r(t) - \rho)$$

with the transversality condition

$$\lim_{t\to\infty} \exp\left(-\int_0^t r(s)ds\right)\int_0^1 V(v,t \mid q)dv = 0$$

for all q, where $V(v,t)$ is nonstochastic.
Definition. An *equilibrium* is an allocation such that (i) the aggregate feasibility constraints for goods and

machines and the transversality condition are satisfied; (ii) the value of the firm and the average quality satisfy optimality of the monopolist and machine demands, and free entry; (iii) prices and quantities of machines are as described in the monopolist's problem; and (iv) the interest rate and wages are consistent with the Euler equation and feasibility in labor markets.

Given that consumption grows at a constant rate on the balanced growth path, feasibility implies that output also grows at a constant rate, and from the Euler equation, the interest rate is constant. If there is positive growth there must be research in at least one sector. Linearity of the value of the firm and innovation costs in quality together imply that free entry holds for all varieties. If free entry holds in all periods, then $\dot{V}(v,t \mid q) = 0$ and R&D for each machine type has the same productivity $z(v,t) = z(t) = z^\star$. Then, the firm value is

$$V(v,y \mid q) = \frac{\beta q(v,t)L}{r^\star + z^\star}$$

so we have an effective discount of $r^\star + z^\star$, and by free entry and the Euler equation respectively, we have:

$$r^\star + z^\star = \eta\beta\lambda L \qquad ; \qquad g^\star = \frac{r^\star - \rho}{\theta} \Longrightarrow r^\star = g^\star \cdot \theta + \rho$$

From the definition of output, we have $\frac{\dot{Y}(t)}{Y(t)} = \frac{\dot{Q}(t)}{Q(t)}$, and dynamics

$$Q(t + \Delta t) = \lambda Q(t)z(t)\Delta t + (1 - z(t)\Delta t)Q(t) + o(\Delta t)$$

Note that the measure of the varieties that experience more than one innovation is second order in t, so $\frac{o(\Delta t)}{\Delta t} \to 0$. Thus, we have that

$$\dot{Q}(t) = (\lambda - 1)z(t)Q(t) \qquad ; \qquad g^\star = (\lambda - 1)z^\star$$

So the equilibrium growth rate is

$$g^\star = \frac{\eta\lambda\beta L - \rho}{\theta + (\lambda - 1)^{-1}}$$