ECON 6100

General Equilibrium Notes

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Spring 2025

Contents

Introduction

"We have all of this administrative crap to do. Let's get that out of the way" – L. Blume

Grading is the least important part of this class – you can just tank it, and as long as you pass the Q you'll be fine! (There is correlation, but unsure if there's causation).

We will begin by talking about linear programming, which is the best way to understand duality. Linear programming is a shockingly useful tool, and duality is essential (see Myerson for a cutting-edge example). We will use the tools Tak taught – specifically, Convex Analysis. Then we will apply these tools to talk about linear production models, including the non-substitution theorem, which drove a lot of macroeconomics in the late 20th and early 21st century. We will go on to talk about some other uses of constant returns to scale production, including in international trade.

We will then go through some of the issues with welfare economics, which Larry believes is an interesting area of research. We will talk about uncertainty, matching, and (if we have time) some mechanism design in market settings.

Grading: Participation is 40%. Problem sets will be 10%. One prelim will be worth 10%, and by implication the final is additionally 40%.

1 Linear Programming

This part of the course will have more proofs than the rest of the course – convexity in general, and linear programming specifically, are about geometry.

Lemma 1.1. Farkas' Lemma *Given a matrix A and a vector b, exactly one of the following is true:*

1. *$Ax = b$, $x \geq 0$ has a solution*
2. *The system $yA \geq 0$, $yb < 0$ has a solution*

Proof. (Intuition) Consider the set

$$\{z : z = Ax, x \geq 0\}$$

This set is convex. Interestingly, it is *not* necessarily closed, though the difference is subtle. More specifically, it is closed, but not for the reason you think it's closed. It's actually a polyhedron, and more specifically a convex polyhedral cone. What Farkas' Lemma says geometrically is that a vector is either in the convex polyhedral cone, or that b can be separated by the cone by a hyperplane – specifically the hyperplane y. □

Remark. Quick notation break – $x \geq 0$ means that each $x_i \geq 0$. $x > 0$ means that x is semipositive, so $x \geq 0$ and some $x_i > 0$. $x \gg 0$ means that each $x_i > 0$. Additionally, if we say that $x \star y$, then we are saying that $x - y \star 0$ for any relationship $\star$.

Definition. A *polyhedron* is the intersection of finite halfspaces. A *polytope* is a bounded polyhedron.

Remark. Any convex set is the intersection of (any number of) halfspaces. Polyhedra have more properties.

Definition. The *canonical form* of a linear program is written

$$\begin{aligned} & \max c \cdot x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

The *standard form* of a linear program is written

$$\begin{aligned} \max\, c \cdot x \\ \text{s.t.} \qquad A'x = b' \\ x \geq 0 \end{aligned}$$

Definition. x is a *vertex* of a polyhedron F if and only if there is no $y \neq 0$ such that $x + y$ and $x - y$ are both in F.

Theorem 1.1. Vertex Theorem *If a linear program in standard form has feasible solutions, then*

1. *It has a feasible vertex*
2. *If $v_P(b) < \infty$ and x is feasible, then there is a feasible vertex x' such that $c \cdot x' \geq c \cdot x$*

Remark. By implication, if a standard form problem has an optimal solution then it has an optimal vertex solution.

Proof.

1. Let F denote the feasible set. We will describe an algorithm for finding a vertex and show that it always succeeds. Choose $x \in F$. If x is a vertex, done! If not, there exists $y \neq 0$ such that $x \pm y \in F$. For any such y, $Ay = 0$. Let $\lambda^\star \geq 0$ solve $\sup\{\lambda : x \pm \lambda y \in F\}$. Since x is not a vertex, $\lambda^\star > 0$, and since F is closed $x \pm \lambda^\star y \in F$. However, this is on a border. At least one of $x \pm \lambda^\star y$ has more zeros than x does. Assign that x_1. If x_1 is a vertex, done! If not, repeat to get x_2, and so on. Eventually, at least x_n will have all zeros and we will be done.
2. Left as exercise, but exact same basic form as (1)

□

Definition. The *support* of a feasible solution x is the set of all indices j such that $x_j > 0$

$$\text{supp}(x) = \{j : x_j > 0\}$$

Definition. The jth column of A is denoted A^j. A feasible solution is *basic* if $\{A^j : j \in \text{supp}(x)\}$ is linearly independent.

Theorem 1.2. *A feasible solution x is a vertex if and only if it is basic.*

Proof. If x is not a vertex, then $\exists\, y \neq 0$ s.t. $x \pm y$ is feasible, and $Ay = 0$ such that if $x_j = 0, y_j = 0$. This implies that Ay is a linear combination of the columns A^j, and since it is equal to zero and $y \neq 0$, then x is not basic since A^j are linearly dependent.

If x is not basic, then A^j are linearly dependent, so there exists $y \neq 0$ such that if $x_j = 0$, $y_j = 0$ and $Ay = 0$. For $\lambda \in \mathbb{R}$ such that $|\lambda|$ sufficiently small, $x \pm \lambda y \geq 0$, meaning that $x \pm \lambda y$ feasible, so x is not a vertex. □

Proposition 1.1. *Suppose x is a feasible solution, y is a feasible solution, $x \neq y$, and $\text{supp}(y) \subseteq \text{supp}(x)$. Then x is not basic.*

Proof. $Ax = b$, $Ay = b$, $A(x - y) = 0$, $x \neq y$, $\implies A^j$ linearly dependent. □

Theorem 1.3. The Fundamental Theorem of Linear Programming *If a problem in standard form has a feasible solution, then it has a basic feasible solution. If it has an optimal solution, it has a basic optimal solution.*

Proof. Left as exercise. □

Definition. We define the *primal problem* as follows:

$$\begin{aligned} v_P(b) = \max\, c \cdot x \\ \text{s.t.} \qquad Ax \leq b \\ x \geq 0 \end{aligned}$$

The *dual problem* is:

$$\begin{aligned} v_D(c) = \min\ & y \cdot b \\ \text{s.t.} \quad & yA \geq c \\ & y \geq 0 \end{aligned}$$

Exercise. Suppose that we have the following primal problem:

$$\begin{aligned} \max\ & c \cdot x \\ \text{s.t.} \quad & Ax \leq b \\ & A'x = b' \\ & x \geq 0 \end{aligned}$$

Prove that the dual can be expressed

$$\begin{aligned} \min\ & y \cdot b + z \cdot b' \\ \text{s.t.} \quad & yA + zA' \geq c \\ & y \geq 0 \end{aligned}$$

Note that there are no sign constraints on the z variables.

Theorem 1.4. Weak Duality *For the primal and dual problems,* $v_P(b) \leq v_D(c)$

Proof. For feasible solutions x and y of the primal and dual respectively, we must have that $(yA - c)x \geq 0$ and $y(b - Ax) \geq 0$, so for all feasible solutions x and y, we must have that $c \cdot x \leq y \cdot b$. □

Theorem 1.5. Duality *For the primal problem and the dual problem, exactly one of the following must hold:*

1. *Both are feasible, both have optimal solutions, and the optimal solutions coincide*
2. *One is unbounded and the other is infeasible*
3. *Both are infeasible*

Proof. Long, left out. Straightforward, but annoying. □

Theorem 1.6. Complimentary Slackness *Suppose that* $x^\star$ *and* $y^\star$ *are feasible for the primal and dual respectively. Then they are optimal solutions if and only if for each constraint* i *in the primal problem and* j *in the dual problem,*

$$y^\star(b - Ax^\star) = 0 \qquad \text{and} \qquad (y^\star A - c)x^\star = 0$$

Proof. In notes, out of time. □

Lemma 1.2. $v_P(b)$ *is concave, and* $v_D(c)$ *is convex, and the domains of each are closed convex sets.*

Consider the following restatement of the Duality Theorem:

Theorem 1.7. *If* $v_P(b)$ *or* $v_D(c)$ *is finite,*

1. *Both are finite*
2. *Both programs have optimal solutions*
3. $\partial v_D(c)$[1] *is the set of solutions to the primal, and* $\partial v_P(b)$ *is the set of solutions to the dual*

Proof. (Just of Part 3, only one part. Other is parallel) ($\Rightarrow$): Assume $y^\star b = v_D(c)$ and $y^\star b = v_P(b)$ by finite. Then for any problem with the same feasible set and objective b', $y^\star$ is feasible and not optimal. Thus, we have

$$y^\star b' - v_P(b') \geq y^\star b - v_P(b)$$

[1] The subgradient of v_D at c.

which implies the subgradient inequality. □

2 Polyhedral Models

Model. *The Open Leontief Model* (sometimes *Input-Output Model*, from Leontief) We have N produced goods, and production is described by a matrix $A = \left[a_{ij}\right]_{N\times N} \in \mathbb{R}^{N\times N}$ where a_{ij} is the amount of good i necessary to produce good j. Specifically, we have Leontief isoquants – if $a_{11} = \frac{1}{2}$ and $a_{12} = 1$, then in the $N = 2$ model the isoquants look like Figure 1.

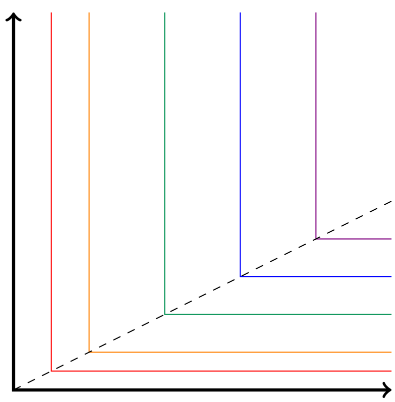

Figure 1: Leontief Isoquants

We have one good that is not produced – $a_0 = (a_{01}, a_{02}, \dots, a_{0N})$. We have a labor endowment of L, and we call gross output x and $y \leq x$ net output.

Call the vector $(a_{0j}, \cdots, a_{Nj})$ a *technique* for producing good j. Note that to produce some vector $x = (x_1, \dots, x_N)$, we need Ax of the inputs. Then, of course, $y = x - Ax$. The question we'll face next time: Given our technology, can we produce anything?

Definition. A is *productive* if $\exists\, x^\star \gg 0$ s.t. $x^\star \gg Ax^\star$ (equivalently: if $y = x^\star - x^\star$, $y \gg 0$).

Theorem 2.1. *If A is productive, any $y \geq 0$ can be produced (*i.e. *for any $y \geq 0, \exists\, x \geq 0$ s.t. $(I - A)x = y$)*

Proof.

Lemma 2.1. *If A is productive, and $x \geq Ax$, then $x \geq 0$.*

Proof. Suppose $\exists\, x \ngeq 0$ s.t. $x \geq Ax$. Define $\lambda' = \inf\{\lambda : x + \lambda x^\star \geq 0\}$, where $x^\star \gg Ax^\star$ exists by productivity, and define $x' = x + \lambda' x^\star$. Then

$$x + \lambda' x^\star \gg Ax + \lambda' Ax^\star = A(x + \lambda' x^\star) \geq 0$$

so λ' is not the infimum. □

Corollary 2.1. *If A is productive, then $I - A$ has full rank.*

Proof. Suppose $(I - A)x = 0$. Then $x \geq Ax$ and $x \geq 0$. Since $(I - A)(-x) = 0$ then $-x \geq A(-x)$ and $-x \geq 0$. Thus, $x = 0$ and $(I - A)$ has a rank 0 null space. □

Since $I - A$ is invertible, for any $y \geq 0$ there is an x such that $(I - A)x = y$, then by the Lemma $x \geq 0$. □

Theorem 2.2. *If $(I - A)^{-1}$ has non-negative columns and is non-singular, then A is productive.*

Proof. For any $y \geq 0$, $(I - A)^{-1}y \geq 0$. Since $(I - A)^{-1}$ is non-singular, it has no zero column, so every column is semi-positive. Therefore $x^\star = (I - A)^{-1}e \gg 0$, and $x^\star \gg Ax^\star$. □

Remark. There are other conditions that work.
Theorem 2.3. Hawkins-Simon *A is productive $\iff$ all leading principal minors are positive*
Theorem 2.4. *If A is productive, then $A^n x \to 0$ (at a geometric rate).*

Proof. Since A is productive, $x^\star \gg Ax^\star$ for some $x^\star \gg 0$, and there is $\lambda \in (0,1)$ such that $\lambda x^\star \gg x^\star$. Then $Ax^\star \ll \lambda A x^\star \ll \lambda^2 x^\star$, and for all n $\lambda^n x^\star \gg A^n x^\star$, so $A^n x^\star \to 0$ and $A^n \to 0$. ☐
Corollary 2.2. *If A is productive, then $\lim_{n\to\infty}(I + A + A^2 + \cdots + A^n) = (I-A)^{-1}$*

Proof. $(I-A)(I + A + A^2 + \cdots + A^n) = I - A^{n+1} \to I$. ☐

Suppose that the economy is endowed with L units of the (non-produced) primary good. What net output bundles can we make? To produce y, we need $(I-A)^{-1}y$ units of gross output, which requires $a_0(I-A)^{-1}y$ of the primary factor. Thus, our production possibility set is

$$P(L) = \{y : a_0(I-A)^{-1}y \leq L\}$$

Definition. A *price vector* $(p_0, p_1, \ldots, p_N) = (p_0, p) \in \mathbb{R}_+^{N+1}$ where p_0 is the price of the primary input and p_i is the price of produced good i. The cost of producing one unit of good j is

$$c_j = p_0 a_{0j} + pA^j$$

and the profit from producing one unit of good j is

$$\pi_j = p_j - c_j = p_j - \sum_m p_m a_{mj} \implies \pi = p(I-A) - p_0 a_0$$

Definition. An *equilibrium* is a tuple $\langle x, y, p, p_0 \rangle$, such that (i) $y \leq (I-A)x$, (ii) $a_0 x < L \implies p_0 = 0$, (iii) $y_m < x_m - a_m x \implies p_m = 0$, (iv) $\pi \leq 0$, (v) $\pi x = 0$, and (vi) $a_0 x \leq L$.
Assumption 2.1. *We set $p_0 = 1$ almost always, and deal with prices relative to labor. The exception is when we have excess labor. See condition* (ii) *of equilibrium above.*
Theorem 2.5. *If A is productive and $a_0 \gg 0$, an equilibrium exists in which $y \gg 0$, $p \gg 0$, and all profits are 0.*

Proof. Per-unit profits are $\pi = p(I-A) - a_0$. If A is productive, then $(I-A)$ is invertible, so take $p = (I-A)^{-1}a_0$, so $\pi = 0$. Next, choose any y. The required labor input is $a_0(I-A)^{-1}y$, and we can scale y directly so that this equals L. Strict positivity of p follows from the fact that $(I-A)^{-1}$ is non-negative and since it is non-singular, it has at least one non-zero element in each column. Conclusion follows from the hypothesis that $a_0 \gg 0$. ☐
Remark. That A is productive is, of course, necessary. The condition that $a_0 \gg 0$ can be relaxed, and that's very contemporary research. Specifically, we need that the product

$$a_0 \Big\{ I + A + A^2 + \cdots + A^{n+1} + \cdots \Big\} \gg 0$$

This would reduce the condition to $a_0 \geq 0$ and $a_0 \neq 0$, and a sufficient condition for that is the graph described by A being irreducible – *i.e.* that if we draw a directed graph where an arrow from i to j means 'i is used in the production of j', that graph being irreducible implies that for sufficiently large m, A^m is always strictly positive, which suffices to show that $(I-A)^{-1}$ is strictly positive. Note that we can reach the entirety of the indirect reach of labor with *only* the first $n-1$ matrix products, where n is the size of the matrix.
Definition. A *convex support function* for a set C is defined by the maximization problem

$$v_C(q) = \max\{q \cdot x : x \in C\}$$

This function is convex and homogeneous of degree 1. The *concave indicator function* of a convex set C is the function

$$\mathbb{1}_C(x) = \begin{cases} 0 & x \in C \\ -\infty & x \notin C \end{cases}$$

(and the *convex indicator function* is defined analogously, with $+\infty$).

Question. What does the set of convex support functions actually look like? It sits in $\mathbf{C}^1$, which is a normed vector space. In fact, it is *precisely* the set of continuous, homogeneous of degree 1, and convex functions, which is a convex cone! We can, in fact, even put a measure on this space and regress over it. This means that we can put a measure over all convex sets, and even prove central limit theorems and laws of large numbers over them.

Example. Consider the production possibility set of our economy, where we have the production matrix A, the labor requirement vector a_0, and labor endowment L. The set is described by the constraints: (i) $y - (I - A)x \leq 0$, (ii) $a_0 x \leq L$, and (iii) $x, y \geq 0$. We can define its convex support function as

$$v_C(q) = \max q \cdot y$$

subject to

$$\begin{aligned} y - (I - A)x &\leq 0 \\ a_0 x &\leq L \\ x, y &\geq 0 \end{aligned}$$

The dual of this problem is

$$\min_{p_0, p} p_0 \cdot L$$

subject to

$$\begin{aligned} p &\geq q \\ -p(I - A) + p_0 a_0 &\geq 0 \\ p_0, p &\geq 0 \end{aligned}$$

Remark. We can interpret q as 'world prices' in a market where only final goods are shipped.

Remark. Complimentary slackness of $p - q$ implies that $p_m > q_m \iff$ we produce 0 of good m, and that either we use a positive amount of good i in production or we make positive profit on good i. Formally:

Corollary 2.3. *At an optimal primal-dual quadruple $(y^\star, x^\star, p^\star, p_0^\star)$, we have that:*

1. *$p^\star y^\star - p^\star(I - A)x^\star = 0$, so if good m is in excess supply then $p_m^\star = 0$.*
2. *$p_0^\star(a_0 x^\star - L) = 0$, so if labor supply is not exhausted then wage $p_0^\star = 0$.*
3. *$(p^\star - q)y^\star = 0$, so if the net output of good m is positive, then $p_m^\star = q_m$.*
4. *$p^\star(I - A)x^\star + p_0^\star a_0 x^\star = 0$, so if good m is produced, profits $\pi_m = 0$.*

Remark. These complementary slackness conditions precisely define the equilibrium we defined above.

Theorem 2.6. *If A is productive and $a_0 \gg 0$, then both the primal and dual have optimal solutions. If $(x^\star, y^\star)$ solves the primal problem and $(p^\star, p_0^\star)$ solves the dual, then $(x^\star, y^\star, p^\star, p_0^\star)$ is an equilibrium.*

Proof. If A is productive, the feasible set is nonempty, as the first primal inequality has at least one solution. If $a_0 \gg 0$, then it is bounded, so the primal problem attains a maximum. Conclusion follows from strong duality. □

Model. *Activity Analysis Model of Production* We have N goods, M activities, $M \geq N$, a matrix $A \in \mathbb{R}^{N \times M}$ where a_{mn} is the amount of good n needed to run activity m at unit level, and a_{0m} the amount of 'labor' required to run activity m at unit level. The only difference, besides A no longer being square, is that we now have $B \in \mathbb{R}^{N \times M}$, where the column B^m is the output vector of goods $1, \ldots, n$ from running activity m

at unit level.

The vector $x \in \mathbb{R}^m_+$ is now the vector of levels at which the different activities are run. For activity vector $x \geq 0$, the input requirements are Ax and the output levels are Bx.

Definition. The model is *productive* if there exists $x^\star \geq 0$ such that $Bx^\star \gg Ax^\star$. The *production possibility set* of the economy is

$$Y = \{y \geq 0 : (B - A)x \geq y, a_0 x \leq L, x \geq 0\}$$

Remark. The general Leontief model is a special case of the Activity Analysis Model, where we assume that there is no joint production.

Definition. A *technology* τ is a set of N activities in $\{1, \dots, M\}$ such that through those activities alone every good is produced.

The problem, therefore, is to characterize the production possibility set. Define the cost functions

$$\lambda(y) = \min\{a_0 x : (B - A)x \geq y, x \geq 0\}$$
$$\lambda^\tau(y) = \min\{a_0 x : (I - A^\tau)x \geq y, x \geq 0\}$$

that gave the minimum amount of the primary factor needed to produce net output y in (first) the general model and (second) the technology τ. Define their respective production possibility sets as

$$P(L) = \{y : (B - A)x \geq y, x \geq 0, a_0 x \leq L\}$$
$$P^\tau(L) = \{y : (I - A^\tau)x \geq y, x \geq 0, a_0 x \leq L\}$$

Note that for any τ, $P^\tau(L) \subseteq P(L)$.

Theorem 2.7. Non-Substitution Theorem *There is a technology $\tau^\star$ such that for all $y \geq 0$, $\lambda(y) = \lambda^{\tau^\star}(y)$.*

Corollary 2.4. $P^{\tau^\star}(L) = P(L)$.

Proof. First, to produce the vector $\mathbb{1}$, we solve the problem

$$\lambda(\mathbb{1}) = \min a_0 x \text{ s.t. } (B - A)x \geq \mathbb{1}, x \geq 0$$

Productivity of A, B implies that the feasible set is nonempty, so this problem has a solution. This means that it has a basic optimal solution, and we call the set of its columns a technology $\tau^\star$.

Recall that for any technology τ, cost is linear in y

$$\lambda^\tau(y) = a_0(I - A^\tau)^{-1} y$$

To show that $\lambda^{\tau^\star}(y) = \lambda(y)$, it suffices to show that $\lambda^{\tau^\star}(y) \leq \lambda^\tau(y)$ for any other τ.

Lemma 2.2. *For each $\mathbb{1}^m$ and for all τ, $\lambda^{\tau^\star}(\mathbb{1}^m) \leq \lambda^\tau(\mathbb{1}^m)$.*

Proof. FSOC, assume that there is a cheaper technology θ for producing $\mathbb{1}^1$. Then

$$\lambda(\mathbb{1}) \leq \lambda^\theta(\mathbb{1}^1) + \sum_{m \geq 2} \lambda^{\tau^\star}(\mathbb{1}^m) < \sum_{m \geq 1} \lambda^{\tau^\star}(\mathbb{1}^m) = \lambda^{\tau^\star}(\mathbb{1})$$

which is a contradiction. □

To conclude the proof, observe that any y can be produced at minimum cost by some technology, and for any technology τ,

$$\lambda^\tau(y) = \sum_m y_m \lambda^\tau(\mathbb{1}^m) \geq \sum_m y_m \lambda^{\tau^\star}(\mathbb{1}^m) = \lambda^{\tau^\star}(y)$$

□

A Brief Aside on Modeling. A model is an abstraction of the world. A model is a set of objects and a set of relationships. In Economics, we have agents, goods, beliefs, preferences as the objects; states such as prices and capital; and relationships such as behavioral relationships (between any objects) and consistency conditions.

3 The Hecksher-Ohlin-Vanek Model

Model. *Leontief Version* Consider a small country with immobile capital stock K and labor endowment L that trades final products clothing c and food f on world markets at prices p_c and p_f. The production technology for good g is described by input requirement coefficients a_{kg} and a_{lg}. Assume:

Assumption 3.1. *Clothing is capital-intensive, food is labor-intensive:* $\frac{a_{kc}}{a_{lc}} > \frac{a_{kf}}{a_{lf}}$

The PPS is the set $\{(x_c, x_f) : a_{kc}x_c + a_{kf}x_f \leq K, a_{lc}x_c + a_{lf}x_f \leq L, x \geq 0\}$. This set is convex, and (as before) we can characterize it with its concave support function. The support function is

$$\begin{aligned} v_P(K, L) = \max_x \, & p_c x_c + p_f x_f \\ \text{s.t.} \qquad a_{ck}x_c + a_{fk}x_f &\leq K \\ a_{cl}x_c + a_{fl}x_f &\leq L \\ x &\geq 0 \end{aligned}$$

The dual is

$$\begin{aligned} v_D(p_c, p_f) = \min_{r,w} \, & rK + wL \\ \text{s.t.} \qquad ra_{ck} + wa_{cl} &\geq p_c \\ ra_{fk} + wa_{fl} &\geq p_f \\ r, w &\geq 0 \end{aligned}$$

The complimentary slackness conditions are

$$\begin{aligned} (r^\star a_{kc} + w^\star a_{lc} - p_c)x_c^\star &= 0 \\ (r^\star a_{kf} + w^\star a_{lf} - p_f)x_f^\star &= 0 \\ r^\star(a_{kc}x_c^\star + a_{kf}x_f^\star - K) &= 0 \\ w^\star(a_{lc}x_c^\star - a_{lf}x_f^\star - L) &= 0 \end{aligned}$$

We can solve this model with a few assumptions on structure. Consider:

Case 1: $x \gg 0$**.** Let A denote the matrix whose rows are input requirements, $A = \begin{pmatrix} a_{kc} & a_{kf} \\ a_{lc} & a_{lf} \end{pmatrix}$. Assumption 3.1 implies that A is non-singular. For a solution where $x_c, x_f \gg 0$, complementary slackness requires that

$$\begin{bmatrix} r^\star & w^\star \end{bmatrix} A = \begin{bmatrix} p_c & p_f \end{bmatrix}$$

meaning that price is equal to marginal cost. A positive solution will exist if and only if:

Assumption 3.2. *Prices are interior:*

$$\frac{a_{kc}}{a_{kf}} > \frac{p_c}{p_f} > \frac{a_{lc}}{a_{lf}}$$

which is satisfiable under the assumptions. If the price ratio equalities are strict, then the dual solution $(r^\star, w^\star)$ is strictly positive. If so, complementary slackness implies that $Ax^\star = \begin{bmatrix} K & L \end{bmatrix}'$. A strictly positive solution requires that:

Assumption 3.3. *(K, L) is in the interior of the span of the columns of A.*

Theorem 3.1. *Suppose Assumptions 3.1, 3.2, and 3.3 hold. Then the primal and dual have unique strictly positive solutions.*

Remark. $x^\star$ maximizes GDP, and $r^\star$ and $w^\star$ are shadow prices for resource constraints. At those prices, all per-unit profits are non-positive and operating industries make 0 profits.

Note that as K and L change within the cone, factor prices do not change.

Theorem 3.2. Factor Price Equalization Theorem *In a diversified equilibrium, for all $(K, L) \in \{y : y = Ax, x \geq 0\}$ factor prices are those prices satisfying Assumption 3.2, which does not depend on (K, L).*

Remark. Two different countries with identical technologies but different capital-labor ratios will have the same factor prices.

Question. What is the effect of an increase in the price of good c?

Answer. Assumption 3.1 implies that the determinant of A is positive, and A^{-1} will have the sign pattern $\text{sgn}A^{-1} = \begin{pmatrix} + & - \\ - & + \end{pmatrix}$. This means that an increase in p_c will increase the rental rate r and lower the wage rate w.

Theorem 3.3. Stolper-Samuelson Theorem *In a diversified equilibrium, an increase in the world price of a commodity raises the price of the factor in which it is intensive and lowers the price of the other factor.*

The Picture. This is entirely illustrated in Figure 2

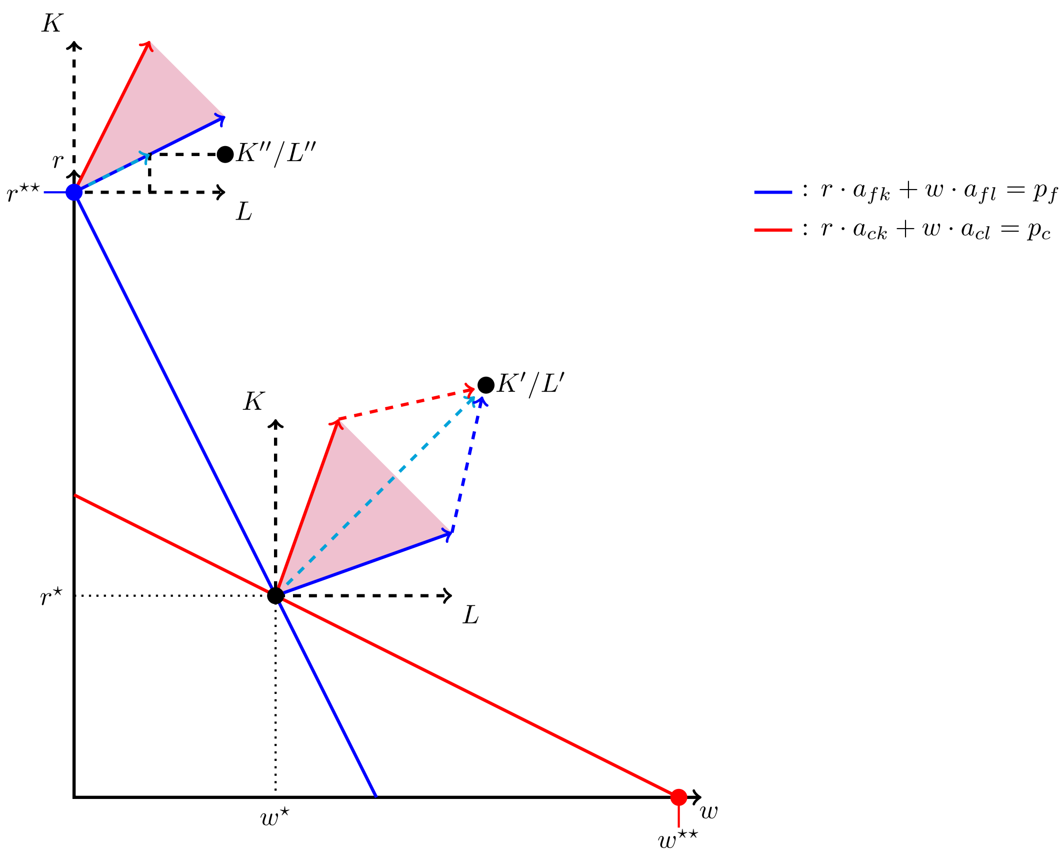

Figure 2: A Diversified Equilibrium

The Story The dot at $(w^\star, r^\star)$ is a *diversified equilibrium* – both goods are produced. The vectors are input requirements describing per-unit cost as a function of r and w. The dual feasible factor prices are those above the food and clothing isocost lines. The $K - L$ axes taking the intersections as their origin measure the primary factor endowment, and the cyan vector is the factor endowment. The (K', L') endowment is in the cone spanned by the input requirement vectors. The requirements for the diversified equilibria are that K', L' are in the cone and that the factor price vector sits on the intersection between the isocost lines for the capital- and labor-intensive industries respectively..

Case 2: $x_c = 0$. Then we have that $x_f > 0$ and $ra_{fk} + wa_{fl} = p_f$. There are three subcases. There is the knife-edge case where the factor endowment vector is the same as the inputs requirement vector for some good. Equilibrium factor prices will be $(w^\star, r^\star)$, but nothing will be produced.

Alternatively, if the red isocost line lies below the blue isocost line everywhere, then only food will be produced, and one factor will be entirely exhausted. If there is excess K, then $r = 0$ and $w = \frac{p_f}{a_{fl}}$. If $\frac{K}{L} = \frac{a_{fl}}{a_{fk}}$, any (r, w) pair on the blue isocost line is optimal. If there is excess L, then $w = 0$ and $r = \frac{p_f}{a_{fk}}$.

The most interesting case is if the red and blue isocost lines cross. Suppose that K is in excess supply, so $a_{fk}x_f < K$. Then $r = 0$, so $w = \frac{p_f}{a_{fl}}$, so $a_{fl}x_f = L$. This solution is the w-intercept of the blue line. However, it's clear that this is infeasible since the solution lies below the red isocost line.
Remark. The blue dot is a specialized equilibrium. The (K'', L'') endowment is below the cone, so equilibrium is at the upper corner. Output x_f is such that capital is just exhausted, and labor is in excess supply. Factor prices are $(0, r^{\star\star})$ and equilibrium factor demand is the other cyan arrow.

Trade. Suppose that we now have two countries with identical technologies. Country A has relatively more labor and country B has relatively more capital. World prices are established in a competitive equilibrium. What is the pattern of trade?
Theorem 3.4. Rybczynski Theorem *The country with a higher ratio of capital to labor will produce relatively more of the capital-intensive good, and the country with a higher ratio of labor to capital will produce relatively more of the labor-intensive good.*

Proof can be seen straightforwardly from the picture. If one country is entirely specialized, then the pattern is even stronger since they'll entirely specialize in the good they have an endowment advantage in.
Model. *Smooth Version* Consider a single small country with immobile capital stock K and labor endowment L, that trades final products a and b on world markets at world prices p_a and p_b. The production technology for good g is described by a production function $f_g(k, \ell)$.
Assumption 3.4. *The production function satisfies the following:*

1. $f_g \in \mathbf{C}^2$
2. f_g *is concave*
3. f_g *has constant returns to scale*
4. f_g *satisfies the Inada Conditions at 0:*

$$\lim_{k \to 0} \nabla_k f_g(k, \ell') = \lim_{\ell \to 0} \nabla_\ell f_g(k', \ell) = \infty \text{ for all } k', \ell' > 0$$

The profit function for industry g is found by the maximization problem

$$\pi_g(p_g, r, w) = \max_{k_g, \ell_g, x_g} p_g x_g - rk_g - w\ell_g \qquad \text{s.t. } x_g \leq f_g(k_g, \ell_g)$$

The solution to this problem gives both the output and factor demands at the output and factor market prices. Equilibrium requires that (i) outputs and factor demands are both profit maximizing, and (ii) all factor markets clear.

Since production is CRS (by Assumptions 3.4), cost functions are of the form $c_g(r,w)x_g$. Profit maximization requires zero profit for producers, so $p_g = c_g(r,w)$. This gives

$$c_f(r,w) = p_f \qquad \text{and } c_c(r,w) = p_c$$

so Shephard's Lemma gives factor demands

$$\frac{\partial c_f(r,w)}{\partial r}x_f + \frac{\partial c_c(r,w)}{\partial r}x_c = K \qquad \text{and} \qquad \frac{\partial c_f(r,w)}{\partial \ell}x_f + \frac{\partial c_c(r,w)}{\partial \ell}x_c = L$$

We have similar results to above: If (K,L) is in the cone spanned by the gradients of the unit cost functions, then a diversified equilibrium would exist. Moving (K,L) around inside the cone changes outputs but does not change factor prices since the first two equations are unperturbed. The gradient $\nabla c(r,w)$ is (by Shephard's Lemma) the input requirement vector. In the smooth model, the picture is Figure 3.

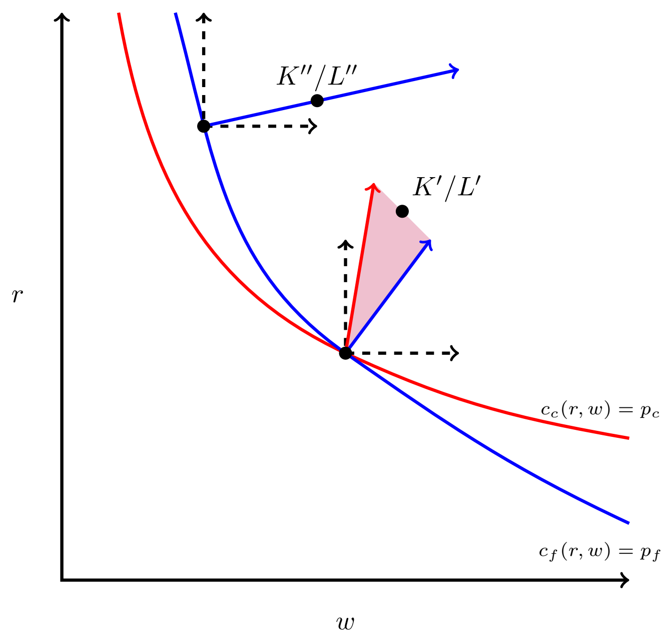

Figure 3: The Smooth Hecksher-Ohlin-Vanek Model

Remark. To demonstrate the Stolper-Samuelson Theorem in this model, we can apply the Implicit Function Theorem to the map

$$F(r,w,p_f,p_c) = \begin{pmatrix} c_f(r,w) - p_f \\ c_c(r,w) - p_c \end{pmatrix}$$

where the equilibria are the tuples for which $F(\cdot) = 0$. The Jacobian of F is

$$DF(\cdot) = \left[D_{r,w}F(\cdot)D_{p_f,p_c}\right]F(\cdot) = \begin{pmatrix} \nabla c_f(\cdot) & -1 & 0 \\ \nabla c_c(\cdot) & 0 & -1 \end{pmatrix}$$

If we assume the hypothesis that $D_{r,w}F(\cdot)$ is non-singular, we have that

$$\begin{pmatrix} \frac{\partial r}{\partial p_f} & \frac{\partial r}{\partial p_c} \\ \\ \frac{\partial w}{\partial p_f} & \frac{\partial w}{\partial p_c} \end{pmatrix} = -\begin{pmatrix} \nabla c_f(\cdot) \\ \nabla c_c(\cdot) \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{\frac{\partial c_f}{\partial r}\frac{\partial c_c}{\partial w} - \frac{\partial c_c}{\partial r}\frac{\partial c_f}{\partial w}} \cdot \begin{pmatrix} \frac{\partial c_c}{\partial w} & -\frac{\partial c_f}{\partial w} \\ \\ -\frac{\partial c_c}{\partial r} & \frac{\partial c_f}{\partial r} \end{pmatrix}$$

The hypothesis that c is capital-intensive implies that the determinant is positive, so an increase in p_c lowers w and raises r, and an increase in p_f raises w and lowers r.

4 Walrasian Equilibrium

Remark. Think of the diamond-water paradox (appears in Smith, due to Plato). Nothing is more useful than water but it's incredibly cheap, nothing is less useful than a diamond but it's amazingly expensive.
Question. What does the value of something actually denote? A lot of people have tried to answer this, and there's a good rundown in Larry's notes.
Definition. The *marginal utility theory* (from Jevons) is that the ratio of prices is equal to the ratio of marginal utilities:

$$\frac{MU_x}{MU_y} = \frac{p_x}{p_y}$$

We can think of the different theories as a difference between classical economists, who tend to think about production and growth; and neoclassical economists, who are more interested in questions of allocation and distribution. We will think of two schools of general equilibrium theory. The *Walras-Cassel* model begins with demand functions, supply functions, and a classical production model. This leads to the two-sector model and the Hecksher-Ohlin-Vanek model. This entire process is about equating supply and demand. On the other hand, *Edgeworth-Pareto* use optimization – utility maximization, profit maximization, welfare economics, etc. This leads to the modern way of conceptualizing general equilibrium theory – especially in macroeconomics.

An aside on the integrability of demand.
Question. Why are indifference surfaces 'more general' than utility functions?

To go from demand to utility, we use a budget balance, indirect utility, and the expenditure function:

$$v^0 = V(p^0, w^0) = U(x^M(p^0, w^0)) \quad ; \quad \mu(p, p^0, w^0) = e(p, V(p^0, m^0)) \quad ; \quad \mu(p^0, p^0, m^0) = m^0$$

where $\mu(p, p^0, w^0) = e(p, V(p^0, m^0))$ is the *income compensation function*. Together, we have that

$$\frac{\partial \mu(p, p^0, m^0)}{\partial p_i} = \frac{\partial e(p, V(p^0, m^0))}{\partial p_i} = x_i^H(p, v^0) = x_i^M(p, e(p, v^0)) = x_i^M(p, \mu(p, p^0, w^0))$$

In summary, we define $e(p) = \mu(p, p^0, w^0)$, which solves the differential equation

$$D\mu(p) = x_i^M(p, e(p)) \quad \text{s.t.} \quad e(p^0) = w^0$$

Fix a $p^\star$ and notice that $\mu(p^\star, p, w)$ is an indirect utility function. We can invert Marshallian demand to get $\chi^m : x \to (p, w)$, and $U(c) = \mu(p^\star, \chi^m(x))$.

Can we carry out this program? If we have two or less goods, definitely! With three or more, it becomes an issue. Suppose we are given a Marshallian demand function x^M. Define the *Slutsky substitution coefficients*

$$\sigma_{ij}(p,w) = \frac{\partial x_i^M}{\partial p_j} + x_j^M \frac{\partial x_i^M}{\partial w}$$

Theorem 4.1. *Let $x^M : \mathbb{R}_+^n \times \mathbb{R}_+ \to \mathbb{R}_+^n$ be a Marshallian demand. If:*

1. *Budgets are exhausted, so $p \cdot x^M(p,w) = w$*
2. *x^M is differentiable throughout its domain*
3. *The Slutsky coefficients are symmetric, so $\sigma_{ij}(p,w) = \sigma_{ji}(p,w)$*
4. *The Slutsky matrix is negative semidefinite*
5. *The magnitude of $D_w x^M$ is bounded on compact subsets of strictly positive prices*

then there is a utility function U on the range of x^M that rationalizes demand.

Behavioral General Equilibrium. Walras and Cassel posit demand functions, firm profit maximization, and search for prices that equilibrate the system. This is behavioral because individual demands are simply decision rules.

Definition. A *behavior* is a rule that maps environments into actions. In GE models, an environment for a consumer is a budget set. An environment for a firm is a price vector and a production possibility set. This is straightforward in an Arrow-Debreu economy, but is more complicated in an exchange economy.

Model. *Market Equilibrium from Demand* We consider an I-person *exchange economy* with N goods. Price vectors are $p \in \mathbb{R}_+^N$. Each individual i is described by an endowment vector $\omega_i \in \mathbb{R}_+^N \setminus \{0\}$ and a demand function $d_i : \mathbb{R}_+^N \times \mathbb{R}_+^N \setminus \{0\} \to \mathbb{R}_+^N$. The *endowment allocation* is $\omega = \{\omega_i\}_{i \in I}$ and *aggregate endowment* is $\boldsymbol{\omega} = \sum_i \omega_i$.

Definition. *Individual excess demand* is $z_i(p,\omega_i) = d_i(p,\omega_i) - \omega_i$ and *aggregate excess demand* is a function $Z : \mathbb{R}_+^N \setminus \{0\} \times \bigtimes_{i \in I} \mathbb{R}_+^N \setminus \{0\} \to \mathbb{R}^N$, where

$$Z(p,\omega) = \sum_i d_i(p,\omega_i) - \boldsymbol{\omega}$$

Equilibrium is *market clearing*, meaning that there is no aggregate excess demand. Formally, a price vector $p \in \mathbb{R}_+^N \setminus \{0\}$ is an *equilibrium price vector* if if $Z(p,\omega) \leq 0$ and $p \cdot Z(p,\omega) = 0$, so no commodity is in excess demand and if a commodity is in excess supply it has price zero.

Assumption 4.1. *We make the following assumptions on the excess demand function:*

1. *$Z(p,\omega)$ is homogeneous of degree 0 in prices*
2. *For all $p \in \mathbb{R}_+^N \setminus \{0\}$, $p \cdot Z(p,\omega) = 0$ (Walras' Law)*
3. *For all ω, $Z(p,\omega)$ is continuous in p*

These can all be justified by reference to individual demand.

Theorem 4.2. *If $Z(p,\omega)$ satisfies Assumption 4.1, an equilibrium price vector exists.*

Corollary 4.1. *If the correspondence $Z(p,\omega)$ is upper hemi-continuous in p and convex-valued, then an equilibrium price vector exists.*

Before we prove these, we first define the two best theorems of all time:

Theorem 4.3. Brouwer *If C is a convex, compact, and non-empty set and $f : C \to C$ is a continuous function, $\exists\, x \in C$ such that $x = f(x)$.*

Theorem 4.4. Kakutani *If C is a convex, compact, and non-empty set and $F : C \rightrightarrows C$ is a nonempty, convex, and closed-valued correspondence, then $\exists\, x \in C$ such that $x \in F(x)$.*

Proof. (Of Theorem 4.2) The concept here is to 'simulate' a price adjustment process and show that it has a fixed point. Homogeneity implies that we can restrict the price space to the unit simplex, $\Delta = \{p \in \mathbb{R}^N_+ : \sum_n p_n = 1\}$. Define $f(p)$ such that $f_i(p) = \max\{-p_i, Z_i(p)\}$. Define the map $\phi : \Delta \to \Delta$ by

$$\phi(p) = \frac{1}{(p + f(p)) \cdot e}(p + f(p))$$

To see what this does, look at

$$\frac{\phi_m(p)}{\phi_n(p)} = \frac{p_m + f_m(p)}{p_n + f_n(p)}$$

If $Z_m \cdot Z_n > 0$, we can't tell the relationship. If $Z_m > 0$ and Z_n... □