Final Exam

ECON 6170

December 09, 2021

Instructions: You have until Dec 16 at 11:59 PM EST to complete this exam and submit a digital copy of the homework on Canvas (typed in latex or handwritten). Late submissions will not be accepted. Feel free to submit any time before the deadline.

1. You are to work alone.
2. You may not discuss the exam with anyone but the instructor.
3. You may only consult materials used in class, including, notes, homeworks, referenced books, and student presentations.
4. You are free to cite results from class or previous homeworks in your answers *unless explicitly stated otherwise*.
5. If you have any questions about the exam, email me at sm2378@cornell.edu.
6. You must submit a title page with the following honor code: "I acknowledge and will adhere to the rules of the exam. I affirm that I will not consult outside sources and that all work will be my own."

The exam is out of 50 points, and there is one extra credit question. The highest possible score is 55/50.

1. (10pts) Prove or disprove each of the following claims. You can draw pictures if it helps when showing a counterexample.

   (a) A sum of two quasiconvex functions is quasiconvex.

   (b) A product of two strictly convex functions is strictly convex.

   (c) The max of two convex functions is convex.

   (d) Let $\phi, \tau : [0,1] \rightrightarrows [0,1]$ be correspondences. Define a new correspondence η as $\eta(x) = \phi(x) \cap \tau(x)$, and suppose this intersection is nonempty for all $x \in [0,1]$. If ϕ and τ are lower-hemicontinuous, then η is also lower-hemicontinuous.

   (e) Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 0$ if x is rational, and $f(x) = x$ if x is irrational. f is discontinuous everywhere.

2. (5pts) Implicit function theorem warm-up.

   (a) Let $F(x_1, x_2, y) = x_1 + x_2 + y - e^{x_1 x_2 y}$ and $(x_1^0, x_2^0, y^0) = (0, 0.5, 0.5)$. Show that the set of (x_1, x_2, y) that solve $F(x_1, x_2, y) = 0$ near (x_1^0, x_2^0, y^0) is the graph of some function $y = h(x_1, x_2)$. Compute Dh.
   Note: If $g(x) = e^x$, $\frac{d}{dx} g(x) = g(x)$.

   (b) Let $F : \mathbb{R}^4 \to \mathbb{R}^2$ be defined as

   $$F(x_1, x_2, y_1, y_2) = (x_1^2 - x_2^2 - y_1^3 + y_2^2 + 4, \;\; 2x_1 x_2 + x_2^2 - 2y_1^2 + 3y_2^4 + 8).$$

   Let $(x_1^0, x_2^0, y_1^0, y_2^0) = (2, -1, 2, 1)$. Show that the set of (x_1, x_2, y_1, y_2) that solve $F(x_1, x_2, y_1, y_2) = 0$ near $(x_1^0, x_2^0, y_1^0, y_2^0)$ is the graph of some function $(y_1, y_2) = h(x_1, x_2)$. Compute Dh.

3. (10 pts) Implicit function theorem, continued.

   (a) Let $F : \mathbb{R}^3 \to \mathbb{R}$. Suppose the conditions for the implicit function theorem are satisfied at all points and that $F(x_1^0, x_2^0, y^0) = 0$. Let $h : \mathbb{R}^2 \to \mathbb{R}$ denote the implicitly defined function for the relation $F(x_1, x_2, y) = 0$ near (x_1^0, x_2^0, y^0). Derive an expression for Dh in terms of F and its partial derivatives.

   (b) Let $F : \mathbb{R}^4 \to \mathbb{R}^2$. Suppose the conditions for the implicit function theorem are satisfied at all points and that $F(x_1^0, x_2^0, y_1^0, y_2^0) = 0$. Let $h = (h^1, h^2)$ denote the implicitly defined function of (x_1, x_2) for the relation $F(x_1, x_2, y_1, y_2) = (0, 0)$ near $(x_1^0, x_2^0, y_1^0, y_2^0)$. Give explicit formulas for $\frac{\partial h^i}{\partial x_j}$ for $i = 1, 2$ and $j = 1, 2$.

4. (5pts) Assume that a monopolist who produces quantity q faces a general inverse-demand function $p = p(q)$, where p is twice continuously differentiable, strictly decreasing, and concave. The monopolist's total cost function is $t(q) = c(q) - sq$, and is twice continuously differentiable, convex, and strictly increasing in q. Assume $s > 0$; you can interpret the $-sq$ part of total cost as a per-unit subsidy of s given to the monopolist by the government to encourage production. The monopolist therefore has a profit function of $\pi(q) = p(q)q - c(q) + sq$. She chooses a quantity level $q \in [0, 100]$ to maximize profit (think of 100 as a capacity cap on what the monopolist can produce in her factory).

   (a) Show that there exists a unique optimal production level for the firm. Assuming that this optimal level is in $(0, 100)$ for some value of the parameters, use the implicit function theorem to sign how the monopolist's optimal quantity choice changes with s.

   (b) Suppose we no longer maintain the assumptions that $p(\cdot)$ is concave and that $c(\cdot)$ is convex; moreover, suppose we only assume that $p(\cdot)$ and $c(\cdot)$ are once differentiable. What can we say about the existence and uniqueness of a solution to the monopolist's problem? If we have or assume existence, what can we conclude about how the solution(s) to the monopolist's problem change with the level of subsidy?

5. (10 pts) Suppose a firm with a constant marginal cost of production, $c > 0$, and a constant elasticity of demand, $\alpha > 0$, chooses a price $p > 0$ to maximize profits: $\pi(p, \alpha) = p^{-\alpha}(p - c)$.

   (a) Show that π has neither increasing nor decreasing differences in (p, α).

   (b) Show that $log \circ \pi$ has decreasing differences in (p, α).

   (c) Conclude (i.e., state and prove the result) how the firm's optimal choice of quantity changes with elasticity of demand. Give a verbal intuition for this comparative static.

   Hint: Prove first that if g is a strictly increasing function, then:

$$\arg\max_{x \in X} f(x, t) = \arg\max_{x \in X} g(f(x, t)).$$

   (d) Now let $\pi(p, \alpha) = D(p, \alpha)(p - c)$, where the demand function D is not necessarily the constant elasticity of demand function, but satisfies the property that elasticity of demand is increasing in the parameter α. Prove that the same conclusion as in part (c) continues to hold for this more general class of demand functions.

6. (10 pts) You have a goose that lays a golden egg with probability $0 < p < 1$, on every day $T = 0, 1, 2 \ldots$. At the start of each day (before seeing whether it will lay an egg or not), if the goose has not yet been slain, you have the option to slay the goose and secure that day's egg with probability 1. Once you slay the goose, it will of course no longer be around to lay eggs in the future. You have a discount factor of 0.9. Every day you receive an egg, you get a flow payoff of 1. If you don't get an egg that day, you get a flow payoff of 0. Your utility for any infinite history is the sum of discounted flow payoffs along that history. You seek a strategy that maximizes your expected discounted payoffs.

   (a) Prove that the utility function is upper and lower convergent. Do not cite general results from class or in presentation notes that tell you this: prove it directly.

   (b) Find an optimal strategy for every $p \in [0, 1]$. Prove that your strategy is indeed optimal.

   (c) Write down the value function (i.e., as a function of the parameter p, keeping fixed the discount rate).

   (c) Is there any $p \in (0, 1)$ for which a strategy of the following form is optimal: slay the goose if and only if it went at least 4 consecutive periods without laying an egg.

7. (Extra Credit: 5 pts)

   Problem 5 highlights a deficiency with relying on the increasing differences property conclude various comparative statics. In particular, though an objective function may not exhibit increasing differences, some strictly monotone transformation of that function might, and our desired comparative statics results would then follow. If we don't identify the right kind of transformation, the desired result may be sitting under our noses and we may still miss it.

   We wouldn't have to miss out on this solution if we instead searched for a certain weaker (!) property than increasing differences that nonetheless, by a result of Milgrom and Shannon (1994), allows us to conclude the same thing. This property is known as *single-crossing*.

   Suppose $X, T \subset \mathbb{R}$. A function $f : X \times T \to \mathbb{R}$ satisfies the *single crossing property* in (x, t) if for all $x' > x$ and $t' > t$:

$$f(x', t) \geq f(x, t) \implies f(x', t') \geq f(x, t'),$$

   and,

$$f(x', t) > f(x, t) \implies f(x', t') > f(x, t').$$

   The interpretation is that if for a smaller parameter value, t, it was a good idea to take a larger action, x', over the smaller one, x, then for a larger parameter value t', it would continue to be a good idea to opt for that larger action.

   Milgrom and Shannon (1994) show that if f is single-crossing in (x, t), then $\arg\max_{x \in X} f(x, t)$ is nondecreasing in t in the strong set order (watch the zoom presentation on this). This is an improvement over the Topkis's theorem we saw in class because we got the same conclusion with a weaker assumption.

   (a) Prove that the single-crossing property is weaker than increasing differences (i.e., that the latter implies the former) (1 point)

   (b) Prove that if f has the single-crossing property in (x, t), then any strictly increasing transformation of f has the single crossing-property in (x, t) (2 points).

   Remark: This is why single-crossing is said to be an ordinal property, unlike increasing differences. To make an analogy, recall that a strictly increasing transformation of a utility function still preserves the same preference ordering over the choice set, which is why we call them ordinal preferences. Recall, similarly, that quasiconvexity is preserved under monotonic transformations while convexity is not, so the former is an ordinal property while the latter is not.

   (c) Give an example of a function that has the single-crossing property but does not satisfy increasing differences; prove your example works either formally or through a picture. (2 points).