Econ 6190 Mid Term Exam: Suggested Solutions

10:10 am - 11:30 am, 8 October 2024

Instructions

*This exam contains **one question** consisting of **nine smaller questions** on **two pages**. Answer all questions. Remember to always explain your answer. Good luck!*

Useful results:

- *If*

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} \sigma_X^2 & \sigma_X\sigma_Y\rho \\ \sigma_X\sigma_Y\rho & \sigma_Y^2 \end{pmatrix}\right),$$

then

$$X \mid Y \sim N\left(\mu_X + \frac{\sigma_X}{\sigma_Y}\rho\,(Y-\mu_Y)\,, (1-\rho^2)\sigma_X^2\right).$$

- *If $X \sim \chi_k^2$, then $E[X] = k$, $Var(X) = 2k$.*

1. We observe a random sample $\{X_1, X_2, \dots X_n\}$ from a normal distribution with unknown mean $\mu \in \mathbb{R}$, unknown variance σ^2 and a pdf

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right), \text{ for } x \in \mathbb{R}.$$

Answer the following questions.

(a) **[10 pts]** Show the first derivative of $f(x)$, $f^{(1)}(x)$, equals $-\frac{1}{\sigma}f(x)\left(\frac{x-\mu}{\sigma}\right)$.

Answer: Standard question.

(b) **[10 pts]** Let $T_1 = \frac{1}{2\sigma^2}(X_2 - X_1)^2$. Prove that $T_1 \sim \chi_1^2$.

Answer: Standard question. See class note.

(c) **[10 pts]** Let $T_2 = T_1 + \frac{2}{3\sigma^2}(X_3 - \bar{X}_2)^2$, where $\bar{X}_2 = \frac{1}{2}(X_1 + X_2)$. Prove that $T_2 \sim \chi_2^2$. For simplicity, you may assume that $\bar{X}_2$ is independent of T_1.

Answer: Standard question. See class note.

(d) **[10 pts]** Let $\hat{\mu}_1 = X_1$ be an estimator of μ. Calculate the bias, variance, and mean square error (MSE) of $\hat{\mu}_1$.

Answer: $\mathbb{E}[\hat{\mu}_1] = \mathbb{E}[X_1] = \mathbb{E}[X] = \mu$ *by random sampling assumption. So*

$$\begin{aligned}
bias(\hat{\mu}_1) &= \mathbb{E}[\hat{\mu}_1] - \mu = 0; \\
var(\hat{\mu}_1) &= \mathbb{E}\left[(\hat{\mu}_1 - \mathbb{E}[\hat{\mu}_1])^2\right] \\
&= \mathbb{E}\left[(X_1 - \mu)^2\right] \\
&= \mathbb{E}\left[(X - \mu)^2\right] \\
&= \sigma^2 \\
MSE(\hat{\mu}_1) &= [bias(\hat{\mu}_1)]^2 + var(\hat{\mu}_1) \\
&= \sigma^2.
\end{aligned}$$

(e) **[15 Pts]** Propose an unbiased estimator for the variance of $\hat{\mu}_1$, say, $\hat{Var}(\hat{\mu}_1)$, and prove its unbiasedness. Then, find the variance of $\hat{Var}(\hat{\mu}_1)$.

Answer: Since $var(\hat{\mu}_1) = \sigma^2$*, an unbiased estimator for* σ^2 *is*

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2, \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

The proof of unbiasedness follows class notes. To find $var(s^2)$*, note since we assumed a normal sampling model, it follows*

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1},$$

and as a result, $var\left(\frac{(n-1)s^2}{\sigma^2}\right) = 2(n-1)$*. Furthermore,*

$$var\left(\frac{(n-1)s^2}{\sigma^2}\right) = \frac{(n-1)^2}{\sigma^4} var\left(s^2\right),$$

we conclude that $var(s^2) = \frac{2(n-1)\sigma^4}{(n-1)^2} = \frac{2\sigma^4}{n-1}$.

For the rest of the questions below, assume that σ^2 is **known**.

(f) **[10 Pts]** Show $T_3 = \frac{1}{n}\sum_{i=1}^n X_i$ is a sufficient statistic for μ using Factorization Theorem.

Answer: Standard question. See class note.

(g) **[15 Pts]** Find the joint distribution of $(\hat{\mu}_1, T_3)$. Carefully state your reasoning.

Answer: Note $\hat{\mu}_1 = X_1$, $T_3 = \frac{1}{n}\sum_{i=1}^n X_i$*, both of which are linear combinations of*

$$(X_1, X_2, \ldots X_n)' \sim \textit{multivariate normal distribution.}$$

As a result, $(\hat{\mu}_1, T_3)$ *also follows a multivariate normal distribution:*

$$\begin{pmatrix} \hat{\mu}_1 \\ T_3 \end{pmatrix} \sim N\left(\begin{pmatrix} \mathbb{E}[\hat{\mu}_1] \\ \mathbb{E}[T_3] \end{pmatrix}, \begin{pmatrix} var(\hat{\mu}_1) & Cov(\hat{\mu}_1, T_3) \\ Cov(\hat{\mu}_1, T_3) & var(T_3) \end{pmatrix}\right),$$

where note $\mathbb{E}[\hat{\mu}_1] = \mu$, $\mathbb{E}[T_3] = \mu$, $var(\hat{\mu}_1) = \sigma^2$, *and* $var(T_3) = \frac{\sigma^2}{n}$. *Now,*

$$\begin{aligned} Cov(\hat{\mu}_1, T_3) &= Cov(X_1, \frac{1}{n}\sum_{i=1}^{n} X_i) \\ &= \frac{1}{n}\sum_{i=1}^{n} Cov(X_1, X_i) \\ &= \frac{1}{n}\sigma^2, \end{aligned}$$

since $Cov(X_1, X_i) = 0$ *for all* $i \neq 1$ *(by independence assumption)and* $Cov(X_1, X_1) = var(X_1) = \sigma^2$. *As a result,*

$$\begin{pmatrix} \hat{\mu}_1 \\ T_3 \end{pmatrix} \sim N\left(\begin{pmatrix} \mu \\ \mu \end{pmatrix}, \begin{pmatrix} \sigma^2 & \frac{1}{n}\sigma^2 \\ \frac{1}{n}\sigma^2 & \frac{1}{n}\sigma^2 \end{pmatrix}\right).$$

(a) **[15 Pts]** Now, consider the following Blackwell-ized estimator of $\hat{\mu}_1$:

$$\hat{\mu}_2 = \mathbb{E}[\hat{\mu}_1 \mid T_3].$$

Derive the analytic form of $\hat{\mu}_2$.

Answer: Since $(\hat{\mu}_1, T_3)'$ *follows a multivariate normal distribution, the conditional distribution* $\hat{\mu}_1 \mid T_3$ *is also normal, and in particular,*

$$\mathbb{E}[\hat{\mu}_1 \mid T_3] = \mathbb{E}[\hat{\mu}_1] + \frac{\sqrt{var(\hat{\mu}_1)}}{\sqrt{var(T_3)}}\rho\left(T_3 - \mathbb{E}[T_3]\right),$$

where

$$\begin{aligned} \rho &= \frac{Cov(\hat{\mu}_1, T_3)}{\sqrt{var(\hat{\mu}_1)}\sqrt{var(T_3)}} \\ &= \frac{\frac{1}{n}\sigma^2}{\sigma\sqrt{\frac{\sigma^2}{n}}} = \frac{1}{\sqrt{n}}. \end{aligned}$$

Hence,

$$\begin{aligned} \mathbb{E}[\hat{\mu}_1 \mid T_3] &= \mu + \frac{\sigma}{\sqrt{\frac{\sigma^2}{n}}}\frac{1}{\sqrt{n}}\left(T_3 - \mu\right) \\ &= \mu + T_3 - \mu \\ &= T_3. \end{aligned}$$

That is, $\hat{\mu}_2 = T_3$.

(b) **[5 Pts]** Compare the MSE of $\hat{\mu}_2$ and T_3. Which one is more efficient?

Answer: Since $\hat{\mu}_2 = T_3$, *they are equally efficient.*