Econ 6190 Problem Set 2

Fall 2024

1. [Hansen 4.9] Suppose that X_i are i.n.i.d. (independent but not necessarily identically distributed) with $\mathbb{E}[X_i] = \mu_i$ and $\text{var}[X_i] = \sigma_i^2$.

   (a) Find $\mathbb{E}[\bar{X}]$;

   (b) Find $\text{var}[\bar{X}]$.

2. [Mid term, 2022] Let $X \sim \text{N}(\mu, \sigma^2)$ for some unknown μ and **known** σ^2. Furthermore, suppose I believe that μ can only take two values, $\frac{1}{2}$ or $-\frac{1}{2}$, and I believe $P\{\mu = \frac{1}{2}\} = \frac{1}{2}$, and $P\{\mu = -\frac{1}{2}\} = \frac{1}{2}$. Now, I draw a single observation X_1 from the distribution of X, and it turns out $X_1 < 0$. Given that I observe $X_1 < 0$, what is my updated probability that $\mu = \frac{1}{2}$? That is, find $P\{\mu = \frac{1}{2} | X_1 < 0\}$. The following notations can be useful: $\Phi(t)$ is the cdf of a standard normal, and $\phi(t)$ is the pdf of a standard normal.

3. [Hansen, 5.2, 5.3] For the standard normal density $\phi(x)$, show that $\phi'(x) = -x\phi(x)$. Then, use integration by parts to show that $\mathbb{E}[Z^2] = 1$ for $Z \sim \text{N}(0, 1)$.

4. [Mid term, 2023] If X is normal with mean μ and variance σ^2, it has the following pdf

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right), \text{ for } x \in \mathbb{R}.$$

   Let X and Y be jointly normal with the joint pdf

$$f(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - 2\frac{\rho xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2}\right)\right), \text{ for } x, y \in \mathbb{R} \qquad (1)$$

   where $\sigma_X > 0, \sigma_Y > 0$ and $-1 \leq \rho \leq 1$ are some constants.

   (a) Without using the properties of jointly normal distributions, show that the marginal distribution of Y is normal with mean 0 and variance σ_Y^2.

   (b) If you cannot work (a) out, assume it is true and move on. Derive the conditional distribution of X given $Y = y$. (Hint: it should be normal with mean $\frac{\sigma_X}{\sigma_Y}\rho y$ and variance $(1-\rho^2)\sigma_X^2$).

(c) Let $Z = \frac{X}{\sigma_X} - \frac{\rho}{\sigma_Y} Y$. Show Y and Z are independent. Clearly state your reasoning. (Hint: For this question, you can use the properties of jointly normal distributions.)

5. [Hansen 5.18, 5.19] Show that:

   (a) If $e \sim \mathrm{N}(0, I_n \sigma^2)$ and $\mathbf{H}'\mathbf{H} = I_n$, then $u = \mathbf{H}'e \sim \mathrm{N}(0, I_n \sigma^2)$.

   (b) If $e \sim \mathrm{N}(0, \Sigma)$ and $\Sigma = \mathbf{A}\mathbf{A}'$, then $u = \mathbf{A}^{-1}e \sim \mathrm{N}(0, I_n)$.

6. [Hansen 6.13] Let $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$. Find the covariance of $\hat{\sigma}^2$ and $\bar{X}$. Under what condition is this zero? [Hint: This exercise shows that the zero correlation between the numerator and the denominator of the t ratio does not always hold when the random sample is not from a normal distribution].