ECON6190 Section 7

Oct. 11, 2024

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Midterm #1

Useful results:

- *If*
$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} \sigma_X^2 & \sigma_X\sigma_Y\rho \\ \sigma_X\sigma_Y\rho & \sigma_Y^2 \end{pmatrix}\right),$$
then
$$X \mid Y \sim N\left(\mu_X + \frac{\sigma_X}{\sigma_Y}\rho\,(Y-\mu_Y)\,,(1-\rho^2)\sigma_X^2\right).$$
- *If $X \sim \chi_k^2$, then $E[X] = k$, $Var(X) = 2k$.*

1. We observe a random (iid) sample $\{X_1, X_2, \ldots X_n\}$ from a normal distribution with unknown mean $\mu \in \mathbb{R}$, unknown variance σ^2 and a pdf
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right), \text{ for } x \in \mathbb{R}.$$
Answer the following questions.
   (a) **[10 pts]** Show the first derivative of $f(x)$, $f^{(1)}(x)$, equals $-\frac{1}{\sigma}f(x)\left(\frac{x-\mu}{\sigma}\right)$.
   Answer: Standard question.
   (b) **[10 pts]** Let $T_1 = \frac{1}{2\sigma^2}(X_2 - X_1)^2$. Prove that $T_1 \sim \chi_1^2$.
   Answer: Standard question. See class note.
   (c) **[10 pts]** Let $T_2 = T_1 + \frac{2}{3\sigma^2}(X_3 - \bar{X}_2)^2$, where $\bar{X}_2 = \frac{1}{2}(X_1 + X_2)$. Prove that $T_2 \sim \chi_2^2$. For simplicity, you may assume that $\bar{X}_2$ is independent of T_1.
   Answer: Standard question. See class note.

(b) Method #1

Since $\{X_1, \ldots X_n\}$ are iid sample from $N(\mu, \sigma^2)$, linear combination of iid normal is also normal.

$E[X_2 - X_1] = E[X_2] - E[X_1] = \mu - \mu = 0.$

$Var(X_2 - X_1) = \underbrace{Var(X_2)}_{\sigma^2} + \underbrace{Var(X_1)}_{\sigma^2} - 2\underbrace{Cov(X_2, X_1)}_{0} = 2\sigma^2$

$\Rightarrow (X_2 - X_1) \sim N(0, 2\sigma^2)$

$\Rightarrow \frac{1}{\sqrt{2\sigma^2}}(X_2 - X_1) \sim \mathcal{N}(0,1)$

Since χ_1^2 is one standard normal squared,

$$T_1 = \frac{1}{2\sigma^2}(X_2 - X_1)^2 = \Big(\underbrace{\tfrac{1}{\sqrt{2\sigma^2}}(X_2 - X_1)}_{\sim \mathcal{N}(0,1)}\Big)^2 \sim \chi_1^2$$

<u>Method #2</u> we know $\frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2$

Consider for $n=2$, $S_2^2 = \frac{1}{2-1}\left((X_1-\bar{X}_2)^2 + (X_2 - \bar{X}_2)^2\right)$, $\bar{X}_2 = \frac{1}{2}(X_1+X_2)$

$$= (X_1 - \tfrac{1}{2}(X_1+X_2))^2 + (X_2 - \tfrac{1}{2}(X_1+X_2))^2$$
$$= \left(\tfrac{1}{2}(X_1-X_2)\right)^2 + \left(\tfrac{1}{2}(X_2-X_1)\right)^2$$
$$= \tfrac{1}{4}(X_1-X_2)^2 + \tfrac{1}{4}(X_2-X_1)^2$$
$$= \tfrac{1}{2}(X_1-X_2)^2$$

$$T_1 = \frac{1}{2\sigma^2}(X_1-X_2)^2 = \frac{(2-1)S_2^2}{\sigma^2} \sim \underbrace{\chi_{2-1}^2}_{\chi_1^2}$$

(c) WTS: $T_2 = T_1 + \frac{2}{3\sigma^2}(X_3 - \bar{X}_2)^2 \sim \chi_2^2$.

Since χ_2^2 is the sum of two independent standard normal squared.

We already showed $T_1 \sim \chi_1^2$, and $\bar{X}_2, X_3$ are independent of T_1,

WTS: $\frac{2}{3\sigma^2}(X_3 - \bar{X}_2)^2 \sim \chi_1^2$.

By similar argument of joint normality,

$$\bar{X}_2 \sim \mathcal{N}\Big(\underbrace{\tfrac{1}{2}E[X_1+X_2]}_{\mu},\ \tfrac{1}{4}\big(\underbrace{var(X_1)}_{\sigma^2} + \underbrace{var(X_2)}_{\sigma^2} + \underbrace{2cov(X_1,X_2)}_{0}\big)\Big)$$

$$\Leftrightarrow \bar{X}_2 \sim \mathcal{N}(\mu, \tfrac{1}{2}\sigma^2)$$

$$X_3 - \bar{X}_2 \sim \mathcal{N}\Big(\mu-\mu,\ \underbrace{var(X_3)}_{\sigma^2} + \underbrace{var(\bar{X}_2)}_{\frac{\sigma^2}{2}} - \underbrace{2cov(X_3,\bar{X}_2)}_{0}\Big)$$

$$\Leftrightarrow (X_3 - \bar{X}_2) \sim \mathcal{N}(0, \tfrac{3}{2}\sigma^2)$$

$$\Leftrightarrow \sqrt{\tfrac{2}{3\sigma^2}}(X_3 - \bar{X}_2) \sim \mathcal{N}(0,1)$$

$$\Rightarrow \frac{2}{3\sigma^2}(X_3 - \bar{X}_2)^2 = \left(\sqrt{\tfrac{2}{3\sigma^2}}(X_3-\bar{X}_2)\right)^2 \sim \chi_1^2$$

(d) **[10 pts]** Let $\hat{\mu}_1 = X_1$ be an estimator of μ. Calculate the bias, variance, and mean square error (MSE) of $\hat{\mu}_1$.

Answer: $\mathbb{E}[\hat{\mu}_1] = \mathbb{E}[X_1] = \mathbb{E}[X] = \mu$ *by random sampling assumption. So*

$$
\begin{aligned}
bias(\hat{\mu}_1) &= \overbrace{\mathbb{E}[\hat{\mu}_1]}^{=\mu} - \mu = 0; \\
var(\hat{\mu}_1) &= \mathbb{E}\left[(\hat{\mu}_1 - \mathbb{E}[\hat{\mu}_1])^2\right] \\
&= \mathbb{E}\left[(X_1 - \mu)^2\right] \\
&= \mathbb{E}\left[(X - \mu)^2\right] \\
&= \sigma^2 \\
MSE(\hat{\mu}_1) &= [bias(\hat{\mu}_1)]^2 + var(\hat{\mu}_1) \\
&= \sigma^2.
\end{aligned}
$$

$var(\hat{\mu}_1) = var(x_1) \overset{iid}{=} \sigma^2$

(e) **[15 Pts]** Propose an unbiased estimator for $\underbrace{\text{the variance of } \hat{\mu}_1}^{\sigma^2}$, say, $\hat{Var}(\hat{\mu}_1)$, and prove its unbiasedness. Then, find the variance of $\hat{Var}(\hat{\mu}_1)$.

Answer: Since $var(\hat{\mu}_1) = \sigma^2$, *an unbiased estimator for* σ^2 *is*

$$
s^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2, \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i.
$$

common mistake: $\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \mu)^2$

parameter

not enough to just call the theorem

The proof of unbiasedness follows class notes. To find $var(s^2)$, *note since we assumed a normal sampling model, it follows*

$$
\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1},
$$

and as a result, $var\left(\frac{(n-1)s^2}{\sigma^2}\right) = 2(n-1)$. *Furthermore,*

$$
var\left(\frac{(n-1)s^2}{\sigma^2}\right) = \frac{(n-1)^2}{\sigma^4} var\left(s^2\right),
$$

trick discussed in section 6

we conclude that $var(s^2) = \frac{2(n-1)\sigma^4}{(n-1)^2} = \frac{2\sigma^4}{n-1}$.

Now assume σ^2 is known

(f) **[10 Pts]** Show $T_3 = \frac{1}{n}\sum_{i=1}^{n} X_i$ is a sufficient statistic for μ using Factorization Theorem.

Answer: Standard question. See class note.

(g) **[15 Pts]** Find the joint distribution of $(\overset{x_1}{\hat{\mu}_1}, \overset{\frac{1}{n}\sum_{i=1}^{n} x_i}{T_3})$. Carefully state your reasoning.

Answer: Note $\hat{\mu}_1 = X_1$, $T_3 = \frac{1}{n}\sum_{i=1}^{n} X_i$, *both of which are linear combinations of*

$$
(X_1, X_2, \ldots X_n)' \sim \textit{multivariate normal distribution.}
$$

As a result, $(\hat{\mu}_1, T_3)$ *also follows a multivariate normal distribution:*

No sufficient just say each of them are normal

$$
\begin{pmatrix} \hat{\mu}_1 \\ T_3 \end{pmatrix} \sim N\left(\begin{pmatrix} \overbrace{\mathbb{E}\left[\hat{\mu}_1\right]}^{\mu} \\ \underbrace{\mathbb{E}\left[T_3\right]}_{\mu} \end{pmatrix}, \begin{pmatrix} \overbrace{var(\hat{\mu}_1)}^{\sigma^2} & Cov(\hat{\mu}_1, T_3) \\ Cov(\hat{\mu}_1, T_3) & \underbrace{var(T_3)}_{\sigma^2/n} \end{pmatrix}\right),
$$

where note $\mathbb{E}[\hat{\mu}_1] = \mu$, $\mathbb{E}[T_3] = \mu$, *var*$(\hat{\mu}_1) = \sigma^2$, *and var*$(T_3) = \frac{\sigma^2}{n}$. *Now,*

$$\begin{aligned} Cov(\hat{\mu}_1, T_3) &= Cov(X_1, \frac{1}{n}\sum_{i=1}^{n} X_i) \\ &= \frac{1}{n}\sum_{i=1}^{n} Cov(X_1, X_i) \\ &= \frac{1}{n}\sigma^2, \end{aligned}$$

b/c iid
$cov(x_1, x_i) = 0, \forall i \neq 1$

since $Cov(X_1, X_i) = 0$ *for all* $i \neq 1$ *(by independence assumption)and* $Cov(X_1, X_1) =$ *var*$(X_1) = \sigma^2$. *As a result,*

$$\begin{pmatrix} \hat{\mu}_1 \\ T_3 \end{pmatrix} \sim N\left(\begin{pmatrix} \mu \\ \mu \end{pmatrix}, \begin{pmatrix} \sigma^2 & \frac{1}{n}\sigma^2 \\ \frac{1}{n}\sigma^2 & \frac{1}{n}\sigma^2 \end{pmatrix}\right).$$

Another way to find covariance:

$$\begin{aligned} cov(x_1, \tfrac{1}{n}\sum_{i=1}^{n} x_i) &= E[(x_1 - \overbrace{E[x_1]}^{\mu})(\bar{x} - \overbrace{E[\bar{x}]}^{\mu})] \\ &= E[x_1\bar{x} - \mu x_1 - \mu\bar{x} + \mu^2] \\ &= E[x_1\bar{x}] - \underbrace{\mu E[x_1]}_{\mu^2} - \underbrace{\mu E[\bar{x}]}_{\mu^2} + \mu^2 \\ &= \underbrace{E[x_1\bar{x}]} - \mu^2 \end{aligned}$$

$$\begin{aligned} E[x_1 \tfrac{1}{n}\sum_{i=1}^{n} x_i] &= \tfrac{1}{n}\sum_{i=1}^{n} E[x_1 x_i] \\ &= \tfrac{1}{n}\Big(\underbrace{E[x_1^2]}_{=\sigma^2+\mu^2} + \underbrace{E[x_1x_2]}_{=E[x_1]E[x_2] = \mu^2} + \cdots\cdots E[x_1x_n]\Big) \\ &= \tfrac{1}{n}(\sigma^2 + n\mu^2) = \frac{\sigma^2}{n} + \mu^2 \end{aligned}$$

b/c $var(x_1) = E[x_1^2] - (E[x_1])^2$, with $\underbrace{var(x_1)}_{\sigma^2}$ and $\underbrace{(E[x_1])^2}_{\mu^2}$

$\Rightarrow E[x_1^2] = \sigma^2 + \mu^2$

$$= \frac{\sigma^2}{n} + \mu^2 - \mu^2 = \sigma^2/n$$

(h) **[15 Pts]** Now, consider the following Blackwell-ized estimator of $\hat{\mu}_1$:

$$\hat{\mu}_2 = \mathbb{E}[\hat{\mu}_1 \mid T_3].$$

Derive the analytic form of $\hat{\mu}_2$.

Answer: Since $(\hat{\mu}_1, T_3)'$ follows a multivariate normal distribution, the conditional distribution $\hat{\mu}_1 \mid T_3$ is also normal, and in particular,

$$\mathbb{E}[\hat{\mu}_1 \mid T_3] = \mathbb{E}[\hat{\mu}_1] + \frac{\sqrt{var(\hat{\mu}_1)}}{\sqrt{var(T_3)}}\rho\left(T_3 - \mathbb{E}[T_3]\right),$$

where

$$\begin{aligned}\rho &= \frac{Cov(\hat{\mu}_1, T_3)}{\sqrt{var(\hat{\mu}_1)}\sqrt{var(T_3)}} \\ &= \frac{\frac{1}{n}\sigma^2}{\sigma\sqrt{\frac{\sigma^2}{n}}} = \frac{1}{\sqrt{n}}.\end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{E}[\hat{\mu}_1 \mid T_3] &= \mu + \frac{\sigma}{\sqrt{\frac{\sigma^2}{n}}}\frac{1}{\sqrt{n}}\left(T_3 - \mu\right) \\ &= \mu + T_3 - \mu \\ &= T_3.\end{aligned}$$

That is, $\hat{\mu}_2 = T_3$.

(i) **[5 Pts]** Compare the MSE of $\hat{\mu}_2$ and T_3. Which one is more efficient?

Answer: Since $\hat{\mu}_2 = T_3$, they are equally efficient.

NOT precise to invoke Rao-Blackwell theorem directly.

Rao-Blackwell Theorem

random sample $\mathbf{X} = \{x_1, \dots, x_n\}$ from a distribution F_θ, $\theta \in \mathbb{R}^k$.

Let $\hat{\theta} := \hat{\theta}(\mathbf{X})$ be a candidate estimator for θ, and $T(\mathbf{X})$ be a s.s.

Let $\tilde{\theta}(\mathbf{x}) = E[\hat{\theta}(\mathbf{x}) \mid T(\mathbf{x})]$,

Then ① $MSE(\tilde{\theta}(\mathbf{x})) \leq MSE(\hat{\theta}(\mathbf{x}))$

② If $\hat{\theta}$ unbiased, so is $\tilde{\theta}$.

4. Let $\{X_1 \dots X_n\}$ be random sample.

$f(x) = \begin{cases} e^{-x+\theta}, & x \geq \theta \\ 0, & o/w \end{cases}$

(a) Suppose X_i has pdf $f(x) = e^{-x+\theta}\mathbf{1}\{x \geq \theta\}$ for some constant θ. Show that

$$\min(X_1, X_2, \dots X_n) \xrightarrow{p} \theta.$$

(b) Suppose X_i is $U[0,\theta]$ for some constant $\theta > 0$. Show that

$$\max(X_1, X_2, \dots X_n) \xrightarrow{p} \theta.$$

$\boxed{\text{DEF}}$ (convergence in probability)

$X_n \xrightarrow{P} X$ if
- $\forall \delta > 0$, $\lim_{n\to\infty} P(|X_n - X| > \delta) = 0$
- $\forall \delta > 0$, $\lim_{n\to\infty} P(|X_n - X| < \delta) = 1$ ✓
- $\forall \delta > 0, \varepsilon > 0$, $\exists n_{\delta,\varepsilon}$ s.t $\forall n \geq n_{\delta,\varepsilon}$,
  $P(|X_n - X| > \delta) < \varepsilon$
  or $P(|X_n - X| < \delta) \geq 1 - \varepsilon$.

(a) WTS: $\min(X_1, \cdots X_n) \xrightarrow{P} \theta$

Denote $\underbrace{X_{min}}_{RV} = \min(X_1, \dots X_n)$

Want write out $P(X_{min} \cdots) \Rightarrow$ need to find CDF of X_{min}.

$$F_{X_{min}}(x) = P(X_{min} < x)$$
$$= 1 - P(X_{min} \geq x)$$
$$\overset{iid}{=} 1 - \underbrace{P(X_1 \geq x)}_{\text{need to find CDF of } X_i} P(X_2 \geq x) \cdots P(X_n \geq x) \quad \cdots ①$$

supp(X) = [θ,∞)

$$\underbrace{F_X(x)}_{P(X \leq x)} = \int_\theta^x e^{-t+\theta} dt = 1 - e^{\theta - x}, \quad \text{for } x \in [\theta, \infty) \Rightarrow \text{plug back to } ①$$

$$F_{X_{min}}(x) = 1 - (1 - (\underbrace{1 - e^{\theta - x}}_{F_X(x)}))^n$$
$$= 1 - (e^{\theta - x})^n. \text{ for } x \in [\theta, \infty)$$

WTS: $\lim_{n\to\infty} P(|X_{min} - \theta| < \delta) = 1$

$$P(|X_{min} - \theta| < \delta) = P(-\delta + \theta < X_{min} < \delta + \theta)$$
$$= F_{X_{min}}(\delta + \theta) - F_{X_{min}}(\underbrace{-\delta + \theta}_{<\theta}) \to 0 \quad \text{b/c supp(X) is } [\theta, \infty)$$
$$= 1 - (e^{\theta - \delta - \theta})^n$$
$$= 1 - (e^{-\delta})^n$$

As $n\to\infty$, $P(|X_{min}-\theta|<\delta) = 1-\underbrace{(e^{-\delta})^n}_{\to 0,\ \text{as } n\to\infty} \to 1$ as $n\to\infty$.

$\Rightarrow$ By def. $\min(X_1, \cdots, X_n) \xrightarrow{P} \theta$.