About TA sections:

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Our plan for today:[1]

[1]Materials adapted from notes provided by a previous Teaching Assistant, Gautier Lenfant.

1 Model setup

Representative household

The representative household has the following utility function,

$$E_t \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma}.$$

Social Planner

We first look at the social planner's problem and the planner's problem is to solve the utility maximization problem subject to the following constraints.

Production:

$$Y_t = A_t K_t^{\alpha} N_t^{1-\alpha}$$

Law of motion for capital:

$$K_{t+1} = (1-\delta_k)K_t + I_t$$

Law of motion for labor:

$$N_t = (1-\delta_n)N_{t-1} + M(V_t, S_t)$$

Resource constraint:

$$Y_t = C_t + I_t + \phi_n V_t$$

Exogenous technology (no need to include this in SP's Lagrangian):

$$\log(A_{t+1}) = \rho \log(A_t) + \sigma_a \epsilon_{t+1}$$

2 Interpreting Euler equations

If we set up our Lagrangian, differentiate with respect to C_t, I_t, V_t, Y_t, N_t and K_{t+1}, and simplify as done in the lecture, we derive the Euler equation for capital and labor.

Euler equation for capital:

$$1 = \beta E_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \left(A_{t+1} \alpha \left(\frac{K_{t+1}}{N_{t+1}} \right)^{\alpha - 1} + 1 - \delta_k \right) \right]$$

This equation states that the cost of purchasing one unit of investment today is equal to the benefits that this unit of investment provides in the future. These benefits include the marginal product of capital and the fraction of capital that carries over to the next period, multiplied by the stochastic discount factor.

We can also rewrite this expression as:

$$C_t^{-\sigma} = \beta E_t \left[C_{t+1}^{-\sigma} \left(A_{t+1} \alpha \left(\frac{K_{t+1}}{N_{t+1}} \right)^{\alpha - 1} + 1 - \delta_k \right) \right]$$

Here, the left-hand side represents the marginal cost of savings or investment in the current period, while the right-hand side represents the marginal benefit in future periods. Both - costs and benefits - are measured in units of the marginal utility of consumption.

Euler equation for labor:

$$\frac{\phi_n}{M_V(V_t, S_t)} = A_t (1 - \alpha) \left(\frac{K_t}{N_t} \right)^{\alpha} + \beta E_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{\phi_n}{M_V(V_{t+1}, S_{t+1})} (1 - \delta_n) \right]$$

The equation states that the social cost of hiring a worker is equal to the marginal product of labor today plus the continuation value, which is the saving from not having to replace the worker in the future. By performing the recursive substitution as discussed in the lecture, we also know that the right-hand side is the discounted sum of future marginal products of labor.

3 Finding the steady state

From the lecture, we know that our economy can be characterized the following equations.

Production:

$$Y_t = A_t K_t^{\alpha} N_t^{1-\alpha}$$

Law of motion for capital:

$$K_{t+1} = (1-\delta_k)K_t + I_t$$

Law of motion for labor:

$$N_t = (1-\delta_n)N_{t-1} + M(V_t, S_t)$$

Resource constraint:

$$Y_t = C_t + I_t + \phi_n V_t$$

Euler equation for capital:

$$1 = \beta E_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \left(A_{t+1}\alpha \left(\frac{K_{t+1}}{N_{t+1}} \right)^{\alpha-1} + 1 - \delta_k \right) \right]$$

Euler equation for labor:

$$\frac{\phi_n}{M_V(V_t, S_t)} = A_t(1-\alpha)\left(\frac{K_t}{N_t}\right)^{\alpha} + \beta E_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{\phi_n}{M_V(V_{t+1}, S_{t+1})}(1-\delta_n) \right]$$

Exogenous technology:

$$\log(A_{t+1}) = \rho \log(A_t) + \sigma_a \epsilon_{t+1}$$

Before we solve for the steady state, let's make some simplifying assumptions.

- Labor supply is exogenous such that $S_t = 1$.
- Matching function is Cobb-Douglas such that $M(V_t, S_t) = \chi V_t^{\varepsilon} S_t^{1-\varepsilon}$. [2]

To solve the steady state, we remove the time subscripts, let $\epsilon_{t+1} = 0$, drop the expectation operators, and arrive at the following equations.

$$Y = AK^{\alpha}N^{1-\alpha}$$

$$\delta_k K = I$$

$$\delta_n N = \chi V^{\varepsilon}$$

$$Y = C + I + \phi_n V$$

$$1 = \beta(\alpha \left(\frac{K}{N}\right)^{\alpha-1} + 1 - \delta_k)$$

$$\frac{\phi_n}{\varepsilon \chi V^{\varepsilon-1}} = (1-\alpha)\left(\frac{K}{N}\right)^{\alpha} + \beta \frac{\phi_n}{\varepsilon \chi V^{\varepsilon-1}}(1-\delta_n).$$

$$A = 1$$

The standard first step is to solve for the steady-state capital-to-labor ratio from the steady-state Euler equation for capital.

$$\frac{K}{N} = \left(\frac{\beta^{-1} - 1 + \delta_k}{\alpha}\right)^{\frac{1}{\alpha-1}}$$

Then we can solve for V from the steady-state Euler equation for labor.

$$V = \left(\frac{\varepsilon\chi}{\phi_n}\frac{1-\alpha}{1-\beta(1-\delta_n)}\left(\frac{K}{N}\right)^{\alpha}\right)^{\frac{1}{1-\varepsilon}}$$

Then we can solve for N from the steady-state law of motion for labor.

$$N = \frac{\chi V^{\varepsilon}}{\delta_n}$$

[2] ϵ is the shock to technology while ε is the curvature of the matching function. They are different variables.

Then we can solve for K from the steady-state capital-to-labor ratio.

$$K = \frac{K}{N} * N$$

Then we can solve for I from the steady-state law of motion for capital.

$$I = \delta K$$

Then we can solve for Y from the production function.

$$Y = K^{\alpha} N^{1-\alpha}$$

Finally we can solve for C from the resource constraint.

$$C = Y - I - \phi_n V$$

4 Log-linearizing around the steady state

4.1 Notation

In this section, we introduce the notation that will be used for log-linearization. Different textbooks may use different notations, but here we adopt the following conventions:

- Lowercase letters will represent log-transformed variables. For example, $n_t = \log(N_t)$ is the log of the variable N_t.
- Hatted letters, such as $\hat{n}_t$, represent deviations of a variable from its steady-state value. For instance, $\hat{n}_t = \log(N_t) - \log(N)$ is the deviation of $log(N_t)$ from its steady state $log(N)$.
- Letters without time subscripts, like N, represent steady-state values. These are the values the system converges to in the long run.

4.2 First-order Taylor approximation

To perform log-linearization, we rely on the first-order Taylor approximation, which allows us to approximate a non-linear function near a point. The general form of the first-order Taylor expansion of a function $f(x)$ around a point a is:

$$f(x) \approx f(a) + f'(a)(x - a)$$

This means that for small deviations of x around a, the function $f(x)$ can be well approximated by a linear expression. We will apply this concept to each of the equations in our model.

4.3 Log-linearizing the equations in our model

4.3.1 Detailed example for production

Recall the production function we use in this model:

$$Y_t = A_t K_t^{\alpha} N_t^{1-\alpha}$$

Using the notations above, this can be rewritten as:

$$e^{y_t} = e^{a_t} e^{\alpha k_t} e^{(1-\alpha)n_t}$$

$$e^{y_t} = e^{a+\alpha k_t+(1-\alpha)n_t}$$

Applying the first-order Taylor approximation to both sides:

$$e^{y_t} \approx e^{y} + e^{y}(y_t - y)$$

$$e^{a+\alpha k_t+(1-\alpha)n_t} \approx e^{a+\alpha k+(1-\alpha)n} + e^{a+\alpha k+(1-\alpha)n}\left((a_t - a) + \alpha(k_t - k) + (1-\alpha)(n_t - n)\right)$$

Combining these, we get:

$$e^{y} + e^{y}(y_t - y) \approx e^{a+\alpha k+(1-\alpha)n} + e^{a+\alpha k+(1-\alpha)n}\left((a_t - a) + \alpha(k_t - k) + (1-\alpha)(n_t - n)\right)$$

Using the fact that

$$e^{y} = e^{a+\alpha k+(1-\alpha)n}$$

the log-linearized production function becomes:

$$\hat{y}_t \approx \hat{a}_t + \alpha \hat{k}_t + (1-\alpha)\hat{n}_t$$

This equation tells us that the output deviation from steady state is approximately a linear combination of deviations in productivity, capital, and labor, weighted by their respective shares in production.

In this approximation, if productivity (A_t) increases, output increases, holding capital and labor constant. Similarly, increasing capital or labor will increase output, but the magnitude of the increase depends on the shares α and $1-\alpha$. The first-order Taylor approximation simplifies the non-linear production function into an easier-to-handle linear form, which is valid for small deviations from the steady state.

4.3.2 Other equations in our model

The same method can be applied to log-linearize other equations in the model. As shown above, the process involves:

1. Taking logarithms of the equation if necessary.
2. Applying a first-order Taylor expansion around the steady state.
3. Expressing the equation in terms of deviations from the steady-state values.

Law of motion for capital

$$\hat{k}_{t+1} = (1-\delta)\hat{k}_t + \delta\hat{i}_t$$

Law of motion for labor

$$N\hat{n}_t = (1-\delta_n)N\hat{n}_{t-1} + \chi\varepsilon V^{\varepsilon}\hat{v}_t$$

Resource constraint

$$Y\hat{y}_t = C\hat{c}_t + I\hat{i}_t + \phi V\hat{v}_t$$

Euler equation for capital

$$0 = E_t[\sigma(\hat{c}_t - \hat{c}_{t+1}) + \alpha\beta\left(\frac{K}{N}\right)^{\alpha-1}(\hat{a}_{t+1} + (\alpha-1)\hat{k}_{t+1} + (1-\alpha)\hat{n}_{t+1})]$$

Euler equation for labor

$$\frac{\phi_n}{\varepsilon\chi}(1-\varepsilon)V^{1-\varepsilon}\hat{v}_t = (1-\alpha)\left(\frac{K}{N}\right)^{\alpha}(\hat{a}_t+\alpha(\hat{k}_t-\hat{n}_t))+\beta E_t\left(\frac{\phi_n}{\varepsilon\chi}V^{1-\varepsilon}(1-\delta_n)(\sigma(\hat{c}_t-\hat{c}_{t+1})+(1-\varepsilon)\hat{v}_{t+1})\right)$$