Econ 6170: Mid-Term 1

19 September 2023

Write your name on every sheet that you answer. There are four questions that are worth 5 points each and one extra credit question worth 2 points. (The highest possible score is 22/20.)

You have the full class time to complete the following problems. You are to work alone. This test is not open book. Please write out your answer neatly below each question, and use a new sheet of paper if you need more space than provided. When using extra sheets, make sure to write out your name and the relevant question number. In your answers, you are free to cite results that you can recall from class or previous problem sets unless explicitly stated otherwise.

Question 1 (5 points) Prove either that the following statements are true or false.

(i) The set $S = \{(x, y) \in \mathbb{R}^2 : x > 0,\, y \geq \frac{1}{x^2}\}$ is open.

(ii) The set $S = \{(x, y) \in \mathbb{R}^2 : x > 0,\, y \geq \frac{1}{x^2}\}$ is closed.

(iii) A closed subset of a compact set $S \subseteq \mathbb{R}^d$ is compact.

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Question 1 continued

Question 2 (5 points) Let $(x_n)_n$ be a sequence in $\mathbb{R}$. A point $s \in \mathbb{R}$ is a *limit point* of $(x_n)_n$ if there exists a subsequence of $(x_n)_n$ that converges to s. Let S be the set of limit points of $(x_n)_n$.

(i) Prove that there is a subsequence $(x_{n_k})_k$ that converges to $\limsup_{n\to\infty} x_n$.

(ii) Prove that $\limsup_{n\to\infty} x_n = \sup S$.

To save on time, you may assume the sequence $(x_n)_n$ is bounded. **Hint:** If you can't prove (i), assume it and use it to prove (ii).

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Question 2 continued

Question 3 (5 points) A *boundary point* of a set $S \subseteq \mathbb{R}^d$ is a point $x \in \mathbb{R}^d$ such that every open ball centred at x intersects both S and S^c. Define

$$\text{bd}\,(S) := \left\{ x \in \mathbb{R}^d : x \text{ is a boundary point of } S \right\}.$$

(i) Show that $\text{bd}(S) = \text{bd}(S^c)$.

(ii) Prove or disprove: If $x \in S$ is an isolated point, then x is a boundary point of S.

(iii) Show that a set $S \subseteq \mathbb{R}^d$ is closed if and only if it contains all its boundary points.

Hint: Recall that a point $x \in S$ is *isolated* if there exists $\epsilon > 0$ such that $B_\epsilon(x) \cap S = \{x\}$.

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Question 3 continued

Question 4 (5 points) Prove that a function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ is continuous if and only if f is upper semi-continuous and lower semi-continuous.

(While this is a proposition from the class, you cannot simply refer to the proposition as your answer. You must provide an explicit proof.)

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Question 4 continued

Question 5 (extra credit: 2 points) Let $u : \mathbb{R}_{+}^{d} \to \mathbb{R}$ be continuous, $\mathbf{p} \in \mathbb{R}_{++}^{d}$ and $m \in \mathbb{R}_{++}$. Define

$$\Gamma := \left\{ \mathbf{x} \in \mathbb{R}_{+}^{d} : \sum_{i=1}^{d} p_i x_i \leq m \right\}.$$

Prove that

$$u^* = \max \left\{ u\left(\mathbf{x}\right) : \mathbf{x} \in \Gamma \right\}$$

is well-defined. What did we just prove?

Hint: Use the Extreme Value Theorem.

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Question 5 continued