Econ 6190 Problem Set 8

Fall 2024

1. [Hansen] A Bernoulli random variable X is

$$P(X = 0) = 1 - p$$
$$P(X = 1) = p$$

Given a random sample $\{X_i, i = 1 \ldots n\}$ from X,

(a) Find the MLE estimator $\hat{p}_{MLE}$ for p.

(b) Find the asymptotic distribution of $\hat{p}_{MLE}$.

(c) Propose an estimator for the asymptotic variance V of $\hat{p}_{MLE}$.

(d) Show the variance estimator you proposed in (c) is consistent.

(e) Calculate the information for p by taking the variance of the efficient score.

(f) Calculate the information for p by taking the expectation of (minus) the second derivative. Did you obtain the same answer?

(g) Thus find the Cramér-Rao lower bound (CRLB) for p.

(h) Let $\mathrm{var}(\hat{p}_{MLE})$ be the asymptotic variance of $\hat{p}_{MLE}$. Compare $\mathrm{var}(\hat{p}_{MLE})$ with the CRLB.

(i) Propose a Method of Moment Estimator $\hat{p}_{MME}$ for p.

2. Suppose X follows a uniform distribution $[0, \theta]$ with $\theta > 0$. Given a random sample $\{X_i, i = 1 \ldots n\}$ drawn from X, find the MLE estimator for θ.

3. Suppose X follows a normal distribution with unknown mean μ and variance $\sigma^2 > 0$. The density of X is

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(x-\mu)^2}{2\sigma^2}).$$

Given a random sample $\{X_i, i = 1 \ldots n\}$ drawn from X, find the MLE estimator for (μ, σ^2).

4. Based on the notation in the slides on *Estimation*, let us prove the Information Matrix Equality

$$\mathbb{E}\left[\frac{\partial^2 \log f(X|\theta_0)}{\partial\theta\partial\theta'}\right] = -\mathbb{E}\left[\frac{\partial \log f(X|\theta_0)}{\partial\theta}\frac{\partial \log f(X|\theta_0)}{\partial\theta'}\right].$$

Let $f = f(x|\theta_0)$, ∇_j means derivative with respect to the j-th element $\theta^{(j)}$, and ∇_{jk} mean 2nd-order derivative with respect to $\theta^{(j)}$ and $\theta^{(k)}$. Suppose we can exchange the integral "$\int$" and derivatives "∇_j".

(a) By differentiating $\int f dx = 1$ with respect to $\theta^{(j)}$, show that $\mathbb{E}[\nabla_j \log f] = 0$.

(b) By differentiating $\mathbb{E}[\nabla_j \log f] = 0$ with respect to $\theta^{(k)}$, show that

$$\mathbb{E}[\nabla_{jk} \log f] + \mathbb{E}\left[(\nabla_j \log f)(\nabla_k \log f)\right] = 0,$$

which yields the Information Matrix Equality.

5. [Hansen 10.16] Let $g(x)$ be a density function of a random variable with mean μ and variance σ^2. Let X be a random variable with density function

$$f(x|\theta) = g(x)(1 + \theta(x - \mu)).$$

Assume $g(x)$, μ and σ^2 are known. The unknown parameter is θ. Assume that X has bounded support so that $f(x|\theta) \geq 0$ for all x.

(a) Verify that $\int_{-\infty}^{\infty} f(x|\theta)dx = 1$.

(b) Calculate $\mathbb{E}[X]$.

(c) Find the information $\mathcal{F}_\theta$ for θ when true parameter is θ_0. Write your expression as an expectation of some function of X

(d) Find a simplified expression for $\mathcal{F}_\theta$ when $\theta_0 = 0$.

(e) Given a random sample $\{X_1, ..., X_n\}$, write the log-likelihood function for θ.

(f) Find the first-order-condition for the MLE $\hat{\theta}$ for θ_0.

(g) Using the known asymptotic distribution for maximum likelihood estimators, find the asymptotic distribution for $\sqrt{n}(\hat{\theta} - \theta_0)$ as $n \to \infty$

(h) How does the asymptotic distribution simplify when $\theta_0 = 0$?

6. Complete the proof of Cramér-Rao Lower Bound on page 20 of the slides on *Estimation* by showing

$$\text{var}\left(\frac{\partial}{\partial \theta} \log f(\mathbf{X}|\theta_0)\right) = n\mathscr{F}_\theta$$

7. Let $\hat{F}_n(x)$ denote the empirical distribution function of a random sample. For each fixed x, show that

$$\sqrt{n}(\hat{F}_n(x) - F(x)) \xrightarrow{d} N(0, F(x)(1 - F(x))),$$

where $F(x) = P\{X \leq x\}$ is the cdf function evaluated at x.

8. [Hansen] Let X follows an exponential distribution with pdf $f(x) = \theta\exp(-\theta x), x \geq 0, \theta > 0$. The expected value of X is given by $\mathbb{E}X = \frac{1}{\theta}$

(a) Find the Cramér-Rao lower bound for θ.

(b) Find the Method of Moment Estimator $\hat{\theta}_{MME}$ for θ.

(c) Find the asymptotic distribution of $\hat{\theta}_{MME}$ by delta method.