ECON 6130

Problem Set 2

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Worked with Robert Betancourt on Problem 1 and Finn Ye on Problem 2. Sarah Greenberg proofread and caught a number of idiotic mistakes.

Problem 1. Exchange economy with three agents

1. The aggregate endowment in each period is $\sum_{i=1}^{3} e_t^i = 3 \; \forall \; t = 0, 1, \ldots.$
2. In an Arrow-Debreu market structure, markets are open before time begins – sometimes denoted as period -1. The agents commit to a stream of trades and prices, which themselves implicitly define a consumption stream for each agent. They will all trade amongst themselves, reallocating resources from the agent with positive endowment in each period towards the agents with 0 endowment.

   A competitive Arrow-Debreu equilibrium is a stream of equilibrium prices $\{\hat{p}_t\}_{t=0}^{\infty}$ and consumptions $\{\hat{c}_t^i\}_{t=0}^{\infty}$ such that given $\{\hat{p}_t\}$, for each $i \in \{1, 2, 3\}$, $\{\hat{c}_t^i\}$ solves

$$\max_{\{\hat{c}_t^i\}} \sum_{t=0}^{\infty} \beta^t \log(c_t^i)$$

   subject to

$$\sum_{t=0}^{\infty} \hat{p}_t c_t^i \leq \sum_{t=0}^{\infty} \hat{p}_t e_t^i$$
$$c_t^i \geq 0 \; \forall \; t = 0, 1, \ldots$$

   Additionally, markets clear, meaning that

$$c_t^1 + c_t^2 + c_t^3 = e_t^1 + e_t^2 + e_t^3 \; \forall \; t = 0, 1, \ldots$$

3. In a sequential market structure, markets are open each period, and trade occurs at each time t. Agents trade claims to consumption in period $t+1$. Specifically, the agent with positive endowment in period $t+1$ will trade some of their endowment to the other two agents.

   A competitive sequential equilibrium is a stream of implied equilibrium consumption claims $\{\tilde{a}Ti_t\}_{t=0}^{\infty}$ and consumptions $\{\tilde{c}_t^i\}_{t=0}^{\infty}$ such that for each $i \in \{1, 2, 3\}$, $\{\tilde{c}_t^i\}$ solves

$$\max_{\{c_t^i\}, \{a_{t+1}^i\}} \sum_{t=0}^{\infty} \beta^t \log(c_t^i)$$

   subject to, for all $t = 0, 1, \ldots,$

$$c_t^i + Q_t a_{t+1}^i \leq e_t^i + a_t^i$$
$$-a_{t+1}^i \leq 3$$
$$c_t^i \geq 0$$

where 3 is the natural debt limit, since the maximum endowment is 3, meaning that borrowing more than 3 in any period will lead to negative consumption. Additionally, we have that markets fully clear, meaning that for each t,

$$\sum_{i=1}^{3} c_t^i = 3 = \sum_{i=1}^{3} e_t^i$$

and

$$\sum_{i=1}^{3} a_{t+1}^i = 0$$

4. The proposition is:
Proposition 1. *The Arrow-Debreu Equilibrium and the Sequential Trading Equilibrium are equivalent as long as both $\hat{c}_t^i = \tilde{c}_t^i$ for all $t = 0, 1, \ldots,$ and $i = \{1, 2, 3\}$ and $\hat{p}_{t+1} = Q_t \hat{p}_t$, where $\hat{c}$ denotes the Arrow-Debreu consumption, $\tilde{c}$ denotes the sequential consumption, $\hat{p}$ denotes the Arrow-Debreu prices, and Q_t is the sequential equilibrium pricing kernel.*

5. We can solve this problem for each agent i, and then solve for the equilibrium. We have that

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \log(c_t^i) - \lambda^i \left(\sum_{t=0}^{\infty} p_t c_t^i - \sum_{t=0}^{\infty} p_t e_t^i \right)$$

The first order conditions give us

$$\frac{\partial \mathcal{L}}{\partial c_t^i} = \frac{\beta^t}{c_t^i} - \lambda^i p_t = 0 \Longrightarrow \beta^t = c_t^i \lambda^i p_t$$

and

$$\frac{\partial \mathcal{L}}{\partial c_{t+1}^i} = \frac{\beta^{t+1}}{c_{t+1}^i} - \lambda^i p_{t+1} = 0 \Longrightarrow \beta^{t+1} = c_{t+1}^i \lambda^i p_{t+1}$$

Combining, we get that

$$p_{t+1}(c_{t+1}^1 + c_{t+1}^2 + c_{t+1}^3) = \beta p_t (c_t^1 + c_t^2 + c_t^3)$$

and using market clearing

$$p_{t+1}(e_{t+1}^1 + e_{t+1}^2 + e_{t+1}^3) = \beta p_t (e_t^1 + e_t^2 + e_t^3) \Longrightarrow p_{t+1} = \beta p_t$$

and assuming that $\hat{p}_0 = 1$, we have that $\hat{p}_t = \beta^t$. Using the first order conditions, we have that

$$\hat{p}_{t+1} c_{t+1}^i = \beta \hat{p}_t c_t^i \Longrightarrow c_{t+1}^i = c_t^i = c^i \;\forall\; t = 0, 1, \ldots$$

The values of the endowments for each consumer are

$$\sum_{t=0}^{\infty} \hat{p}_t e_t^1 = 3 \sum_{t=0}^{\infty} \beta^{3t} = \frac{3}{1-\beta^3}$$

$$\sum_{t=0}^{\infty} \hat{p}_t e_t^2 = 3\beta \sum_{t=0}^{\infty} \beta^{3t} = \frac{3\beta}{1-\beta^3}$$

$$\sum_{t=0}^{\infty} \hat{p}_t e_t^3 = 3\beta^2 \sum_{t=0}^{\infty} \beta^{3t} = \frac{3\beta^2}{1-\beta^3}$$

Thus, from the budget constraint, we have that

$$(c_t^1, c_t^2, c_t^3) = \left(\frac{3-3\beta}{1-\beta^3}, \frac{3\beta - 3\beta^2}{1-\beta^3}, \frac{3\beta^2 - 3\beta^3}{1-\beta^3} \right) \qquad \text{for all } t = 0, 1, \ldots$$

Recall also from above that

$$\hat{p}_t = \beta^t \qquad \text{for all } t = 0, 1, \ldots$$

6. The agents are better off! If there was no trade, agent 1 would receive utility of $\log(3)$ in period 1, utility of $-\infty$ in period 2, utility of $-\infty$ in period 3, utility of $\beta^4 \log(3)$ in period 4, and so on, and their total utility would be $-\infty$. Instead, they receive total utility

$$\frac{3-3\beta}{1-\beta^3}\sum_{t=0}^{\infty}\beta^t = \frac{3}{1-\beta}$$

Similarly, agent 2 receives $\frac{3\beta}{1-\beta^3}$ instead of $-\infty$, and agent 3 receives $\frac{3\beta^2}{1-\beta^3}$ instead of $-\infty$. All agents strictly benefit.

7. As shown above from the first order conditions, the equilibrium consumption sequences are constant over time because $\hat{p}_t = \beta^t$ for all t, meaning that since

$$\hat{p}_{t+1}c_{t+1}^i = \beta\hat{p}_t c_t^i$$

we have that

$$c_{t+1}^i = c_t^i \qquad \text{for all } i \in \{1,2,3\}, t = 0,1,\ldots$$

Thus, consumption streams are constant over time. A plot of the consumption streams and prices over time is:

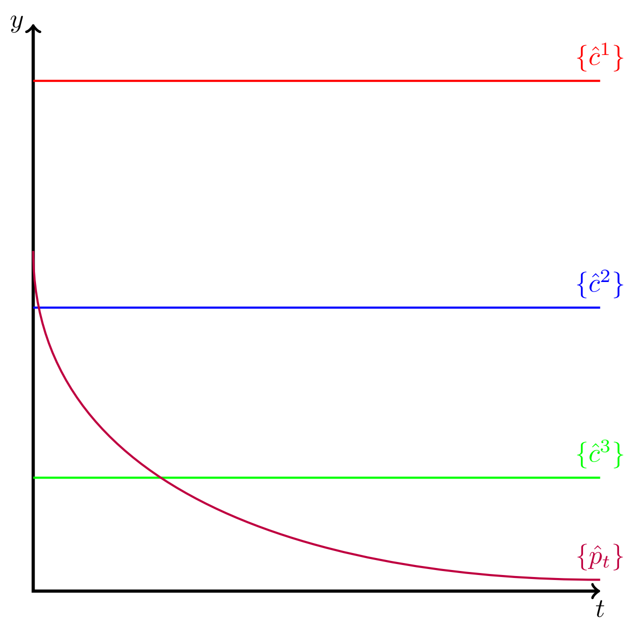

Figure 1: Consumption Streams and Prices Over Time

We can see that, for any $\beta \in (0,1)$ and any $t \in \mathbb{N} \cup \{0\}$, $u(\hat{c}_t^1) > u(\hat{c}_t^2) > u(\hat{c}_t^3)$, since utility is strictly increasing and $\beta^2\hat{c}_t^1 = \beta\hat{c}_t^2 = \hat{c}_t^3 \implies \hat{c}_t^1 > \hat{c}_t^2 > \hat{c}_t^3$. Note that they are not equal. Even though the endowments are similarly structured, agent 1 has a distinct advantage by having the endowment in the first period, meaning that they have a higher discounted total endowment than agents 2 or 3. Agent 2 also has a higher discounted total endowment than agent 3. The higher total discounted endowment, the more bargaining power, so the higher consumption in the Arrow-Debreu structure.

8. We have that an asset a_0 gives a stream $\{a_t\}$ of dividends, where $a_t = 0.05$ for all $t = 0,1,\ldots$. The

total discounted value of that asset under the equilibrium prices $\hat{p}_t$ is

$$\sum_{t=0}^{\infty} \hat{p}_t a_t = 0.05 \sum_{t=0}^{\infty} \hat{p}_t = 0.05 \sum_{t=0}^{\infty} \beta^t = \frac{0.05}{1-\beta}$$

9. We have that the social planner is maximizing weighted utility subject to a budget of the total endowments. She is solving

$$\max_{\{c_t^i\}} \sum_{i=1}^{3} \lambda^i \sum_{t=0}^{\infty} \beta^t \log(c_t^i)$$

subject to

$$\sum_{i=1}^{3} c_t^i \leq \sum_{i=1}^{3} e_t^i = 3 \quad \text{for all } t = 0, 1, \ldots$$

where $c_t^i \geq 0 \; \forall \; i, t$, and $\lambda^3 = 1$.

10. We have that the social planner solves the Lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} \left[\sum_{i=1}^{3} \lambda^i \beta^t \log(c_t^i) - \theta_t \left(\sum_{i=1}^{3} c_t^i - \sum_{i=1}^{3} e_t^i \right) \right]$$

The first order conditions are

$$\frac{\partial \mathcal{L}}{\partial c_t^i} = \frac{\lambda^i \beta^t}{c_t^i} - \theta_t = 0 \Longrightarrow \beta^t = \frac{\theta_t c_t^i}{\lambda^i}$$

The same condition holds for each agent i, so for agents 1 and 2 we have that

$$\lambda^1 = \frac{c_t^1}{c_t^3} \Longrightarrow c_t^1 = \lambda^1 c_t^3$$

and

$$\lambda^2 = \frac{c_t^2}{c_t^3} \Longrightarrow c_t^2 = \lambda^2 c_t^3$$

This means that their equilibrium consumption is strictly increasing in their social planning weights. Additionally, we have that

$$\frac{\partial \mathcal{L}}{\partial \theta_t} = \sum_{i=1}^{3} c_t^i - \sum_{i=1}^{3} e_t^i = 0 \Longrightarrow c_t^1 + c_t^2 + c_t^3 = 3$$

which means we can rewrite the first order conditions as

$$c_t^1 = \lambda^1 (3 - c_t^1 - c_t^2) \Longrightarrow c_t^1 = \lambda^1 \frac{3 - c_t^2}{1 + \lambda^1}$$

and then

$$\begin{aligned} c_t^2 &= \lambda^2 (3 - c_t^1 - c_t^2) \\ &= \lambda^2 \left(3 - \lambda^1 \frac{3 - c_t^2}{1 + \lambda^1} - c_t^2 \right) \\ \Rightarrow c_t^2 &= \frac{3\lambda^2}{1 + \lambda^1 + \lambda^2} \end{aligned}$$

Substituting back into the expression for c_t^1, we get

$$c_t^1 = \lambda^1 \left(3 - c_t^1 - \frac{3\lambda^2}{1+\lambda^1+\lambda^2}\right)$$
$$\Rightarrow c_t^1 = \frac{3\lambda^1}{1+\lambda^1+\lambda^2}$$

and finally, substituting into the (binding) budget constraint, we get

$$c_t^3 = 3 - \frac{3\lambda^1}{1+\lambda^1+\lambda^2} - \frac{3\lambda^2}{1+\lambda^1+\lambda^2} = \frac{3}{1+\lambda^1+\lambda^2}$$

Since the social planner's weights do not change, we can say that $c_t^i = c_{t+1}^i = \cdots = c^i$ for all $i \in \{1,2,3\}$ and $t = 0, 1, \ldots$. Thus, we have that

$$(c^1, c^2, c^3) = \left(\frac{3\lambda^1}{1+\lambda^1+\lambda^2}, \frac{3\lambda^2}{1+\lambda^1+\lambda^2}, \frac{3}{1+\lambda^1+\lambda^2}\right)$$

For the first order conditions to be identical, it must be the case that the consumption streams that solve them be identical. Thus, we must have that

$$\frac{3-3\beta}{1-\beta^3} = \frac{3\lambda^1}{1+\lambda^1+\lambda^2}$$
$$\frac{3\beta-3\beta^2}{1-\beta^3} = \frac{3\lambda^2}{1+\lambda^1+\lambda^2}$$
$$\frac{3\beta^2-3\beta^3}{1-\beta^3} = \frac{3}{1+\lambda^1+\lambda^2}$$

Solving using Wolfram Alpha, we get that this is true when

$$\lambda^1 = \frac{1}{\beta^2} \quad \text{and} \quad \lambda^2 = \frac{1}{\beta}$$

11. Consumption streams are not constant. In our original problem, we found that consumption streams were constant because the first order conditions simplified to

$$p_{t+1}(c_{t+1}^1 + c_{t+1}^2 + c_{t+1}^3) = \beta p_t(c_t^1 + c_t^2 + c_t^3)$$

and using market clearing, we get that

$$p_{t+1}(e_{t+1}^1 + e_{t+1}^2 + e_{t+1}^3) = \beta p_t(e_t^1 + e_t^2 + e_t^3)$$

From there, we were able to divide out the total endowments to get a simple recursive relationship on prices. However, in this case, if $t \pmod 3 = 0$, then $\sum_{i=1}^3 e_t^i = 3$, but otherwise $\sum_{i=1}^3 e_{t+1}^i = 4$. Since aggregate endowments are not equal in each period, the simple recursive relationship on prices does not exist and consumption streams are not constant. Formally, since $\hat{p}_{t+1}$ is not necessarily equal to $\beta\hat{p}_t$, the relationship $\hat{p}_{t+1}c_{t+1}^i = \beta\hat{p}_t c_t^i$ does not simplify down to $c_{t+1}^i = c_t^i$.

Problem 2. One-period Pareto problem

1. Take some $c, c' \in \mathbb{R}$. We have that for fixed θ,

$$v_\theta(c) = \max_{c^1,c^2} \theta u(c^1) + (1-\theta)w(c^2) \text{ subject to } c^1 + c^2 = c$$

and

$$v_\theta(c') = \max_{c^1,c^2} \theta u(c^1) + (1-\theta)w(c^2) \text{ subject to } c^1 + c^2 = c'$$

To show that v_θ is concave, it suffices to show that for $c, c' \in \mathbb{R}^2_+$ and $\alpha \in (0,1)$,

$$\alpha v_\theta(c) + (1-\alpha)v_\theta(c') \leq v_\theta(\alpha c + (1-\alpha)c')$$

We have that there exist maximizing bundles (c^1, c^2) such that $c^1 + c^2 = c$, and $(c^{1'}, c^{2'})$ such that $c^{1'} + c^{2'} = c'$. This means that

$$\begin{aligned}
\alpha v_\theta(c) + (1-\alpha)v_\theta(c') &= \alpha(\theta u(c^1) + (1-\theta)w(c^2)) + (1-\alpha)(\theta u(c^{1'}) + (1-\theta)w(c^{2'})) \\
&= \theta(\alpha u(c^1) + (1-\alpha)u(c^{1'})) + (1-\theta)(\alpha w(c^2) + (1-\alpha)w(c^{2'})) \\
&< \theta u(\alpha c^1 + (1-\alpha)c^{1'}) + (1-\theta)w(\alpha c^2 + (1-\alpha)c^{2'}) \\
&\leq v_\theta(\alpha c + (1-\alpha)c')
\end{aligned}$$

Where the strict inequality uses the strict concavity of u and w, and the weak inequality uses the fact that since $\alpha(c^1 + c^2) + (1-\alpha)(c^{1'} + c^{2'}) = \alpha c + (1-\alpha)c'$, the bundle is feasible and thus weakly less than the maximizing bundle. Thus, v_θ is strictly concave, and the solution to this problem is a concave utility function dependent on θ.

2. We have that the social planner is optimizing over the Lagrangian

$$\mathcal{L} = \theta u(c^1) + (1-\theta)w(c^2) + \lambda(c - c^1 - c^2)$$

For some c, $v_\theta(c)$ admits a maximizing pair (c^1, c^2). At this maximizing pair (and the original c), it must be the case that the first order conditions are equal to zero. Formally, we have that

$$\frac{\partial \mathcal{L}}{\partial c^1} = \theta u'(c^1) - \lambda = 0 \Longrightarrow \theta u'(c^1) = \lambda$$

and

$$\frac{\partial \mathcal{L}}{\partial c^2} = (1-\theta)w'(c^2) - \lambda = 0 \Longrightarrow (1-\theta)w'(c^2) = \lambda$$

So at a maximizing c, it must be the case that $\theta u'(c^1) = (1-\theta)w'(c^2) = \lambda$. Additionally, from our definition of the Lagrangian, we know that $v'_\theta(c) = \lambda$. Thus, we have that at any c,

$$v'_\theta(c) = \theta u'(c^1) = (1-\theta)w'(c^2)$$

where (c^1, c^2) is the maximizing consumption pair.

Problem 3. Proving the First Welfare Theorem.

1. We have that there exists a feasible allocation $\{\tilde{c}^t_i\}_{t=0}^\infty$ such that $U(\tilde{c}^1) > U(\hat{c}^1)$. Since we are assuming (as in all Arrow-Debreu equilibria) that $U'(c) > 0 \ \forall\ c$, we can say that $U(\tilde{c}^1) > U(\hat{c}^1) \iff \tilde{c}^1 > \hat{c}^1$. Taking the left-inner-product with the price vector $\hat{p}$, we get that

$$\hat{p} \cdot \tilde{c}^1 > \hat{p} \cdot \hat{c}^1 \Longrightarrow \sum_{t=0}^\infty \hat{p}_t \tilde{c}^1_t > \sum_{t=0}^\infty \hat{p}_t \hat{c}^1_t$$

2. We have that there exists feasible allocation $\{\tilde{c}^t_i\}_{t=0}^\infty$ such that $U(\tilde{c}^i) \geq U(\hat{c}^i)$ for all $i \neq 1$. Consider two cases for arbitrary i. First, assume that $U(\tilde{c}^i) > U(\hat{c}^i)$. Then, as in part (1), we have that $\sum_{t=0}^\infty \hat{p}_t \tilde{c}^i_t > \sum_{t=0}^\infty \hat{p}_t \hat{c}^i_t \Longrightarrow \sum_{t=0}^\infty \hat{p}_t \tilde{c}^i_t \geq \sum_{t=0}^\infty \hat{p}_t \hat{c}^i_t$.

Next, assume that $U(\tilde{c}^i) = U(\hat{c}^i)$. Since we have that $U'(c) > 0 \; \forall \; c$, this implies that $\tilde{c}^i = \hat{c}^i$. Taking the left-inner-product with the price vector $\hat{p}$, we get that

$$\hat{p} \cdot \tilde{c}^i = \hat{p} \cdot \hat{c}^i \implies \sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^i = \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^i \implies \sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^i \geq \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^i$$

3. The two above conditions combine to imply that $\tilde{c}^i \geq \hat{c}^i$ for all i, and that $\tilde{c}^1 > \hat{c}^1$. This implies that $\sum_i \sum_{t=0}^{\infty} \tilde{c}_t^i > \sum_i \sum_{t=0}^{\infty} \hat{c}_t^i$. However, recall that markets clear in any Arrow-Debreu equilibrium. This means that since we assumed that $\{\hat{c}\}$ was an Arrow-Debreu allocation, that $\sum_i \sum_{t=0}^{\infty} \hat{c}_t^i = \sum_i \sum_{t=0}^{\infty} e_t^i$. That implies that $\sum_i \sum_{t=0}^{\infty} \tilde{c}_t^i > \sum_i \sum_{t=0}^{\infty} e_t^i$, which violates our earlier assumption that $\{\tilde{c}\}$ was a feasible allocation. This is a contradiction, so it must be the case that $(\{\hat{c}_t^i\}_{t=0}^{\infty})_{i \in I}$ is Pareto efficient.