2. Euclidean Topology

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1 Basic topology of $\mathbb{R}^d$

For any $d \in \mathbb{N}$, let $\mathbb{R}^d$ denote the product set:

$$\mathbb{R}^d := \times_{k=1}^{d} \mathbb{R}^k.$$

In what follows, it should be understood that $d \in \mathbb{N}$, when I write $\mathbb{R}^d$.

Definition 1. If $a \in \mathbb{R}^d$, write $a = (a_1, \ldots, a_k)$. The *Euclidean distance* between $a, b \in \mathbb{R}^d$ is given by

$$\|b - a\| = \sqrt{\sum_{i=1}^{d} (b_i - a_i)^2}$$

In $\mathbb{R}^2$, this is the familiar distance given by the Pythagorean theorem. In $\mathbb{R}$, $\|b - a\| = |b - a|$.

Remark 1. There are many other "distances" or *metrics* in $\mathbb{R}^d$. Indeed, a more natural way to study topology (and analysis generally) is abstracting away from $\mathbb{R}$ or $\mathbb{R}^d$ and working directly in "metric spaces". In this class, we stick to $\mathbb{R}^d$ with the Euclidean distance metric.

Definition 2. For every $\mathbf{x}_0 \in \mathbb{R}^d$ and $\epsilon > 0$, the *open ball* (sometimes *neighbourhood*) of radius ϵ centred at $\mathbf{x}_0$ is the set

$$B_\epsilon(\mathbf{x}_0) := \left\{ \mathbf{x} \in \mathbb{R}^d : \|\mathbf{x} - \mathbf{x}_0\| < \epsilon \right\}.$$

Definition 3. A set $S \subseteq \mathbb{R}^d$ is *open* if for every $\mathbf{x} \in S$ there exists some $\epsilon > 0$ such that $B_\epsilon(\mathbf{x}) \subseteq S$.

Exercise 1. Verify that open intervals $(a, b) \subset \mathbb{R}$ are indeed open according to the definition above. Do the same for the interval $(a, +\infty) \subset \mathbb{R}$.

Definition 4. A set $S \subseteq \mathbb{R}^d$ is *closed* if its complement $S^c = \mathbb{R}^d \backslash S$ is open.

Remark 2. By a strict reading of the definition, we can see that the empty set $\emptyset$ and the entire space $\mathbb{R}^d$ are open. Hence, they are also closed (and are the only sets in $\mathbb{R}^d$ that are both open and closed, in fact).

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Remark 3. Many sets in $\mathbb{R}^d$ are neither open nor closed. For instance, in $\mathbb{R}$, the set $(0,1] = \{x \in \mathbb{R} : 0 < x \leq 1\}$ is neither open nor closed (why?).

Proposition 1. *The (arbitrary) union of open sets is open. The intersection of finitely many open sets is open.*

Exercise 2 (PS3)**.** Prove this. What about arbitrary intersections of open sets?

Exercise 3 (PS3)**.** Prove that the closed interval $[a,b]$ is indeed closed. (Feel free to use Exercise 1.)

Proposition 2. *The arbitrary intersection of closed sets is closed. The union of finitely many closed sets is closed.*

Exercise 4 (PS3)**.** Prove this.[1] What about arbitrary unions of closed sets?

Proposition 3. *Any finite subset of $\mathbb{R}^d$ is closed.*

Proof. It suffices to show that a singleton set is closed. (why?) Fix $\mathbf{x} \in \mathbb{R}^d$. Then, $\{\mathbf{x}\}^c = \times_{i=1}^n (-\infty, x_i) \cup (x_i, +\infty)$. Since product of open sets are open, and unions of open sets are union, $\{\mathbf{x}\}^c$ must be open; i.e., $\{\mathbf{x}\}$ is closed. ■

Proposition 4. *A set $S \subseteq \mathbb{R}^d$ is closed if and only if for every sequence $(\mathbf{x}_n)$ in S, $\mathbf{x}_n \to \mathbf{x}$ implies $\mathbf{x} \in S$.*

Proof. Let $S \subseteq \mathbb{R}^d$ be closed and let $(\mathbf{x}_n)$ be a sequence in S such that $\mathbf{x}_n \to \mathbf{x} \in \mathbb{R}^d$. By way of contraction, suppose that $\mathbf{x} \notin S$; i.e., $\mathbf{x} \in S^c$. Since S is closed, S^c is open (why?), so there is some $\epsilon > 0$ such that $B_\epsilon(\mathbf{x}) \subseteq S^c$ (why?). Now since $\mathbf{x}_n \to \mathbf{x}$, there exist some $N \in \mathbb{N}$ such that $\|\mathbf{x} - \mathbf{x}_n\| < \epsilon$ for all $n > N$. Therefore, for $n > N$, $\mathbf{x}_n \in B_\epsilon(\mathbf{x}) \subseteq S^c$, which implies $x_n \notin S$; a contradiction. Thus, $\mathbf{x} \in S$.

Suppose now that every convergent sequence in S has limits in S. We want to show that S is closed. Toward a contradiction, suppose S is not closed; i.e., S^c is not open. Then, there must exist an $\mathbf{x} \in S^c$ such that, for every $\epsilon > 0$, the open ball $B_\epsilon(\mathbf{x})$ is not entirely contained in S^c meaning that $B_\epsilon(\mathbf{x})$ must intersect $(S^c)^c = S$. Consider the sequence of open balls generated by choosing $\epsilon = \frac{1}{n}$ for every $n \in \mathbb{N}$. Since all of these intersect S, we can select one $\mathbf{x}_n \in S$ from every $B_{\frac{1}{n}}(\mathbf{x})$. Further, since $x_n \in B_{\frac{1}{n}}(\mathbf{x})$ and $\frac{1}{n} \to 0$, $\mathbf{x}_n \to \mathbf{x}$. Thus, we have constructed a sequence $(\mathbf{x}_n)$ in S that converges. By hypothesis, it must be that its limit $\mathbf{x}$ is in S. But this contradicts $\mathbf{x} \in S^c$. ■

Definition 5. A set S is *compact* if every open cover has a finite subcover. That is, if for *any* collection $\mathcal{U}$ of open sets such that $S \subseteq \bigcup_{U \in \mathcal{U}} U$, there is a *finite* subset $\{U_1, U_2, \dots U_m\}$ that still covers S, i.e., $S \subseteq \bigcup_{i=1}^m U_i$.

Remark 4. Compactness captures the idea of finiteness. To see this, take any subset $S \subseteq \mathbb{R}$ and let us ask if the set is is completely contained in an open ball, which is in fact equivalent to asking if S is bounded (pause and make sure you understand this). When S is finite, $\epsilon \coloneqq \max\{|x - x'| : x' \in S\} + 1$ is well-defined for any $x \in S$. Thus, we may pick any $x \in S$ and observe that $S \subset B_\epsilon(x)$; i.e., S is bounded. However, if S is infinite, ϵ need not be well-defined. But suppose we know that S is compact. First observation is that $\{B_n(x) : n \in \mathbb{N}\}$ is an open cover of S. Thus, by compactness,

[1] *Hint:* Recall De Morgan's laws.

there exists finitely many $B_{\epsilon_{n_1}}(x)$, $\dots$, $B_{\epsilon_{n_m}}(x)$ such that $S \subseteq \bigcup_{i=1,\dots,} B_{m_i}(x)$. Thus, we can now set $\epsilon := \max\{\epsilon_{n_1}, \dots, \epsilon_{n_m}\} + 1$ so that $S \subset B_\epsilon(x)$. This should give you a sense in which compactness provides a finite structure for infinite sets.

Theorem 1 (Heine-Borel)**.** *A set $S \subseteq \mathbb{R}^d$ is compact if and only if it is closed and bounded.*

Exercise 5. Give an example of open cover that does not admit a finite subcover for $(0,1] \subset \mathbb{R}$ (which is not closed) and for $[0,+\infty)$ (which is not bounded).

Remark 5. In PS3, you'll see a related notion of compactness called *sequential compactness.* The two notions are equivalent in Euclidean spaces but need not be in more general spaces.

2 Continuous real functions

Definition 6 (Continuity using sequences)**.** A real-valued function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ is *continuous at* $x_0 \in S$ if, for every sequence $(x_n)_n$ in S converging to x_0, we have $f(x_n) \to f(x_0)$. The function is *continuous on* S if it is continuous at every $x_0 \in S$.

Exercise 6 (PS4)**.** A point x is an *isolated point* in $S \subseteq \mathbb{R}$ if $x \in S$ and there exists $\epsilon > 0$ such that $B_\epsilon(x) \cap S = \{x\}$. For example, $\{1\}$ is an isolated point in $S = \{1\} \cup [2,3]$. What real-valued functions $f : S \to \mathbb{R}$ is continuous at 1?

Definition 7 (Continuity using neighbourhoods)**.** A real-valued function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ is *continuous at* $x_0 \in S$ if, for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(x_0)| < \epsilon \ \forall x \in S : |x - x_0| < \delta.$$

Remark 6. In PS4, you will see yet another equivalent definition of a continuous function that uses its preimage.

Proposition 5. *The two definitions of continuity are equivalent (in $\mathbb{R}$). That is, a function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in S$ according to Definition 6 if and only if it is continuous according to Definition 7.*

Proof. Suppose that f is continuous at x_0 according to Definition 7. Let $(x_n)_n$ be a sequence in S such that $x_n \to x_0$; we want to prove that $f(x_n) \to f(x_0)$. Let $\epsilon > 0$. By definition, there exists $\delta > 0$ such that $x \in S$ and $|x - x_0| < \delta$ imply $|f(x) - f(x_0)| < \epsilon$. Since $x_n \to x_0$, there exists $N \in N$ such that $n > N$ implies $|x_n - x_0| < \delta$. Therefore, for all $n > N$, $|f(x_n) - f(x_0)| < \epsilon$; i.e., that is, $f(x_n) \to f(x_0)$.

Conversely, suppose that f is continuous at x_0 according to Definition 6. By contradiction, assume that it is not continuous at x_0 according to Definition 7. Then there exists $\epsilon > 0$ such that for every $\delta > 0$, there exists a $x \in S$ with $|x - x_0| < \delta$ and $|f(x) - f(x_0)| \geq \epsilon$ (note the effect of negation on all the quantifiers). In particular, we can choose $\delta = \frac{1}{n}$, for each $n \geq 1$, and then get a sequence $(x_n)_n$ in S such that $|x_n - x_0| < \frac{1}{n}$ and $|f(x_n) - f(x_0)| \geq \epsilon$. Thus, $x_n \to x_0$ yet the limit of $f(x_n)$ is not $f(x_0)$. Contradiction. ■

Proposition 6. *Let f and g be real-valued functions that are continuous at x_0, and let $k \in \mathbb{R}$. Then, the following functions are all continuous at x_0: (i) $|f|$; (ii) $k \cdot f$; (iii) $f + g$; (iv) $f \cdot g$; (v) f/g, if $g(x_0) \neq 0$.*

Proof. If this kind of proof is new to you, you can try proving (i) and (iii) as an exercise. ■

Proposition 7. *If $f : S \to \mathbb{R}$ is continuous at x_0, $f(x_0) \in T \subseteq \mathbb{R}$ and $g : T \to \mathbb{R}$ is continuous at $f(x_0)$, then the composite function $g \circ f$ is continuous at x_0.*

Proof. Let $(x_n)_n$ be a sequence in $\{x \in S : f(x) \in T\}$ (why here?) converging to x_0. Since f is continuous at x_0, $f(x_n) \to f(x_0)$. Using this, since g is continuous at $f(x_0)$, we have $g(f(x_n)) \to g(f(x_0)) = (g \circ f)(x_0)$ (why is $g(f(x_n))$ always well-defined?). ■

Exercise 7 (PS4)**.** Prove Proposition 7 using ϵ-δ definition of continuity.

Exercise 8 (PS4)**.** Let f and g be continuous at x_0. Prove or disprove: $\max(f, g)$ is continuous at x_0.[2]

Exercise 9 (PS4)**.** Prove or disprove: $f : S \to \mathbb{R}$ is continuous at x_0 if and only if, for every *monotonic* sequence $(x_n)_n$ in S converging to x_0, $f(x_n) \to f(x_0)$. Hint: You can use Exercise 30 from "1. Real Sequences.pdf".

2.1 Other kinds of of continuity

Definition 8. A function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ is *uniformly continuous on* $Z \subseteq S$ if, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(z)| < \epsilon \ \forall x, z \in Z : |x - z| < \delta.$$

If f is uniformly continuous on S, then f is *uniformly continuous.*

Exercise 10. What is the difference between uniform continuity and continuity?

Proposition 8. *Let $S \subseteq \mathbb{R}$ contain a closed and bounded interval $[a, b]$ and $f : S \to \mathbb{R}$. Then, f is continuous on $[a, b]$ if and only if f is uniformly continuous on $[a, b]$.*

Proof. Given the previous exercise, it suffices to show that if f is continuous on $[a, b]$, then f is uniformly continuous on $[a, b]$. Toward a contradiction, suppose that f is continuous on $[a, b]$ but f is not uniformly continuous on $[a, b]$; i.e., for some $\epsilon > 0$, for any $\delta > 0$, there exists $x, y \in [a, b]$ such that $|x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$. This implies that there exist sequences $(x_n)_n$ and $(y_n)_n$ in $[a, b]$ such that

$$|x_n - y_n| < \frac{1}{n} \text{ and } |f(x_n) - f(y_n)| \geq \epsilon \ \forall n \in \mathbb{N}. \tag{1}$$

By the Bolzano-Weierstrass Theorem (because $[a, b]$ is bounded), there exists a convergent subsequence $(x_{n_k})_k$ of $(x_n)_n$. Let $x := \lim_{k\to\infty} x_{n_k}$. Since $a \leq x_{n_k} \leq b$ for all $k \in \mathbb{N}$, the Sandwich rule tells us that $a \leq x \leq b$ (because $[a, b]$ is closed). Thus, we have $f(x_{n_k}) \to f(x)$ since f is continuous on $[a, b]$. In particular, f is continuous at x. Then, since $x_n - y_n \to 0$ by the first part of (1), we must have that $y_{n_k} \to x$. Therefore, $\lim_{k\to\infty} f(x_{n_k}) = f(x) = \lim_{k\to\infty} f(y_{n_k})$; i.e., $|f(x_m) - f(y_m)| < \epsilon$ for some $m \in \mathbb{N}$ sufficiently large. But this contradicts the second part of (1). ■

[2]**Hint:** $\max\{f, g\} = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|$

Example 1. The function $f(x) = x^2$ is continuous but not uniformly continuous on $(0, \infty)$. To see this, consider two sequences $(x_n)_n$ and $(y_n)_n$ defined as $x_n = n$ and $y_n = n + \frac{1}{n}$ for all $n \in \mathbb{N}$. Note that

$$\lim_{n\to\infty} |x_n - y_n| = \lim_{n\to\infty} \left|\frac{1}{n}\right| = 0.$$

However,

$$|f(x_n) - f(y_n)| = \left(n + \frac{1}{n}\right)^2 - n^2 = 2 + \frac{1}{n^2} \geq 2 \; \forall n \in N.$$

Nevertheless, by the proposition above, f is both continuous and uniformly continuous on any closed and bounded interval $[a, b] \subset \mathbb{R}$.

Definition 9. Let $S \subseteq \mathbb{R}$ contain the closed interval $[a, b]$. A function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ is *absolutely continuous on* $[a, b]$ *if, for any* $\epsilon > 0$, *there exists* $\delta > 0$ *such that, for every finite disjoint collection* $\{(a_k, b_k)\}_{k=1}^n$ *of open intervals in* (a, b),

$$\sum_{k=1}^{n} (b_k - a_k) < \delta \Rightarrow \sum_{k=1}^{n} |f(b_k) - f(a_k)| < \epsilon.$$

Remark 7. The criterion for absolute continuity in the case the finite collection of intervals consists of the interval (a, b) itself is the criterion for uniform continuity on (a, b). Thus, while uniform continuity says that a function cannot vary too much over one small interval, absolute continuity is a stronger requirement that the function cannot vary too much over any union of intervals (whose total length is less than δ). The difference between continuity and absolute continuity is that the δ can be "divided".

Example 2. An example of a uniformly continuous function that is not absolutely continuous is the following:

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases}.$$

It is a bit of work to show that it is not absolutely continuous, but note that this is uniformly continuous because it is continuous on a closed and bounded interval $[0, 1]$.

Definition 10. A function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ is *Lipschitz continuous on* $Z \subseteq S$ if there exists a *Lipschitz constant* $M > 0$ such that

$$|f(x) - f(y)| \leq M|x - y| \;\; \forall x, y \in Z.$$

If f is Lipschitz continuous on S, then f *is Lipschitz continuous.*

Remark 8. If f has Lipschitz constant M, then we can set $\delta = \frac{\epsilon}{M}$ in the criterion for absolute/uniform continuity. Hence, we have Lipschitz continuity $\Rightarrow$ Absolute continuity $\Rightarrow$ Uniform continuity $\Rightarrow$ Continuity.

Definition 11 (Semi-continuity using neighbourhoods)**.** A function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ is *upper semi-continuous (*resp. *lower semi-continuous) at* $x_0 \in S$ if, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$f(x) \leq f(x_0) + \epsilon \; \forall x \in S : |x - x_0| < \delta.$$

(resp. $f(x) \geq f(x_0) - \epsilon \ \forall x \in S : |x - x_0| < \delta$.)

Exercise 11. Draw a function that is continuous on $[0, 1]$. Modify the function so that, at some $x \in [0, 1]$, the function is (i) upper semi-continuous but not lower semi-continuous at x; (ii) lower semi-continuous but not upper semi-continuous at x; (iii) neither upper semi-continuous nor lower semi-continuous at x.

Definition 12 (Semi-continuity using sequences)**.** A function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ is *upper semi-continuous (* resp. *lower semi-continuous) at* $x_0 \in S$ if, for any sequence x_n that converges to x_0, $\limsup_{n\to\infty} f(x_n) \leq f(x_0)$ (resp. $\liminf_{n\to\infty} f(x_n) \geq f(x_0)$).

Exercise 12. Prove that definitions of semi-continuity using neighbourhoods and using sequences are equivalent.

Proposition 9. *A function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ is continuous if and only if it is upper semi-continuous and lower semi-continuous.*

Exercise 13. Prove Proposition above by appealing to the ϵ-δ definition of semi-continuity.

3 Extreme value theorem

Definition 13. A real-valued function $f : S \to \mathbb{R}$ is *bounded* if $f(S) \equiv \{f(x) : x \in S\}$ is bounded.

Theorem 2 (Boundedness theorem)**.** *Let $f : S \to \mathbb{R}$ be a continuous function on a nonempty, compact set $S \subseteq \mathbb{R}$. Then, f is bounded.*

Proof. We want to show that for any $x \in S$, there exists $M \in \mathbb{R}$ such that $|f(x)| \leq M$. Fix any $x \in S$. Since f is continuous, for $\epsilon = 1$, there exists $\delta_x > 0$ such that, for any $x' \in S$ such that $|x' - x| < \delta$, we have $|f(x) - f(x')| < 1$. Define $I_x \coloneqq (x - \delta_x, x + \delta_x)$. Since $|x' - x| < \delta_x$ for any $x' \in I_x \cap S$, we have

$$1 > |f(x') - f(x)| \geq ||f(x')| - |f(x)|| \geq |f(x')| - |f(x)|,$$

where the second line uses the reverse triangle inequality. Thus, we have $1 + |f(x)| > |f(x')|$ for any $x' \in I_x \cap S$.

Since S is compact and $\{I_x\}_{x\in S}$ is an open cover of S, there exists a finite subcover; i.e., there exists a finite number of points $\{x_1, \ldots, x_n\} \subseteq S$ such that $\{I_{x_i}\}_{i=1}^n$ is also a cover of S. Define

$$M \coloneqq 1 + \max\{|f(x_i)| : i \in \{1, \ldots, n\}\}.$$

Then, M is a bound on $|f(x)|$ for any $x \in S$. To see this, for any $x' \in S$, since $S \subseteq \bigcup_{i=1}^n I_{x_i}$, there is an index $i \in \{1, \ldots, n\}$ such that $x' \in I_{x_i} \cap S$ so that $|f(x')| < 1 + |f(x)| \leq M$. ■

Remark 9. In the proof above, we first find a local bound using continuity of f; i.e., we find $M_x \geq |f(x)$ for each $x \in S$. We then use compactness to find a bound that does not depend on x. Thus, you can think of compactness as allowing us to take a local property and making it into a global property.

Theorem 3 (Extreme value theorem)**.** *Let $f : S \to \mathbb{R}$ be a continuous function on a nonempty, compact set $S \subseteq \mathbb{R}$. Then, f attains its maximum and minimum on S; i.e., there exists $u, \ell \in S$ such that $f(u) = \sup f(S)$ and $f(\ell) = \inf f(S)$.*

Proof. By the Boundedness theorem, we know that $M \coloneqq \sup f(S) < \infty$. By definition of supremum, there exists a sequence $(x_n)_n$ in S such that $f(x_n) \to M$.[3] By the Bolzano-Weierstrass theorem, $(x_n)_n$ has a subsequence $(x_{n_k})_k$ that converges to $u \in S$. Since f is continuous on S, $f(u) = \lim_{k\to\infty} f(x_{n_k})$. But $\lim_{k\to\infty} f(x_{n_k}) = \lim_n f(x_n) = M$ (why?), hence $f(u) = M$. Thus, the supremum M is attained.

Finally, since $-f$ is continuous, $-f$ takes a maximum value on S; i.e., f takes a minimum value on S. ∎

Exercise 14. Prove or disprove: The extreme value theorem still holds if f is defined on (a, b).

Exercise 15. Prove or disprove: The extreme value theorem still holds if f is defined on (a, b) and we add the assumption that f is bounded.

Remark 10. Neither continuity nor the compactness of the domain of the function is necessary for existence of extreme values. For example, consider $f : \mathbb{R}_{++} \to \mathbb{R}$ defined as $f(x) \coloneqq \mathbf{1}_{\{x\in\mathbb{Q}\}}$. Observe that $\mathbb{R}_{++}$ is neither closed nor bounded so that the domain of f is not compact. Moreover, f is everywhere discontinuous. However, f clearly attains a maximum (at every rational number) and a minimum (at every irrational number).

Theorem 4 (A more general extreme value theorem)**.** *Let $f : S \to \mathbb{R}$ be a function on a nonempty compact set $S \subseteq \mathbb{R}$. If f is upper semi-continuous, then there exists $u \in S$ such that $f(u) = \sup f(S)$. If f is lower semi-continuous, then there exists $\ell \in S$ such that $f(\ell) = \inf f(S)$.*

Proof. We consider the case for when f is upper semi-continuous. Suppose that $f : S \to \mathbb{R}$ is an upper semi-continuous function on a nonempty compact set $S \subseteq \mathbb{R}$. By the Boundedness theorem, $f(S)$ is bounded. Thus, there exists a sequence $(x_n)_n$ in S such that $f(x_n) \to \sup f(S)$. Observe that, while the sequence $(f(x_n))_n$ is convergent, the sequence $(x_n)_n$ need not be convergent. However, by sequential compactness (Problem Set 3),[4] there exists a subsequence $(x_{n_k})_k$ that converges to some $u \in S$. Since f is upper semi-continuous,

$$\limsup_{k\to\infty} f\left(x_{n_k}\right) \leq f\left(u\right) \leq \sup f\left(S\right).$$

Since $((f(x_{n_k}))_k$ is a subsequence of a convergent sequence $(f(x_n))_n$, the subsequence $((f(x_{n_k}))_k$ converges to the same limit as $(f(x_n))_n$ (why?). Hence,

$$\limsup_{k\to\infty} f\left(x_{n_k}\right) = \lim_{n\to\infty} f\left(x_n\right) = \sup f\left(S\right).$$

It follows that $f(u) = \sup f(S)$. ∎

[3] Since for each $n \in \mathbb{N}$, $M - \frac{1}{n}$ is not an upper bound for $f(S)$ so that there exists $y_n \in f(S)$ such that $M - \frac{1}{n} < y_n \leq M$. By the Sandwich rule, $y_n \to M$.

[4] Recall: A set S is sequentially compact if every sequence in S has a subsequence converging to an element in S. In PS3, you prove(d) that sequential compactness and compactness are equivalent (in $\mathbb{R}^d$).

Remark 11. A simple takeaway is as follows: if you want to solve a maximisation (resp. minimisation problem), you should check that (i) domain is compact; (ii) objective function is upper semi-continuous (resp. lower semi-continuous).

4 Intermediate value theorem

Theorem 5. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. For every y between $f(a)$ and $f(b)$, there exists an $x_0 \in [a, b]$ such that $f(x_0) = y$.*

Proof. Without loss of generality assume $f(a) < y < f(b)$. Let $S = \{x \in [a, b] : f(x) < y\}$. S is nonempty (why?) so by the Completeness axiom $x_0 = \sup S$ exists, and $x_0 \in [a, b]$ (why?).[5]

Since x_0 is the supremum of S, there exists a sequence $(x_n)_n$ in S such that $x_n \to x_0$. By continuity of f and the fact that $f(x_n) < y$ for all $n \in \mathbb{N}$, $f(x_0) = \lim_{n\to\infty} f(x_n) \leq y$.

Now consider the sequence $t_n = \min\{b, x_0 + \frac{1}{n}\}$ (why do we need the min?). Clearly, $t_n \to x_0$ (why?). Since $t_n \notin S$ (why?), $f(t_n) \geq y$ for every n. Using this and continuity, $f(x_0) = \lim f(t_n) \geq y$. Putting the two inequalities together we get $f(x_0) = y$. ■

Proposition 10. *Suppose $f : [0, 1] \to [0, 1]$ is continuous. Then, f has a* fixed point*; i.e., a point $x_0 \in [0, 1]$ such that $f(x_0) = x_0$.*

Proof. The function $g(x) = f(x) - x$ is also continuous, and $g(0) = f(0) - 0 = f(0) \geq 0$, and $g(1) = f(1) - 1 \leq 1 - 1 = 0$. By the Intermediate Value Theorem applied to $[g(1), g(0)]$, there exists $x_0 \in [0, 1]$ such that $g(x_0) = 0$, that is, $f(x_0) = x_0$. ■

[5] There exists a sequence in $[a, b]$ that converges to x_0. Since $[a, b]$ is closed, it must contain x_0.