ECON 6170 Module 7 and Problem Set 11 Answers

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Exercise 1. The Lagrangian of this problem is

$$\mathcal{L}(x, \mu, \theta) = f(x, \theta) + \sum_{k=1}^{K} \mu_k h_k(x, \theta)$$

Because the constraint qualification is satisfied and all function are continuously differentiable, the hypotheses for the theorem of Lagrange are satisfied. Fix $\theta = \theta_0$.

$$\nabla_{(x,\mu)} \mathcal{L}(x^*, \mu^*, \theta_0) = 0$$

at any solution x^* for some μ^*. Define a new function, $\mathscr{L} : \mathbb{R}^m \times \mathbb{R}^{d+K} \to \mathbb{R}^{d+K}$ defined by

$$\mathscr{L}(\theta, x, \mu) := \nabla_{(x,\mu)} \mathcal{L}(x, \mu, \theta) = \left(\frac{\partial \mathcal{L}(x, \mu, \theta)}{\partial x_1}, \ldots, \frac{\partial \mathcal{L}(x, \mu, \theta)}{\partial x_d}, \frac{\partial \mathcal{L}(x, \mu, \theta)}{\partial \mu_1}, \cdots, \frac{\partial \mathcal{L}(x, \mu, \theta)}{\partial \mu_K} \right)$$

If we define $y := (x, \mu)$, then we can write $\mathscr{L}(\theta_0, y^*) = 0$. Suppose we know that $D_y \mathscr{L}(\theta_0, y^*)$ is invertible (which is the case when each equation that defines the critical point of the Lagrangian has content), we can then appeal to the IFT to conclude that there exists a continuously differentiable function $g : B_{\varepsilon_\theta}(\theta_0) \subseteq \mathbb{R}^m \to B_{\varepsilon_y}(y^*) \subseteq \mathbb{R}^{d+K}$ such that

$$g(\theta) = y \iff \nabla_y \mathcal{L}(y, \theta) = 0$$

And

$$Dg(\theta) = \left[D_y^2 \mathcal{L}(x, \mu, \theta) \right]^{-1} D_\theta D_y \mathcal{L}(x, \mu, \theta)$$

This tells us how the critical points, (x^*, μ^*) change locally with the parameter θ. If these critical points define solutions to the maximisation problem, then $Dg(\theta)$ tells us how those solutions change locally with θ.

Exercise 2. [1] The Kuhn-Tucker theorem tells us that for any given value of θ, $x^*(\theta)$ and $\lambda^*(\theta)$ must satisfy

$$\nabla_x f(x^*(\theta), \theta) + \lambda^*(\theta) \cdot \nabla_x h(x^*(\theta), \theta) = 0 \tag{1}$$

$$h(x^*(\theta), \theta) = 0 \tag{2}$$

The latter implies

$$f^*(\theta) = f(x^*(\theta), \theta) = f(x^*(\theta), \theta) + \lambda^*(\theta) \cdot h(x^*(\theta), \theta)$$

*Based on solutions provided by Professor Takuma Habu.

[1] Solution modified from that written by Peter Ireland.

Differentiating both sides with respect to θ yields

$$\nabla f^*(\theta) = \nabla_x f(x^*(\theta),\theta)\cdot\nabla x^*(\theta) + \nabla_\theta f(x^*(\theta),\theta)$$
$$+\lambda^*(\theta)\cdot[\nabla_x h(x^*(\theta),\theta)\cdot\nabla x^*(\theta) + \nabla_\theta h(x^*(\theta),\theta)] + \nabla\lambda^*(\theta)\cdot h(x^*(\theta),\theta)$$

Applying (1), we get

$$\nabla f^*(\theta) = \nabla_\theta f(x^*(\theta),\theta) + \lambda^*(\theta)\cdot\nabla_\theta h(x^*(\theta),\theta) + \nabla\lambda^*(\theta)\cdot h(x^*(\theta),\theta)$$

And then applying (2), we get

$$\nabla f^*(\theta) = \nabla_\theta f(x^*(\theta),\theta) + \lambda^*(\theta)\cdot\nabla_\theta h(x^*(\theta),\theta)$$

Exercise 3. A maximum of S is a supremum of S that lies in S. Suppose x', x'' are maxima of S. Then both are upper bounds for S. In particular, $x' \geq x''$ and $x'' \geq x'$, so by antisymmetry, $x' = x''$.

Exercise 4.

$$\begin{aligned}
&x \vee y \\
&\iff x \geq x \;\&\; x \geq y \;\&\; (z \geq x \;\&\; z \geq y \implies z \geq x) \\
&\iff x \geq y
\end{aligned}$$

$$\begin{aligned}
&x \wedge y \\
&\iff x \leq x \;\&\; x \leq y \;\&\; (z \leq x \;\&\; z \leq y \implies z \leq x) \\
&\iff x \leq y
\end{aligned}$$

$$\begin{aligned}
&\neg(x \geq y) \\
&\iff x \vee y \neq x \\
&\iff x \vee y \neq x \;\&\; x \vee y \geq x \\
&\iff x \vee y > x
\end{aligned}$$

$$\begin{aligned}
&\neg(x \leq y) \\
&\iff x \wedge y \neq x \\
&\iff x \wedge y \neq x \;\&\; x \wedge y \leq x \\
&\iff x \wedge y < x
\end{aligned}$$

Exercise 5.

$$\{(0,1),(1,0)\}$$

We will use the following Lemma in Exercise 6:

Lemma 6. *f has increasing differences in (x,θ) if and only if f has increasing differences in $(x_i,\theta_j;x_{-i},\theta_{-j})$ for all $i \in \{1,\dots,d\}$ and all $j \in \{1,\dots,m\}$.*

Proof. Suppose f has increasing differences in (x, θ). In particular, suppose $x_i' \geq x_i$, $\theta_j' \geq \theta_j$, x' is x with x_i replaced by x_i', and θ' is θ with θ_j replaced by θ_j'. Then $x' \geq x$ and $\theta' \geq \theta$, so

$$f(x', \theta') - f(x, \theta') \geq f(x', \theta) - f(x, \theta)$$

or equivalently

$$f(x_i', \theta_j'; x_{-i}, \theta_{-j}) - f(x_i, \theta_j'; x_{-i}, \theta_{-j}) \geq f(x_i', \theta_j; x_{-i}, \theta_{-j}) - f(x_i, \theta_j; x_{-i}, \theta_{-j})$$

Conversely, suppose f has increasing differences in $(x_i, \theta_j; x_{-i}, \theta_{-j})$ for all $i \in \{1, \ldots, d\}$ and all $j \in \{1, \ldots, m\}$. Suppose $x' \geq x$ and $\theta' \geq \theta$. Then $x_i' \geq x_i$ for all i and $\theta_j' \geq \theta_j$ for all j. Let $i \in \{1, \ldots, d\}$ and $x^i := (x_1, \ldots, x_{i-1}, x_i', \ldots, x_d)$. Then

$$\begin{aligned}
&f(x^i, \theta') - f(x^{i+1}, \theta') \\
&\geq f(x^i, \theta_1, \theta_2', \ldots, \theta_m') - f(x^{i+1}, \theta_1, \theta_2', \ldots, \theta_m') \\
&\geq f(x^i, \theta_1, \theta_2, \theta_3', \ldots, \theta_m') - f(x^{i+1}, \theta_1, \theta_2, \theta_3', \ldots, \theta_m') \\
&\geq \cdots \\
&\geq f(x^i, \theta) - f(x^{i+1}, \theta)
\end{aligned}$$

Each step j follows from increasing differences in $(x_i, \theta_j; x_{-i}, \theta_{-j})$. We can rewrite:

$$f(x^i, \theta') - f(x^i, \theta) \geq f(x^{i+1}, \theta') - f(x^{i+1}, \theta)$$

Applying this iteratively to $i = 1, 2, \ldots, d$, we have

$$\begin{aligned}
f(x', \theta') - f(x', \theta) &\geq f(x^2, \theta') - f(x^2, \theta) \\
&\geq f(x^3, \theta') - f(x^3, \theta) \\
&\geq \cdots \\
&\geq f(x, \theta') - f(x, \theta)
\end{aligned}$$

Therefore, f has increasing differences in (x, θ). □

Exercise 6. By the lemma above, f has increasing differences in (x, θ) if and only if, for all distinct i, j and all $\varepsilon, \delta > 0$,

$$f(x_i + \varepsilon, \theta_j + \delta; x_{-i}, \theta_{-j}) - f(x_i + \varepsilon, \theta_j; x_{-i}, \theta_{-j}) \geq f(x_i, \theta_j + \delta; x_{-i}, \theta_{-j}) - f(x_i, \theta_j; x_{-i}, \theta_{-j}) \tag{3}$$

Dividing both sides by δ and taking limits as $\delta \searrow 0$, we have that

$$\frac{\partial f}{\partial \theta_j}(x_i + \varepsilon, \theta_j; x_{-i}, \theta_{-j}) \geq \frac{\partial f}{\partial \theta_j}(x_i, \theta_j; x_{-i}, \theta_{-j}) \tag{4}$$

Rewrite (4) as

$$\frac{\partial f}{\partial \theta_j}(x_i + \varepsilon, \theta_j; x_{-i}, \theta_{-j}) - \frac{\partial f}{\partial \theta_j}(x_i, \theta_j; x_{-i}, \theta_{-j}) \geq 0 \tag{5}$$

Dividing both sides of (5) by ε and taking limits as $\varepsilon \searrow 0$, we get

$$\frac{\partial^2 f}{\partial x_i \partial \theta_j}(x, \theta) \geq 0 \tag{6}$$

Conversely, (6) implies that

$$\frac{\partial f}{\partial \theta_j}(x, \theta)$$

is increasing in x_i, which implies (5). Then (5) implies

$$f(x_i + \varepsilon, \theta_j; x_{-i}, \theta_{-j}) - f(x_i, \theta_j; x_{-i}, \theta_{-j})$$

is increasing in θ_j, implying

$$f(x_i + \varepsilon, \theta_j + \delta; x_{-i}, \theta_{-j}) - f(x_i, \theta_j + \delta; x_{-i}, \theta_{-j}) \geq f(x_i + \varepsilon, \theta_j; x_{-i}, \theta_{-j}) - f(x_i, \theta_j; x_{-i}, \theta_{-j})$$

which is just a rearrangement of (3).

Exercise 7. Suppose $f : X \times \Theta \to \mathbb{R}$ has single-crossing differences so that, for any $\theta' > \theta$,

$$\begin{aligned} f\left(x'', \theta\right) \geq f\left(x', \theta\right) &\implies f\left(x'', \theta'\right) \geq f\left(x', \theta'\right) \\ f\left(x'', \theta\right) > f\left(x', \theta\right) &\implies f\left(x'', \theta'\right) > f\left(x', \theta'\right). \end{aligned}$$

Because φ is strictly increasing in $f(x, \theta)$, each inequality continues to hold when we take $\varphi(\cdot, \theta)$ of both sides. Therefore,

$$\begin{aligned} \varphi(f\left(x'', \theta\right), \theta) \geq \varphi(f\left(x', \theta\right), \theta) &\implies \varphi(f\left(x'', \theta'\right), \theta') \geq \varphi(f\left(x', \theta'\right), \theta') \\ \varphi(f\left(x'', \theta\right), \theta) > \varphi(f\left(x', \theta\right), \theta) &\implies \varphi(f\left(x'', \theta'\right), \theta') > \varphi(f\left(x', \theta'\right), \theta') \end{aligned}$$

as required.

Additional Exercises

Exercise 2. We assume $f > 0$ (this is necessary for $\log f$ to exist). If f is log-supermodular, then $\log f$ is supermodular, so

$$\log f(z) + \log f(z') \leq \log f(z \vee z') + \log f(z \wedge z')$$

or equivalently

$$f(z)f(z') \leq f(z \vee z')f(z \wedge z')$$

Suppose z, z' satisfy $f(z) \geq f(z \wedge z')$. Then

$$f(z \vee z')f(z \wedge z') \geq f(z)f(z') \geq f(z \wedge z')f(z')$$

from which it follows that $f(z \vee z') \geq f(z')$. Similarly, if $f(z) > f(z \wedge z')$, then the same argument yields $f(z \vee z') > f(z')$. Therefore, f is quasi-supermodular.

Exercise 3. Write $\pi(y, p, -q) := pf(y) - q \cdot y$. Then π has increasing differences in y, $(p, -q)$: given $y' \geq y$, $p' \geq p$, and $q' \leq q$,

$$p'f(y') - q' \cdot y - p'f(y) + q' \cdot y = p'[f(y') - f(y)] \geq p[f(y') - f(y)] = pf(y') - q \cdot y - pf(y) + q \cdot y$$

And π is supermodular in y for each $(p, -q)$:

$$pf(y) - q \cdot y + pf(y') - q \cdot y' = p[f(y) + f(y')] - q \cdot (y + y')$$
$$= p[f(y) + f(y')] - q \cdot (y \vee y' + y \wedge y') \leq p[f(y \vee y') + f(y \wedge y')] - q \cdot (y \vee y' + y \wedge y')$$
$$= pf(y \vee y') - q \cdot (y \vee y') + pf(y \wedge y') - q \cdot (y \wedge y')$$

Supermodularity implies quasi-supermodularity and increasing differences implies single-crossing differences, so we can apply the Theorem of Milgrom and Shannon to obtain that $X^*(p, -q) := \arg\max_y \pi(y, p, -q)$ is nondecreasing in the strong set order.

Exercise 4. Let $(x', p') \geq (x, p)$. Then

$$p'x' - c(x') - px' + c(x') = (p' - p)x' \geq (p' - p)x = p'x - c(x) - px + c(x)$$

so $px - c(x)$ has increasing differences in (x, p). It is also supermodular in x for any p: assume WLOG that $x' \geq x$

$$px - c(x) + px' - c(x') = p(x \vee x') - c(x \vee x') + p(x \wedge x') - c(x \wedge x')$$

Therefore, we can again apply the Theorem of Milgrom and Shannon.

Exercise 5. By the Theorem of Milgrom and Shannon,

$$Z^{**}(\theta) := \arg\max\{F(x, y, \theta) \mid x \in \mathbb{R}^{d_1}_{++} \text{ and } y \in \mathbb{R}^{d_2}_{++}\}$$

is nondecreasing in the strong set order. Then by Proposition 6, given $\theta'' > \theta'$,

$$\sup Z^{**}(\theta'') \geq \sup Z^{**}(\theta')$$

$Z^{**}(\theta'')$ is a nonempty and compact sublattice of $\mathbb{R}^{d_1}_{++} \times \mathbb{R}^{d_2}_{++}$, so by Proposition 1, it is a subcomplete sublattice. By Corollary 1, it contains its supremum, so we can define

$$(x^{**}, y^{**}) := \max Z^{**}(\theta'')$$

Then

$$(x^{**}, y^{**}) \geq \sup Z^{**}(\theta') \geq (x', y')$$

Note that $F(\cdot, \theta)$ being supermodular implies that $F(\cdot, y, \theta)$ is supermodular for any $y \in \mathbb{R}^{d_2}_{++}$. Moreover, Lemma 6 above implies that F having increasing differences in $((x, y), \theta)$ implies f has increasing differences in $(x, \theta; y)$. Furthermore, Lemma 6 above in combination with Lemma 1 from the lecture notes implies that if $F(\cdot, \theta)$ is supermodular then F has increasing differences in $(x, y; \theta)$. Applying Lemma 6 a final time, we obtain that F has increasing differences in $(x, (y, \theta))$. The Theorem of Milgrom and Shannon then implies that

$$Z^*(y, \theta) := \arg\max\{F(x, y, \theta) \mid x \in \mathbb{R}^{d_1}_{++}\}$$

is nondecreasing in the strong set order. Therefore, $\theta'' > \theta'$ implies

$$x^* := \max Z^*(y', \theta'') \geq \sup Z^*(y', \theta') \geq x'$$

Because $y^{**} \geq y'$, we also have that

$$x^{**} = \sup Z^*(y^{**}, \theta'') \geq \max Z^*(y', \theta'') = x^*$$