Midterm 2

ECON 6170

October 28, 2021

Instructions: You have the full class time to complete the following problems. You are to work alone. This test is not open book. In your answers, you are free to cite results that you can recall from class or previous homeworks *unless explicitly stated otherwise*. The exam is out of 20 points, and there is one extra credit question. The highest possible score is 22/20.

1. (5pts) Consider the correspondence $\phi : \mathbb{R}_+ \to \mathbb{R}^2_+$, where $\phi(p_a) = \{(x_a, x_b) \in \mathbb{R}^2_+ | p_a x_a + x_b \leq 1\}$. This is the budget correspondence of a consumer with wealth 1 in an economy with two goods, a and b, with per-unit prices of p_a and 1, respectively. Call $\phi(p_a)$ the set of affordable consumption bundles when the price of a is p_a.

   (a) *Warm-up*: Draw a graph of the set of affordable bundles when $p_a = 2$. Separately draw a graph of the set of affordable bundles when $p_a = 0$.

   (b) Prove or disprove: ϕ is upper-hemicontinuous at $p_a = 0$.

   (c) Is it **true** or **false** that ϕ is lower-hemicontinuous at $p_a = 0$? Justify your answer **informally but completely** in words and/or pictures. No need for a formal proof.

2. (5pts) Let $f : \mathbb{R} \to \mathbb{R}$ be smooth. Let $x_0 \in \mathbb{R}$. Prove that

$$f''(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) + f(x_0 - h) - 2f(x_0)}{h^2}.$$

Hint: Take the second order Taylor approximation of f centered at x_0 (so with a third order remainder term) and evaluate it first at $x_0 - h$ and then at $x_0 + h$. Sum the two equations, rearrange, and take limits.

3. (5pts) Let $L : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. Using the definition of a derivative, prove that L is differentiable everywhere.

4. (5pts) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = \frac{x}{y+1}$.

   Is f twice differentiable at $(x, y) = (0, 0)$? If yes, compute f'' at $(0, 0)$. If no, prove it.

   Hint 1: First check if f is differentiable at $(x, y) = (0, 0)$.

   Hint 2: If it exists, the derivative of f evaluated at a point (x, y) is some 1×2 matrix. But viewed as a function, it maps an arbitrary (x, y) to a point in $\mathbb{R}^2$. The second derivative of f, if it exists, is simply the first derivative of that function.

5. (Extra Credit: 2 pts) Suppose $f : \mathbb{R} \to \mathbb{R}$ is convex and twice differentiable on $\mathbb{R}$. Prove that $f''(x) \geq 0$ for all $x \in \mathbb{R}$.

   Hint 1: Feel free to use the result stated in Problem 2.

   Hint 2: f convex in particular implies that $f(0.5x + 0.5y) \leq 0.5f(x) + 0.5f(y)$.