Problem Set 8

Due: TA Discussion, 25 October 2023.

1 Exercises from class notes

All from "5. Differentiation.pdf".

Exercise 8. Prove the following: Suppose $f : X \subseteq \mathbb{R}^d \to \mathbb{R}^m$ is differentiable at $\mathbf{x}_0 \in \text{int}(X)$. Then, $\frac{\partial f_i}{\partial x_j}(\mathbf{x}_0)$ exists for any $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, d\}$ and

$$Df(\mathbf{x}_0) = \left[\frac{\partial f_i}{\partial x_j}(\mathbf{x}_0)\right]_{ij} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}_0) & \cdots & \frac{\partial f_1}{\partial x_d}(\mathbf{x}_0) \\ \cdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}_0) & \cdots & \frac{\partial f_m}{\partial x_d}(\mathbf{x}_0) \end{bmatrix}_{m \times d}.$$

Exercise 9. Let $f(x, y) = \frac{xy}{x^2+y^2}$, if $(x, y) \neq (0, 0)$, and let $f(0, 0) = 0$. Show that the partial derivatives of f exist at $(0, 0)$, but that f is not differentiable at $(0, 0)$.

Exercise 10. Let f be a differentiable function from $(a, b) \subset \mathbb{R}$ into an open subset $Y \subset \mathbb{R}^d$. Let $g : Y \to \mathbb{R}$ be differentiable at $f(x_0)$ for $x_0 \in (a, b)$. Express $D(g \circ f)$ in terms of the partial derivatives of f and g.

Remark 1. Exercise 10 is a very common use of the chain rule and is typically what people mean when they say *totally differentiate {function} by {parameter}*. For example, t may be price, γ gives the optimal m-good consumption bundle for the agent with ten dollars at any price, and f gives the utility of the agent for a given consumption bundle. In this case g is the agent's value function or indirect utility function. It represents the agent's utility, assuming he makes optimal purchases, as a function of price. $g'(t)$ is how the agent's utility changes with price.

Exercise 11. Prove Young's Theorem for the case when $d = 2$. **Hint**: Consider a rectangle formed with vertices at $\mathbf{x}_0$, $(x_{0,1} + h_1, x_{0,2})$, $(x_{0,1}, x_{0,2} + h_2)$, $(x_{0,1} + h_1, x_{0,2} + h_2)$. Let

$$\begin{aligned} r(\mathbf{h}) &:= f(x_{0,1} + h_1, x_{0,2} + h_2) - f(x_{0,1} + h_1, x_{0,2}), \\ t(\mathbf{h}) &:= f(x_{0,1} + h_1, x_{0,2} + h_2) - f(x_{0,1}, x_{0,2} + h_2) \end{aligned}$$

so that $r(\cdot)$ is the difference in f along the "right edge" of the rectangle and $t(\cdot)$ is the difference in f along the "top edge" of the rectangle. Let

$$d(\mathbf{h}) := f(x_{0,1}+h_1, x_{0,2}+h_2) - f(x_{0,1}+h_1, x_{0,2}) - [f(x_{0,1}, x_{0,2}+h_2) - f(\mathbf{x}_0)],$$

which is the difference in f along the right edge minus the difference along the left edge. Note that

$$d(\mathbf{h}) = r(h_1, h_2) - r(0, h_1) = t(h_1, h_2) - t(h_1, 0).$$

To proceed, apply the mean value theorem, re-express everything in terms of partials of f rather than partials of r and t, and then apply mean value theorem again. Divide both sides by $h_1 h_2$ to get almost what you want. Now take the limit of h_1 and h_2 to 0 and use continuity of the cross partials at $\mathbf{x}_0$ to conclude the result.

Exercise 14. Let $f : X \subseteq \mathbb{R}^d \to \mathbb{R}$, where X is nonempty, open and convex. For any $\mathbf{x}, \mathbf{v} \in \mathbb{R}^d$, let $S_{\mathbf{x},\mathbf{v}} := \{t \in \mathbb{R} : \mathbf{x} + t\mathbf{v} \in X\}$ and define $g_{\mathbf{x},\mathbf{v}} : S_{\mathbf{x},\mathbf{v}} \to \mathbb{R}$ as $g_{\mathbf{x},\mathbf{v}}(t) := f(\mathbf{x} + t\mathbf{v})$. Then, f is (resp. strictly) concave on X if and only if $g_{\mathbf{x},\mathbf{v}}(\cdot)$ is (resp. strictly) concave for all $\mathbf{x}, \mathbf{v} \in \mathbb{R}^d$ with $\mathbf{v} \neq \mathbf{0}$.

Exercise 17. Let $f : \mathbb{R}^2_{++} \to \mathbb{R}$ be defined by $f(x,y) := x^\alpha y^\beta$ for some $\alpha, \beta > 0$. Compute the Hessian of f at $(x,y) \in \mathbb{R}^2_{++}$. Find conditions on α and β such that f is (i) strictly concave, (ii) f is concave but not strictly concave, (iii) f is neither concave nor convex. How do your answers change if the domain of f was $\mathbb{R}^2_+$?

2 Additional Exercises

Definition 1. A function $f : \mathbb{R}^d \to \mathbb{R}$ is *homogenous of degree* k if

$$f(\lambda \mathbf{x}) \equiv \lambda^k f(\mathbf{x}) \ \forall \lambda > 0.$$

All linear functions are homogenous of degree 1 but homogeneity of degree one is weaker than linearity; e.g., $f(x,y) = \sqrt{xy}$.

If f is homogenous of degree 1, then f has constant returns to scale. If f has homogeneity of degree $k > 1$ (resp. $k < 1$), then f has increasing (resp. decreasing) returns to scale.

Theorem 1 (Euler's Theorem). *If $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{x} \in \text{int}(X)$ and homogenous of degree k, then*

$$\nabla f(\mathbf{x})\, \mathbf{x} = k f(\mathbf{x}).$$

Exercise 1. Prove Euler's Theorem.