Econ 6190 Second Exam

7:30-9:00 pm, November 7

Instructions

This exam consists of two questions, not of equal length or difficulty. Answer all questions. Remember to always explain your answer. Good luck!

1. **[60 pts]** Let $X \sim F$ be a random variable that follows an unknown distribution F with finite second moment. A random sample $\{X_i\}_{i=1}^n$ of size n is drawn from F. Let $Var(X) = \sigma^2$. Consider the following estimator for $\mu = \mathbb{E}[X]$:

$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} w_i X_i + b,$$

where $\{w_i\}_{i=1}^n$ and b are some constants that can potentially depend on n.

(a) **[5 pts]** What do we mean by saying an estimator is unbiased for μ? Under what conditions is $\hat{\theta}$ unbiased?

(b) **[5 pts]** Calculate the bias and variance of $\hat{\theta}$.

(c) **[10 pts]** Suppose you care about the mean square error (MSE) of an estimator. Would you ever prefer an estimator $\hat{\theta}$ for which $w_1 = w_2 \ldots = w_n > 1$? Why or why not? (hint: can you find an alternative estimator that has a smaller MSE irrespective of the values of μ and σ^2?)

(d) **[10 pts]** Suppose you care about the mean square error (MSE) of an estimator. Would you ever prefer an estimator $\hat{\theta}$ for which $w_1 = w_2 \ldots = w_n < 0$? Why or why not? (hint: can you find an alternative estimator that has a smaller MSE irrespective of the values of μ and σ^2?)

(e) **[10 pts]** Now, let $w_1 = w_2 \ldots = w_n = 1 - \frac{1}{n}$, $b = \frac{1}{n}$. Suppose $\sigma^2 = 1$. For a fixed n, find the values of the μ for which $\hat{\theta}$ is more efficient than $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$.

(f) **[10 pts]** Show that a sufficient condition for $\hat{\theta} \xrightarrow{p} \mu$ is

$$b = o(1), \frac{\sum_{i=1}^n w_i}{n} - 1 = o(1), \frac{1}{n^2}\sum_{i=1}^{n} w_i^2 = o(1).$$

(g) **[10 pts]** Is the estimator proposed in (e) consistent? If yes, prove it and find its asymptotic distribution after suitable normalization; if not, explain why.

2. **[40 pts]** Let $\{X_1 \dots X_n\}$ be a sequence of i.i.d random variables with mean μ and and variance σ^2. Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

   (a) **[10 pts]** If $\mu \neq 0$, how you would approximate the distribution of $(\bar{X}_n)^2$ (after suitable normalization) in large samples as $n \to \infty$?

   (b) **[10 pts]** If $\mu \neq 0$, derive the stochastic order of magnitude for $(\bar{X}_n)^2$.

   (c) **[10 pts]** If $\mu = 0$, how would you approximate the distribution of $(\bar{X}_n)^2$ (after suitable normalization) in large samples as $n \to \infty$?

   (d) **[10 pts]** If $\mu = 0$, derive the stochastic order of magnitude for $(\bar{X}_n)^2$.