Problem Set 5

Due: TA Discussion, 27 September 2024.

1 Exercises from class notes

All from “3. Convexity.pdf”.

Exercise 3. Let S be a subset of $\mathbb{R}^d$. Prove that $\text{co}(S)$ is the collection of all finite convex combinations of elements in S.

Exercise 7. Prove that the closed convex hull of a set S is the closure of the convex hull of S; i.e., $\overline{\text{co}}(S) = \text{cl}(\text{co}(S))$.

Exercise 10. Prove that a function is concave (resp. convex) if and only if its subgraph (resp. epigraph) is convex.

Remark 1. That subgraph is convex implies that the upper contour sets, $\{\mathbf{x} \in S : f(\mathbf{x}) \geq y\}$, are convex for any $y \in \mathbb{R}$ (check!). Think back to your intermediate-micro class, when you were maximising a concave utility function over cheese and apples with respect to a budget constraint. Recall the shape of the indifference curves. When the set of bundles giving utility greater than some $\bar{u} \in \mathbb{R}$ is strictly concave, the constrained optimal bundle is the *unique* point where the budget curve is tangent to some indifference curve. Once you know this, you can easily do things like plot out the curve of optimal bundles as your budget grows or shrinks, you can show that the marginal rate of substitution is decreasing (so when I have tons of cheese and no apples, I'm willing to trade away a lot of cheese to get even one bite of apple), etc. The previous problem tells you that concavity of u is a sufficient assumption under which the indifference curves (which are the boundary points of convex sets) take this nice shape.

Exercise 11. Prove that an affine function is both convex and concave.

Exercise 12. Prove the following: Let $X \subseteq \mathbb{R}^d$ be convex and let $f : X \to \mathbb{R}$. Then, f is quasiconcave (resp. quasi-convex) if and only if the upper (resp. lower) contour sets are convex; i.e., $\{\mathbf{x} \in X : f(\mathbf{x}) \geq y\}$ (resp. $\{\mathbf{x} \in X : f(\mathbf{x}) \leq y\}$) are convex for any $y \in \mathbb{R}$.

Remark 2. We now realise that for the indifference curves to have the nice properties, an even weaker assumption on u give us the same result. The previous exercise shows you that it is necessary and sufficient to assume u is quasi-concave. Your instinct should always be to make as few assumptions on preferences as necessary to make the point that you want to make, *especially* if data on choices can't validate those assumptions.

Exercise 13 TFU: If f is a quasi-concave function and $h : \mathbb{R} \to \mathbb{R}$ is nondecreasing function then $h \circ f$ is quasi-concave. Do the same replacing quasi-concave with concave (in both places).

2 Additional Exercises

Exercise 1. Let $X \subseteq \mathbb{R}^d$ be convex. Prove or give a counter-example:

(i) If $f, g : X \to \mathbb{R}$ are convex on X, then $f + g$ is convex on X.

(ii) If $f, g : X \to \mathbb{R}$ are quasi-convex on X, then $f + g$ is quasi-convex on X.

(iii) A concave function $f : X \to \mathbb{R}$ is quasi-concave. **Hint:** If you're not using earlier exercises, you're working too hard.

(iv) A strictly concave function $f : X \to \mathbb{R}$ is strictly quasi-concave.