1. Model

(a) jump : C_t, I_t, A_t ; endog states; K_t ; exog states: $\varnothing$

(b) $$\max_{\{C_t, I_t, K_{t+1}\}} E \sum_{t=0}^{\infty} \beta^t \left\{ \log(C_t) + \lambda_{1,t} \left(C_t - A_t K_t^{\alpha} + I_t \right) + \lambda_{2,t} \left(K_{t+1} - (1-\delta) K_t - I_t \right) \right\}$$

FOC

C_t) $\frac{1}{C_t} = \lambda_{1,t}$

+ constraints & LOM for A

I_t) $\lambda_{1,t} = \lambda_{2,t}$

K_{t+1}) $\lambda_{2,t} = \beta E_t \left[\lambda_{2,t+1} \left(\alpha A_{t+1} K_{t+1}^{\alpha-1} + 1 - \delta \right) \right]$

(c) $$\max_{\{C_t, I_t, A_t, K_{t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \log(C_t) + \theta_{1,t} \left(C_t + I_t - A_t K_t^{\alpha} \right) + \theta_{2,t} \left(K_{t+1} - (1-\delta) K_t - I_t \right) + \theta_{3,t} \left(A_t - \left(K_t \right)^{a_0} \right) \right\}$$

FOC

C_t) $\frac{1}{C_t} = \theta_{1,t}$

I_t) $\theta_{1,t} = \theta_{2,t}$

A_t) $\theta_{1,t} \left(-K_t^{\alpha} \right) + \theta_{3,t} = 0$ (1)

K_{t+1}) $\lambda_{2,t} = \beta E_t \left[\lambda_{2,t+1} \left(\alpha A_{t+1} K_{t+1}^{\alpha-1} + 1 - \delta \right) + \theta_{3,t+1} \left(a_0 K_{t+1}^{a_0 - 1} \right) \right]$ (2)

(d) $V(K_t) = \max_{K_{t+1}} \Big\{ \log\big(K_t^{a_0} K_t^{\alpha} + (1-\delta) K_t - K_{t+1} \big)$

$+ \beta E\, V(K_{t+1}) \Big\}$

FOC

K_{t+1} $\quad \frac{1}{C_t} = \beta E_t V'(K_{t+1})$

But evelope says

$$V'(\;) = \frac{1}{C_t}\left((a_0 + \alpha) K_{t+1}^{a_0+\alpha-1} + (1-\delta) \right)$$

Combine to get

$$1 = E_t\left[\frac{C_t}{C_{t+1}} \left((a_0+\alpha) K_{t+1}^{a_0+\alpha-1} + 1-\delta \right) \right]$$

Now use ① & ② from (C) . . .

$$\theta_{3t} = \frac{1}{C_t} K_t^{\alpha}$$

So ② implies

$$\frac{1}{C_t} = \beta E_t\left[\frac{1}{C_{t+1}} \alpha K_{t+1}^{a_0} K_{t+1}^{\alpha-1} + 1-\delta + \frac{1}{C_{t+1}} K_{t+1}^{\alpha} a_0 K_{t+1}^{a_0-1} \right]$$

(Combine terms)

$$= \beta E_t\left[\frac{1}{C_{t+1}} (\alpha + a_0) K_{t+1}^{a_0+\alpha-1} + 1-\delta \right]$$

✓ same

e) In S.P. economy, steady state implies

$$\frac{1}{\beta} = (\alpha + a_0) K^{\alpha + a_0 - 1} \Rightarrow K = \left(\frac{1}{\beta(\alpha + a_0)}\right)^{\frac{1}{\alpha + a_0 - 1}}$$

$$\Rightarrow K = \left(\beta(\alpha + a_0)\right)^{\frac{1}{1-\alpha-a_0}}$$

don't forget

In decentralined economy

$$\frac{1}{\beta} = \alpha K^{\alpha + \boxed{a_0} - 1} \Rightarrow K = \left(\beta(\alpha)\right)^{\frac{1}{1-\alpha-a_0}}$$

Since the exponent is positive, and $\beta\alpha < \beta(\alpha + a_0)$

it's clear the latter is smaller.

2. (a) $$1 = \beta E_t\left[\frac{C_t}{C_{t+1}}\left(\alpha K_{t+1}^{\alpha+a_0-1} + 1-\delta\right)\right] \quad ①$$

$$C_t = K_t^{\alpha+\alpha_0} + (1-\delta)K_t - K_{t+1} \quad ②$$

2 (b) we have $\frac{1}{\beta} = \alpha K^{\alpha+a_0-1} + 1-\delta$

$$\Rightarrow \left[\frac{\left(\frac{1}{\beta}-1+\delta\right)}{\alpha}\right]^{\frac{1}{\alpha+a_0-1}} = K_{ss}$$ frome euler equation

from RC.

$$C = K^{\alpha+a_0} + (1-\delta)K - K = K_{ss}^{\alpha+\alpha_0} - \delta K_{ss}$$

(c)

log linerize ①

$$\frac{1}{\beta} = E_t\left[\frac{\exp(c_t)}{\exp(c_{t+1})}\left(\alpha \exp((\alpha+a_0-1)k_t) + 1-\delta\right)\right]$$

let's call $f^1 = \frac{\exp(c_t)}{\exp(c_{t+1})}\left(\alpha \exp((\alpha+a_0-1)k_t) + 1-\delta\right)$

optional simplification

$$f^1_{c_t}\Big|_{ss} = \frac{\exp(c)}{\exp(c)}\left(\alpha \exp((\alpha+a-1)k) + 1-\delta\right) = \frac{1}{\beta}$$

$$f^1_{c_{t+1}}\Big|_{ss} = -\frac{\exp(c)}{\exp(c)^2}\left(\qquad \right)\exp(c) = \frac{1}{\beta}$$

$$f^1_{k_{t+1}}\Big|_{ss} = \frac{\exp(c)}{\exp(c)}\alpha \exp((\alpha+a_0-1)k)(\alpha+a_0-1)$$

so

$$f^1 \approx f^1_{ss} + E_t\left[\frac{1}{\beta}\left(\hat{c}_t - \hat{c}_{t+1}\right) + \alpha K^{\alpha+a_0-1}(\alpha+a_0-1)\hat{k}_{t+1}\right]$$

$$C_t - K_t^{\alpha+\alpha_0} - (1-\delta)K_t + K_{t+1} = 0 \leftarrow \text{call } f^2$$

$$\Rightarrow \exp(c_t) - \exp((\alpha+\alpha_0)k_t) - (1-\delta)\exp(k_t) + \exp(k_{t+1}) = 0$$

$$f^2_{c_t} = \exp(c)$$

$$f^2_{k_{t+1}} = \exp(k)$$

$$f^2_{k_t} = -\exp((\alpha+\alpha_0)k)(\alpha+\alpha_0) - (1-\delta)\exp(k)$$

So $f^2 \approx f^2_{ss} + C\,\hat{c}_t + K\,\hat{k}_{t+1} - \left(K^{\alpha+\alpha_0}(\alpha+\alpha_0) - (1-\delta)K\right)\hat{k}_t$

2(d) using my equation ordering

$$F_x = \begin{bmatrix} 0 \\ -\left(K^{\alpha+\alpha_0} (\alpha+\alpha_0) - (1-\delta)K \right) \end{bmatrix}$$

$$f_y = \begin{bmatrix} \frac{1}{B} \\ C \end{bmatrix}$$

$$f_{x'} = \begin{bmatrix} \alpha K^{\alpha+a_0-1} (\alpha+a_0-1) \\ K \end{bmatrix}$$

$$f_{y'} = \begin{bmatrix} -\frac{1}{\beta} \\ 0 \end{bmatrix}$$

Q4.

I would start by noting that the only aggregate state variable (the only way that agents are connected, effectively) is through equation (4). Second, I would note that we are only looking for a steady-state: once we know steady-state \bar K, we can solve the agent's individual problem without referring to other agent's choices. With those observations, I would

1. Conjecture a value for \bar K, and assume that individual capital stocks all start at \bar K

2. Solve for individual optimal investment decision as a function of K(i,t) taking as given \bar K. For this step, I could use a variety of approaches, but the one I would naturally use is to use the "hat" basis functions to conjecture a policy k(i,t+1) = h(k(i,t), eps(i,t)) and then use projection method to find the best approximate individual policies.

In step 2, I would approximate expectations by using the Gaussian-Hermite quadratiture to approximate the distribution of eps(i,t+1) faced by the agent.

3. Using this policy function, I would draw a large random sample of eps(i,t) and compute the optimal capital choice k(i,t) = h(k(i,t), eps(i,t)). (In the first iteration, k(i,t) = \bar K.) Using this large population, could update by guess of \bar K.

4. I would iterate on steps (1)-(3) updating both individual and aggregate capital at each step, until convergence.