ECON 6170 Section 8

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October 25, 2024

Exercise 8. Prove the following: Suppose $f : X \subseteq \mathbb{R}^d \to \mathbb{R}^m$ is differentiable at $x_0 \in \operatorname{int} X$. Then $\frac{\partial f_i}{\partial x_j}(x_0)$ exists for any $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, d\}$ and

$$Df(x_0) = \left[\frac{\partial f_i}{\partial x_j}(x_0)\right]_{ij} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_0) & \cdots & \frac{\partial f_1}{\partial x_d}(x_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x_0) & \cdots & \frac{\partial f_m}{\partial x_d}(x_0) \end{bmatrix}$$

Differentiability of $f : X \subseteq \mathbb{R}^d \to \mathbb{R}^m$ at x_0 means that

$$\frac{\|f(x_0 + \vec{h}) - f(x_0) - A\vec{h}\|_m}{\|\vec{h}\|_d} \to 0$$

as $\vec{h} \to 0$, for some $A \in \mathbb{R}^{m \times d}$. For $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, d\}$, we want to show that[1]

$$\lim_{h \to 0} \frac{f_i(x_0 + he_j) - f_i(x_0)}{h}$$

exists, and equals the (i, j)-th entry of A. To do so, it suffices to show that

$$\left|\frac{f_i(x_0 + he_j) - f_i(x_0)}{h} - a_{ij}\right|$$

is bounded above by by some function that converges to zero with h. Letting $A_{i\bullet}$ be the i-th row of A as a vector in $\mathbb{R}^d$, we can rewrite this as

$$\begin{aligned} \left|\frac{f_i(x_0 + he_j) - f_i(x_0) - A_{i\bullet}^\top he_j}{h}\right| &= \left|\frac{f_i(x_0 + he_j) - f_i(x_0) - A_{i\bullet}^\top he_j}{\|he_j\|_d}\right| \\ &\le \frac{1}{\|he_j\|_d}\sqrt{\sum_{i=1}^m \left[f_i(x_0 + he_j) - f_i(x_0) - A_{i\bullet}^\top he_j\right]^2} \\ &= \frac{\|f(x_0 + he_j) - f(x_0) - Ahe_j\|_m}{\|he_j\|_d} \qquad (*) \end{aligned}$$

But he_j is a sequence of d-vectors converging to zero with h, so $(*)$ converges to zero as $h \to 0$.

[1] Note that $\vec{h}$ above is a d-vector, whereas here it is a scalar.

Exercise 11. Prove Young's Theorem for the case when $d = 2$.

To simplify notation, I write $x := (x_1, x_2)$ (i.e., drop the 0 subscript). Define

$$r(h_1, h_2) := f(x_1 + h_1, x_2 + h_2) - f(x_1 + h_1, x_2)$$
$$t(h_1, h_2) := f(x_1 + h_1, x_2 + h_2) - f(x_1, x_2 + h_2)$$

Then

$$\begin{aligned} r(h_1, h_2) - r(0, h_2) &= f(x_1 + h_1, x_2 + h_2) - f(x_1 + h_1, x_2) - f(x_1, x_2 + h_2) + f(x_1, x_2) \\ &= t(h_1, h_2) - t(h_1, 0) \end{aligned}$$

By the mean-value theorem applied to $r(\cdot, h_2)$ and $t(h_1, \cdot)$

$$\frac{\partial r(c_1, h_2)}{\partial x_1} h_1 = \frac{\partial t(h_1, c_2)}{\partial x_2} h_2$$

for some $c_1 \in (0, h_1)$ and $c_2 \in (0, h_2)$. Rewriting in terms of f,

$$h_1 \left(\frac{\partial f(x_1 + c_1, x_2 + h_2)}{\partial x_1} - \frac{\partial f(x_1 + c_1, x_2)}{\partial x_1} \right) = h_2 \left(\frac{\partial f(x_1 + h_1, x_2 + c_2)}{\partial x_2} - \frac{\partial f(x_1, x_2 + c_2)}{\partial x_2} \right)$$

Applying the mean value theorem to $\frac{\partial f(x_1 + c_1, \cdot)}{\partial x_1}$ and $\frac{\partial f(\cdot, x_2 + c_2)}{\partial x_2}$,

$$h_1 h_2 \frac{\partial^2 f(x_1 + c_1, \gamma_2)}{\partial x_2 \partial x_1} = h_2 h_1 \frac{\partial^2 f(\gamma_1, x_2 + c_2)}{\partial x_1 \partial x_2}$$

for some $\gamma_1 \in (x_1, x_1 + h_1)$, $\gamma_2 \in (x_2, x_2 + h_2)$. We can divide both sides by $h_1 h_2$ to get

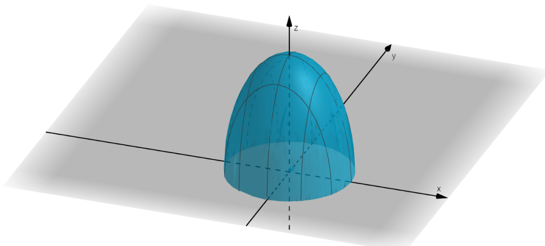
$$\frac{\partial^2 f(x_1 + c_1, \gamma_2)}{\partial x_2 \partial x_1} = \frac{\partial^2 f(\gamma_1, x_2 + c_2)}{\partial x_1 \partial x_2}$$

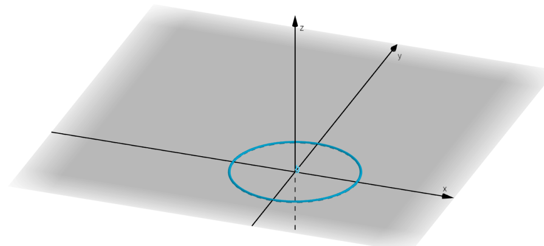
Note that as $h_1 \to 0$, $c_1 \to 0$ and $\gamma_1 \to x_1$; and as $h_2 \to 0$, $c_2 \to 0$ and $\gamma_2 \to x_2$. Taking the limit of both sides as $h_1, h_2 \to 0$ and using that $f \in C^2$,

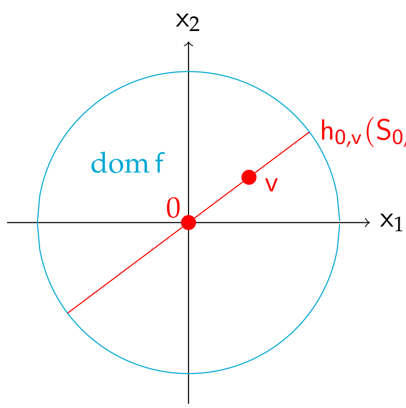
$$\frac{\partial^2 f(x_1, x_2)}{\partial x_2 \partial x_1} = \frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2}$$

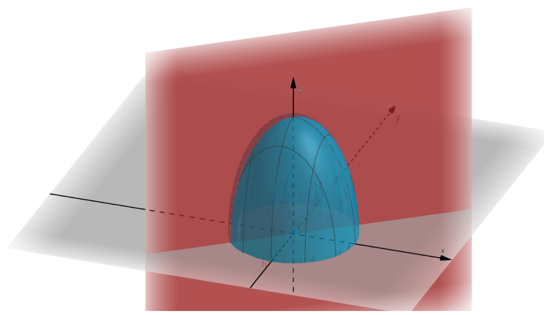
Exercise 12. Let $f : X \subseteq \mathbb{R}^d \to \mathbb{R}$, where X is nonempty, open and convex. For any $x, v \in \mathbb{R}^d$, let $S_{x,v} := \{t \in \mathbb{R} \mid x + tv \in X\}$ and define $g_{x,v} : S_{x,v} \to \mathbb{R}$ as $g_{x,v}(t) := f(x + tv)$. Then, f is concave (resp. strictly concave) on X if and only if $g_{x,v}$ is concave (resp. strictly concave) for all $x, v \in \mathbb{R}^d$ with $v \neq 0$. Prove this.

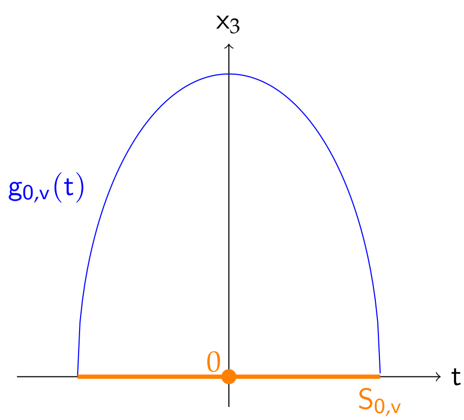
The following diagrams illustrate this question visually, for the function $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$, given by $f(x_1, x_2) := 2\sqrt{1 - x_1^2 - x_2^2}$, where $X := \{(x_1, x_2) \mid \|(x_1, x_2)\| \leq 1\}$ is the closed unit ball in $\mathbb{R}^2$. The mapping $h : S_{x,v} \to X$ is given by $h_{x,v}(t) := x + tv$











We can see that $g_{0,v}$ inherits the concavity of f.

Suppose first that f is concave on X. Fix any $x, v \in \mathbb{R}^d$ with $v \neq 0$. For any $t, t' \in S_{x,v}$ and any $\alpha \in [0,1]$,

$$\begin{aligned} g_{x,v}\left(\alpha t + (1-\alpha)\, t'\right) &= f\left(x + \left(\alpha t + (1-\alpha)\, t'\right) v\right) \\ &= f\left(\alpha\left(x + tv\right) + (1-\alpha)\left(x + t'v\right)\right) \\ &\geq \alpha f\left(x + tv\right) + (1-\alpha)\, f\left(x + t'v\right) \\ &= \alpha g_{x,v}\left(t\right) + (1-\alpha)\, g_{x,v}\left(t'\right) \end{aligned}$$

Hence, $g_{x,v}(\cdot)$ is concave. Conversely, suppose that for any $x, v \in \mathbb{R}^d$ with $v \neq 0$, $g_{x,v}(\cdot)$ is concave. Pick any $z_1, z_2 \in X$ and any $\alpha \in [0,1]$. Letting $x = z_1$ and $v = z_2 - z_1$, observe that $g_{x,v}(0) = f(z_1)$,

$g_{x,v}(1) = f(z_2)$, and

$$g_{x,v}(\alpha) = f(z_1 + \alpha(z_2 - z_1)) = f((1-\alpha)z_1 + \alpha z_2)$$

Since $g_{x,v}(\cdot)$ is concave, for any $\alpha \in (0,1)$,

$$\begin{aligned} f((1-\alpha)z_1 + \alpha z_2) &= g_{x,v}(\alpha) \\ &= g_{x,v}((1-\alpha)\cdot 0 + \alpha \cdot 1) \\ &\geq (1-\alpha)g_{x,v}(0) + \alpha g_{x,v}(1) \\ &= (1-\alpha)f(z_1) + \alpha f(z_2) \end{aligned}$$

i.e., f is concave. The proof case for strict concavity is analogous.