ECON 6170 Module 2 Additional Answers

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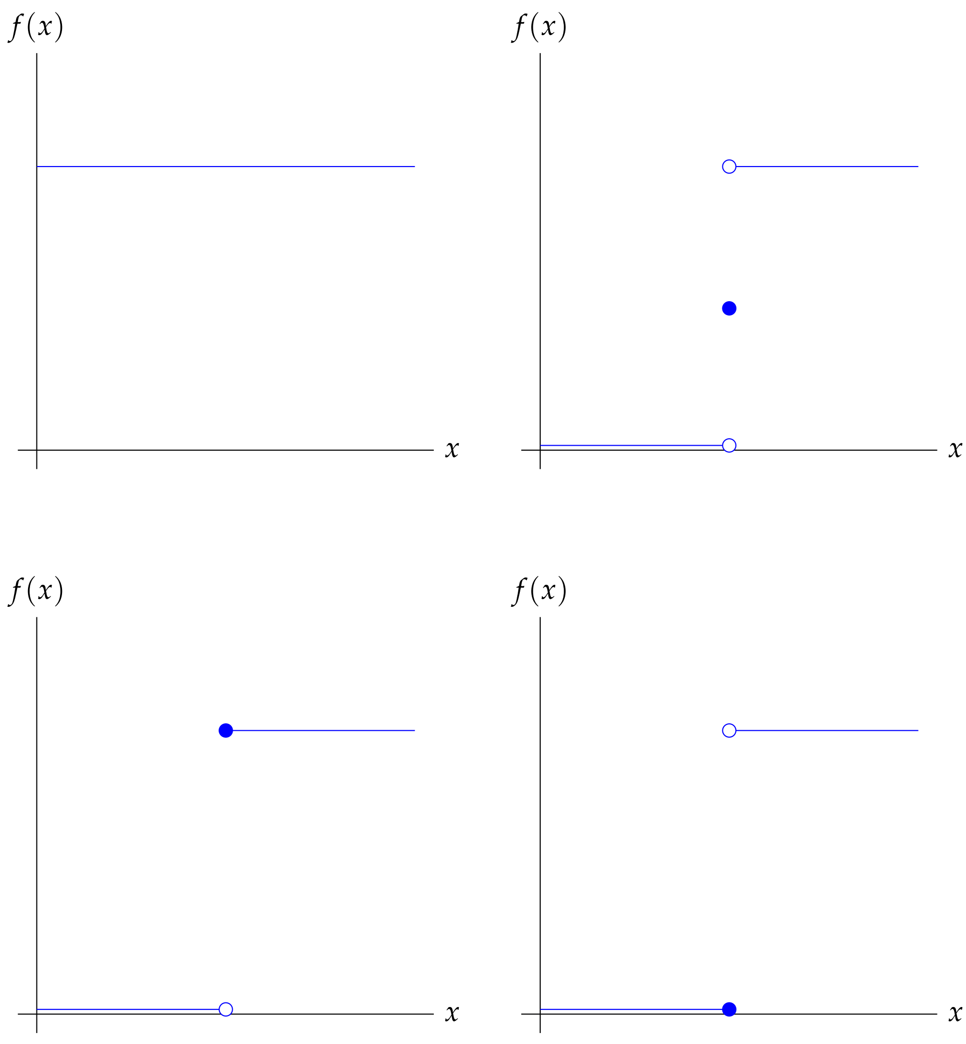
Exercise 11.



Clockwise from top left: $f(x) := 1$ is continuous; $f(x) := 1/2 \cdot \mathbf{1}\{x = 1/2\} + \mathbf{1}\{x > 1/2\}$ is neither upper nor lower continuous; $f(x) := \mathbf{1}\{x \geq 1/2\}$ is upper but not lower continuous; $f(x) := \mathbf{1}\{x > 1/2\}$ is lower but not upper continuous.

Exercise 12. We will show that the two definitions of upper semicontinuity coincide (the proof for lower semicontinuity is analogous).

Suppose f is upper semicontinuous at x_0 using neighbourhoods, and $x_n \to x_0$. BWOC, suppose $\limsup f(x_n) > f(x_0)$. Choose ε small enough that $\limsup f(x_n) - \varepsilon > f(x_0)$. By upper semicontinuity using neighbourhoods, there exists $\delta > 0$ such that $|x_n - x| < \delta$ implies $f(x_n) \leq \limsup f(x_n) - \varepsilon$. By convergence, for $n \geq N$, $|x_n - x| < \delta$. But then

$$\limsup f(x_n) \leq \sup\{x_n \mid n \geq N\} \leq \limsup f(x_n) - \varepsilon < \limsup f(x_n)$$

which is a contradiction.

Suppose f is upper semicontinuous at x_0 using sequences. Fix $\varepsilon > 0$. BWOC suppose f is not upper semicontinuous using neighbourhoods. Then there exists $\varepsilon > 0$ such that for all $\delta > 0$, there exists x having $|x - x_0| < \delta$ and $f(x) > f(x_0) + \varepsilon$. Choosing $\delta_n := 1/n$, this defines a sequence $x_n \to x$ such that $f(x_n) > f(x) + \varepsilon$ for all n. Clearly, $\limsup f(x_n) \geq f(x_0) + \varepsilon > f(x_0)$, contradicting upper semicontinuity using sequences.

Exercise 13. Suppose f is continuous at x_0. Then for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$. But this is equivalent to $-\varepsilon < f(x) - f(x_0) < \varepsilon$, which is equivalent to $f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon$. Thus, f is both upper and lower semicontinuous at x_0.

Suppose f is both upper and lower semicontinuous at x_0. Then for all $\varepsilon > 0$ there exists $\delta_1, \delta_2 > 0$ such that $|x - x_0| < \min\{\delta_1, \delta_2\}$ implies $f(x) \leq f(x_0) + \varepsilon$ and $f(x) \geq f(x_0) - \varepsilon$. Combining, we have $|f(x) - f(x_0)| < \varepsilon$. It follows that f is continuous at x_0.