ECON 6090

Problem Set 5

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9. The consumer solves the problem

$$\max_x u(w-x) + \mathbb{E}[v(x+y)]$$

where $y \sim F(\cdot)$. Denote the solution to this problem as $x^\star$ and the solution to the problem where y is degenerate with mean 0 as x_0.

(a) Recall that in the degenerate problem, since u and v are concave, we have that $v'(x_0) - u'(w-x_0) = 0$. If $\mathbb{E}[v'(x_0+y)] > v'(x_0)$, we have that $\mathbb{E}[v'(x_0+y)] - u'(w-x_0) > 0$, so x_0 is not a maximizer of the problem. It remains to show that the true maximizer is greater than x_0. At x_0, we have that $\mathbb{E}[v'(x_0+y)] > u'(w-x_0)$. At the true maximizer $x^\star$, we have that $\mathbb{E}[v'(x^\star+y)] = u'(w-x^\star)$. Conclusion follows by noting that u and v are concave, so u' and v' are decreasing in the argument. Thus, $x^\star > x_0$.

(b) We have that for v_1 and v_2, $-v_1'''(x)/v_1''(x) \leq -v_2'''(x)/v_2''(x)$ for all x, and that $\mathbb{E}[v_1'(x_0+y)] > v_1'(x_0)$. Note that the coefficient of absolute risk aversion of v_i' is equivalent to the coefficient of absolute prudence of v_i. Thus, from Proposition 6.C.2 in Mas-Colell, we have that since v_1' has a coefficient of absolute risk aversion that is not greater than v_2', v_2' has a greater certainty equivalent than v_1', meaning that $\mathbb{E}[v_2'(x_0+y)] > v_2'(x_0)$. In the context of part (a), this implies that if one individual decides to invest in a risky lottery, a second individual with a not-greater coefficient of absolute prudence will also invest, and they will not invest less.

(c) We have that $v'''(x) > 0$ for all x, then v' is convex, meaning that v' exhibits risk-loving behavior. Since $\mathbb{E}[y] = 0$, we have that $\mathbb{E}[v'(x+y)] > v'(x)$ for all x.

(d) We have that the coefficient of absolute risk aversion is decreasing in wealth, meaning that

$$\frac{\partial}{\partial w}\left[-\frac{v''(x)}{v'(x)}\right] < 0 \Longrightarrow -\frac{v'''(x)v'(x) - (v''(x))^2}{(v'(x))^2} = \frac{v''(x)}{v'(x)}\left(\frac{v''(x)}{v'(x)} - \frac{v'''(x)}{v''(x)}\right) < 0$$

Thus, we have that $-\frac{v''(x)}{v'(x)} < -\frac{v'''(x)}{v''(x)}$.

14. We have that $u^\star(\cdot)$ is strongly more risk-averse than $u(\cdot)$ if and only if there exists a positive constant k and a nonincreasing, concave function $v(\cdot)$ such that $u^\star(x) = ku(x) + v(x)$ for all x.

(a) We have that the coefficient of absolute risk aversion for $u^\star$ at some x is

$$r(x, u^\star) = -\frac{ku''(x) + v''(x)}{ku'(x) + v'(x)}$$

we want to show that

$$-\frac{ku''(x) + v''(x)}{ku'(x) + v'(x)} \geq -\frac{u''(x)}{u'(x)} \Longrightarrow u'(x)(ku''(x) + v''(x)) \leq u''(x)(ku'(x) + v'(x))$$

This simplifies to

$$ku'(x)u''(x) + u'(x)v''(x) \leq ku'(x)u''(x) + u''(x)v'(x) \Longrightarrow u'(x)v''(x) \leq u''(x)v'(x)$$

Which holds as long as

$$-\frac{v''(x)}{v'(x)} \geq -\frac{u''(x)}{u'(x)}$$

Since, by assumption, u is increasing and concave, and v is non-increasing and concave, the left side is non-negative and the right side is non-positive. Conclusion follows.

(b) Suppose FSOC that there exists $u^\star(x) = ku(x) + v(x)$, where v is non-constant, non-increasing, and concave. Define M such that $M = \inf\{C \in \mathbb{R} : u(x) \leq C \; \forall \; x\}$. Since u is increasing, as $x \to \infty$, $u(x) \to M$. However, since v is non-constant and non-increasing, $\exists \; x \in \mathbb{R}$ sufficiently large such that $u^\star(x) > u^\star(x + \varepsilon)$ for some $\varepsilon > 0$. This contradicts the assumption that $u^\star$ must be increasing.

(c) We have from (a) that strong risk aversion implies Arrow-Pratt risk aversion. It remains to show that the converse is not true. Consider the functions $u(x) = -\exp(-\alpha x)$ and $v(x) = -\exp(-\beta x)$, where $\beta > \alpha$. Both functions exhibit constant absolute risk aversion, so v is more risk-averse than u in the Arrow-Pratt sense. However, since they are each bounded above, by (b) v is not strongly more risk-averse than u.

15. We have a risk-averse decision maker, investing x_1 in a riskless asset and x_2 in a risky asset that pays a with probability π and b with probability $1 - \pi$. They begin with $w = 1$.

(a) Since the decision-maker is risk-averse, they will invest strictly positive levels in the riskless asset if there is a probability of loss with respect to the risky asset. Thus, the necessary condition is that at least one of a, b is strictly less than 1.

(b) Again, since the decision-maker is risk-averse, they will invest in the risky asset only if its expected value is greater than that of the riskless asset, *i.e.*when $\pi a + (1 - \pi)b > 1$.

(c) The decision-maker is maximizing the problem

$$\max_{x_1, x_2} \pi u(x_1 + ax_2) + (1 - \pi)u(x_1 + bx_2) \text{ s.t. } x_1, x_2 \in [0, 1], x_1 + x_2 = 1$$

The first condition falls away because we're assuming that the conditions from (a) and (b) hold, so the Lagrangian this admits is

$$\mathcal{L} = \pi u(x_1 + ax_2) + (1 - \pi)u(x_1 + bx_2) + \lambda(1 - x_1 - x_2)$$

The first order conditions are

$$\frac{\partial \mathcal{L}}{\partial x_1} = \pi u'(x_1 + ax_2) + (1 - \pi)u'(x_1 + bx_2) - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = a\pi u'(x_1 + ax_2) + b(1 - \pi)u'(x_1 + bx_2) - \lambda = 0$$

which, combining, get

$$\pi u'(x_1 + ax_2) + (1 - \pi)u'(x_1 + bx_2) = a\pi u'(x_1 + ax_2) + b(1 - \pi)u'(x_1 + bx_2)$$

which imply

$$\pi(1 - a)u'(x_1 + ax_2) + (1 - \pi)(1 - b)u'(x_1 + bx_2) = 0$$

The final first order condition is

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 1 - x_1 - x_2 = 0 \Longrightarrow x_1 + x_2 = 1$$

(d) Using the implicit function theorem, and holding b constant, define

$$g(x_1, a, \pi) = \pi(1-a)u'(x_1 + a(1-x_1)) + (1-\pi)(1-b)u'(x_1 + b(1-x_1))$$

We have that

$$\frac{\partial x_1}{\partial a} = -\frac{\frac{\partial g}{\partial a}}{\frac{\partial g}{\partial x_1}} = -\frac{-\pi u'(x_1 + a(1-x_1)) + \pi(1-a)(1-x_1)u''(x_1 + a(1-x_1))}{\pi(1-a)(1-a)u''(x_1 + a(1-x_1)) + (1-\pi)(1-b)(1-b)u''(x_1 + b(1-x_1))}$$

where all terms in the numerator and denominator are negative, so $\frac{\partial x_1}{\partial a} \leq 0$.

(e) If we are assuming, like in (d), that $a < 1$, it follows that $b > 1$. Thus, as π increases, the lottery gets worse, so the decision maker would invest more in the riskless asset. Thus, I conjecture that $\frac{\partial x_1}{\partial \pi} > 0$.

(f) From the first order conditions and the implicit function theorem, we have that

$$\frac{\partial x_1}{\partial \pi} = -\frac{\partial g/\partial \pi}{\partial g/\partial x_1}$$

We know that the denominator is negative, from part (d). It remains to show that the numerator is positive, and conclusion will follow. We have that

$$\frac{\partial g}{\partial \pi} = \underbrace{(1-a)u'(x_1 + a(1-x_1))}_{>0} - \underbrace{(1-b)u'(x_1 + b(1-x_1))}_{<0} > 0$$

16. An individual has Bernoulli utility function $u(\cdot)$ and initial wealth w. Let lottery L offer a payoff of G with probability p and a payoff of B with probability $(1-p)$.

(a) The individual would sell the lottery for no less than the amount that would guarantee the same expected utility – *i.e.*, a price y such that

$$pu(w+G) + (1-p)u(w+B) = u(w+y)$$

(b) They would purchase the lottery for an amount x such that they would have the same expected utility whether they had the lottery or not – *i.e.*, a price x such that

$$pu(w-x+G) + (1-p)u(w-x+B) = u(w)$$

(c) In general, $x \neq y$, as the different levels of wealth will change how much the lottery is 'worth' to the decision maker. However, if u exhibits constant absolute risk aversion, then they will coincide. If u exhibits CARA, then the above conditions imply that

$$w - c_w = (w-x) - c_{w-x}$$

where c_w is the certainty equivalent of the lottery with wealth w and c_{w-x} is the certainty equivalent of the lottery with wealth $w-x$.

(d) Directly calculating (using Wolfram), we get that y solves

$$p\sqrt{20} + (1-p)\sqrt{15} = \sqrt{10+y} \implies y = -5\left(4\sqrt{3}p^2 - 7p^2 - 4\sqrt{3}p + 6p - 1\right)$$

and x solves

$$p\sqrt{20-x} + (1-p)\sqrt{15-x} = \sqrt{10} \implies x = \frac{5\left(2p^3 + 7p^2 \pm 2\sqrt{2}\sqrt{-2p^5 + 7p^4 - 8p^3 + 3p^2} - 8p + 1\right)}{4p^2 - 4p + 1}$$

17. We have that an individual faces a two-period portfolio allocation problem, dividing her wealth between a risky asset with return x and a safe asset with return R. They have initial wealth w_0, and in period $t \in \{1, 2\}$ their wealth depends on the portfolio α_{t-1} chosen previously, defined by

$$w_t = ((1 - \alpha_{t-1})R + \alpha_{t-1}x_t)w_{t-1}$$

The individual is maximizing w_2, where we assume that x_1, x_2 are i.i.d.

Proof. First, assume that u has CRRA preferences. The wealth at the end of each period is

$$w_1 = ((1 - \alpha_0)R + \alpha_0 x_1)w_0 \quad \text{and} \quad w_2 = ((1 - \alpha_1)R + \alpha_1 x_2)w_1$$

Combining, we get that

$$w_2 = ((1 - \alpha_1)R + \alpha_1 x_2)((1 - \alpha_0)R + \alpha_0 x_1)w_0$$

Since CRRA preferences are scale-invariant, for any λ we have that $u(\lambda x) = \lambda^{1-\sigma}u(x)$, where σ is the coefficient of relative risk aversion. When the consumer is maximizing the expected utility, we have that

$$\mathbb{E}[u(w_2)] = \mathbb{E}\left[((1 - \alpha_1)R + \alpha_1 x_2)^{1-\sigma}u(w_1)\right] = \mathbb{E}[u(w_1)] \cdot ((1 - \alpha_1)R + \alpha_1\,\mathbb{E}[x_2])^{1-\sigma}$$

Thus, the choice of α that maximizes w_1 will also maximize w_2, since x_i are i.i.d., and $\alpha_0 = \alpha_1$.

Next, assume that u has CARA preferences. We know that u has the form $u(x) = -\exp(-\gamma x)$, where $\gamma > 0$ is the coefficient of absolute risk aversion. Thus,

$$\mathbb{E}[u(w_2)] = \mathbb{E}\left[u(w_1)\exp(-\gamma(((1 - \alpha_1)R + \alpha_1 x_2)))\right]$$

However, we cannot split the expectation here as above, since we do not know that the relevant moments for x necessarily exist. Thus, the choice of α_1 depends on x_1, so it will not necessarily hold that $\alpha_0 = \alpha_1$. □

18. Suppose that a decision maker has utility $u(x) = \sqrt{x}$.

(a) We have that wealth $w = 5$. The coefficient of absolute risk aversion is

$$-\frac{u''(w)}{u'(w)} = -\frac{(-0.25)w^{-1.5}}{(0.5)w^{-0.5}} = \frac{1}{2}\frac{\sqrt{5}}{\sqrt{125}} = \frac{1}{2} \cdot \frac{1}{5} = 0.1$$

The coefficient of relative risk aversion is

$$-w\frac{u''(w)}{u'(w)} = 5 \cdot \frac{1}{10} = 0.5$$

(b) The certainty equivalent of this lottery is

$$u^{-1}(0.5u(16) + 0.5u(4)) = u^{-1}(2 + 1) = u^{-1}(3) = 9$$

The probability premium is π such that

$$u(10) = (0.5 + \pi)u(16) + (0.5 - \pi)u(4) \implies \sqrt{10} = 2 + 4\pi + 1 - 2\pi \implies \pi = \frac{\sqrt{10} - 3}{2}$$

(c) The certainty equivalent of this lottery is

$$u^{-1}(0.5u(36) + 0.5u(16)) = u^{-1}(3 + 2) = u^{-1}(5) = 25$$

The probability premium is π such that

$$u(26) = (0.5 + \pi)u(36) + (0.5 - \pi)u(16) \Rightarrow \sqrt{26} = 3 + 6\pi + 2 - 4\pi \Rightarrow \pi = \frac{\sqrt{26} - 5}{2}$$

The probability premium is higher in the first lottery, which implies that u has decreasing absolute risk aversion, implied by the fact that it has constant relative risk aversion.

19. We have that an individual has utility $u(x) = -\exp(-\alpha x)$ with $\alpha > 0$, and initial wealth w. He invests in a riskless asset with return r and N jointly normally distributed random assets with means $\mu = (\mu_1, \ldots, \mu_N)$ and variance V. We assume that V is full rank.

    Denote by x_i the amount invested in risky asset i, and by y_i its return. The agent's realized wealth is

$$w' = \left(w - \sum_{i=1}^{N} x_i\right) r + \sum_{i=1}^{N} x_i y_i$$

    By the properties of jointly normal distributions, $w' \sim \mathcal{N}\left(\left(w - \sum_{i=1}^{N} x_i\right) r + \sum_{i=1}^{N} x_i \mu_i, x^T V x\right)$. The expected utility of this is

$$\mathbb{E}[u(w')] = \mathbb{E}[-\exp(-\alpha w')]$$

    Using the properties of the moment generating function of a normal random variable, we have that

$$\mathbb{E}[u(w')] = -\exp\left[\left(\left(w - \sum_{i=1}^{N} x_i\right) r + \sum_{i=1}^{N} x_i \mu_i\right)(-\alpha) - (x^T V x)\frac{\alpha^2}{2}\right]$$

    Monotonically transforming this by $\ln(\cdot)$, we get that expected utility is maximized when

$$-\alpha(\mu - r) - \alpha^2 V x = 0 \Longrightarrow x = \frac{\mu - r}{\alpha V}$$

    where the $-$ in the numerator denotes elementwise subtraction.