ECON 6190 Section 2

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4. [Hansen 6.6] Show that $\mathbb{E}[s] \le \sigma$. where $s = \sqrt{s^2}$ and s^2 is the sample variance.

sample variance $s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2$ has the property that $E[s^2] = \sigma^2$.

$\Rightarrow$ unbiased estimator of σ^2

$\sigma = \sqrt{\sigma^2} = \sqrt{E[s^2]}$

General Jensen's inequality

- If $g(\cdot) : \mathbb{R}^m \to \mathbb{R}$ is convex, then for any random vector X for which $\mathbb{E}\,\|X\| < \infty$ and $\mathbb{E}|g(X)| < \infty$

$$g(\mathbb{E}[X]) \le \mathbb{E}[g(X)]$$

  If $g(\cdot) : \mathbb{R}^m \to \mathbb{R}$ is concave, then the inequality is reversed

- Jensen's inequality holds conditionally as well

By Jensen's inequality, $g(x) = \sqrt{x}$ is concave.

$$g(E[s^2]) \geq E[g(s^2)]$$

$$\Leftrightarrow \underbrace{\sqrt{E[s^2]}}_{\sigma} \geq \underbrace{E[\sqrt{s^2}]}_{E[s]}$$

$$\Rightarrow E[s] \le \sigma.$$

5. [Hong 3.31] Show that if X is a continuous random variable, then

$$\min_a \mathbb{E}|X - a| = \mathbb{E}|X - m|,$$

where m is the median of X.

$E[g(x)] = \int g(x) f(x) dx$

$$E[|x-a|] = \int_{-\infty}^{\infty} |x-a| f(x) dx$$

$$= \underbrace{\int_{-\infty}^{a} (a-x) f(x) dx}_{\text{“}f(x,t)\text{”}} + \int_{a}^{\infty} (x-a) f(x) dx$$

- **Leibniz integral rule**

  Suppose $g(x) = \int_{\alpha(x)}^{\beta(x)} f(x,t)dt$, then

$$\frac{d}{dx}g(x) = \int_{\alpha(x)}^{\beta(x)} \frac{\partial f(x,t)}{\partial x}dt + \left(\frac{d}{dx}\beta(x)\right) f(x,\beta(x)) - \left(\frac{d}{dx}\alpha(x)\right) f(x,\alpha(x))$$

$x \to a$
$t \to x$

By Leibniz integral rule:

$$\underbrace{\frac{dE[|X-a|]}{da}}_{\text{function of } a} = \int_{-\infty}^{a} \underbrace{\frac{d\,(a-x)f(x)}{da}}_{f(x)}dx + \overbrace{\left(\frac{da}{da}\right)\underbrace{\cancel{(a-a)}f(a)}_{(a-x)f(x)\big|_{x=a}}}^{=0} - 0$$

$$+ \int_{a}^{\infty} \underbrace{\frac{d\,(x-a)f(x)}{da}}_{-f(x)}dx + 0 - \underbrace{\left(\frac{da}{da}\right)\cancel{(a-a)}f(a)}_{=0}$$

$$= \int_{-\infty}^{a} f(x)dx - \int_{a}^{\infty} f(x)dx \quad (*)$$

Set $(*) = 0$. $\Rightarrow \int_{-\infty}^{a} f(x)dx = \int_{a}^{\infty} f(x)dx$

$$\Rightarrow P(X \le a) = P(X > a)$$

Since $P(X \le a) + P(X > a) = 1 \Rightarrow P(X \le a) = P(X > a) = 1/2$.

By def of the median, $a = m$.

To check at $a = m$ is a minimum, $\varepsilon > 0$,

Consider to the left of m, $\underbrace{\int_{-\infty}^{m-\varepsilon} f(x)dx}_{<1/2} - \underbrace{\int_{m-\varepsilon}^{\infty} f(x)dx}_{>1/2} < 0$

Consider to the right of m, $\underbrace{\int_{-\infty}^{m+\varepsilon} f(x)dx}_{>1/2} - \underbrace{\int_{m+\varepsilon}^{\infty} f(x)dx}_{<1/2} > 0$

$\Rightarrow$ min

6. [Mid-term, Fall 2021] Let X be a random variable with conditional density

$$f(x|\theta) = \begin{cases} \frac{1}{\theta} & \text{if } 0 < x < \theta \\ 0 & \text{otherwise} \end{cases}.$$

Usually we treat parameter θ as a constant. Now suppose $\theta > 0$ is treated as a random variable with density

$$g(\theta) = \begin{cases} \theta e^{-\theta} & \text{if } \theta > 0 \\ 0 & \text{if } \theta \le 0 \end{cases},$$

where we use notation θ as both the random variable and the specific values it can take. Answer this following questions. (This question does not require any prior knowledge on Bayesian statistics, but is a test of your understanding of the key notions introduced in class.)

(a) Find $f(x)$, the marginal density of X.

(b) Find $g(\theta|x)$, the conditional density of θ given $X = x$.

(c) Find $\mathbb{E}[(\theta - a)^2|X = x]$ for some given constant a. (You are NOT required to work out the final integration.)

(a) 1) find joint distribution 2) find marginal of the other variable

$$h(x,\theta) = f(x|\theta)\,g(\theta) = \begin{cases} \frac{1}{\theta}\theta e^{-\theta} = e^{-\theta}, & 0<x<\theta \\ 0, & \text{o/w} \end{cases}$$

$$f(x) = \int_{-\infty}^{\infty} h(x,\theta)\,d\theta = \int_{x}^{\infty} e^{-\theta}\,d\theta = -e^{-\theta}\Big|_{x}^{\infty}$$

$$= 0 - (-e^{-x})$$

$$= e^{-x}, \quad x>0$$

$$\Rightarrow f(x) = \begin{cases} e^{-x}, & x>0 \\ 0, & \text{o/w} \end{cases}$$

(b) For $x>0$,

$$g(\theta|x) = \frac{h(\theta,x)}{f(x)} = \begin{cases} \frac{e^{-\theta}}{e^{-x}} = e^{x-\theta}, & 0<x<\theta \\ \frac{0}{e^{-x}} = 0, & x \ge \theta \end{cases}$$

$g(\theta|x)$ is not defined for $x \le 0$.

(c) $$E\left[\underbrace{(\theta-a)^2}_{\text{function of } \theta} \,\middle|\, X=x\right] = \int (\theta-a)^2 g(\theta|x)\, d\theta$$

$g(\theta|x)$

$$= \int_x^\infty (\theta-a)^2 e^{x-\theta}\, d\theta$$

$$= e^x \int_x^\infty (\theta-a)^2 e^{-\theta}\, d\theta .$$

4. [Hong 5.47] Suppose X and Y are random variables such that $\mathbb{E}[Y|X] = 7-(1/4)X$ and $\mathbb{E}[X|Y] = 10-Y$. Determine the correlation between X and Y.

Recall: $$\text{corr}(X,Y) = \frac{\text{cov}(X,Y)}{\sqrt{\text{var}(X)\text{var}(Y)}}$$

$$\text{cov}(X,Y) = E\left[(X-E[X])(Y-E[Y])\right]$$
$$= \underbrace{E[XY]}_{?} - E[X]E[Y]$$

$$E[X] \overset{\text{LIE}}{=} E[E[X|Y]] = E[10-Y] = 10-E[Y] \quad (1)$$
$$E[Y] = E[E[Y|X]] = E[7-1/4X] = 7-1/4E[X] \quad (2)$$

Combine (1) (2):

$$\begin{cases} E[X] = 10-E[Y] \\ E[Y] = 7-1/4E[X] \end{cases} \Rightarrow \begin{cases} E[X] = 4 \\ E[Y] = 6. \end{cases}$$

$$E[XY] \overset{\text{LIE}}{=} E[\underbrace{E[XY|X]}_{Y|X}] = E[X\,E[Y|X]]$$
$$= E[X(7-1/4X)]$$
$$= 7E[X] - 1/4E[X^2]$$
$$= 28 - 1/4E[X^2]$$

$$E[XY] = E[\underbrace{E[XY|Y]}_{X|Y}] = E[Y\,E[X|Y]]$$
$$= E[Y(10-Y)]$$
$$= 10E[Y] - E[Y^2]$$
$$= 60 - E[Y^2]$$

$$\Rightarrow \begin{cases} E[X^2] = 112 - 4E[XY] \\ E[Y^2] = 60 - E[XY] \end{cases}$$

$$\mathrm{var}(X) = E[X^2] - (\underbrace{E[X]}_{4})^2 = 112 - 4E[XY] - 16 = 4(24 - E[XY])$$

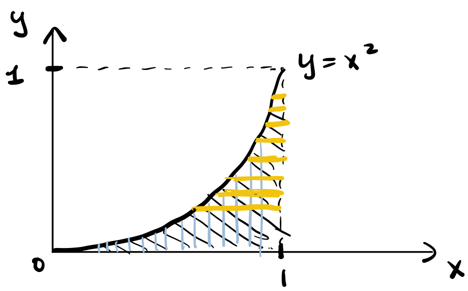
$$\mathrm{var}(Y) = E[Y^2] - (E[Y])^2 = 60 - E[XY] - 36 = 24 - E[XY]$$

$$\mathrm{corr}(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{var}(X)\mathrm{var}(Y)}} = \frac{E[XY] - 24}{\sqrt{4(24-E[XY])(24-E[XY])}} = -1/2$$

2. [Hong 5.4]

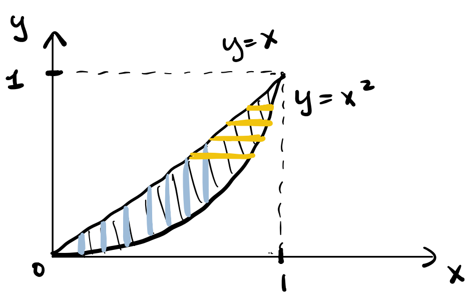
(a) Find $P(X > \sqrt{Y})$ if X and Y are jointly distributed with pdf $f_{X,Y}(x, y) = x + y$ for $0 \le x \le 1, 0 \le y \le 1$.

(b) Find $P(X^2 < Y < X)$ if X and Y are jointly distributed with pdf $f_{X,Y}(x, y) = 2x$ for $0 \le x \le 1, 0 \le y \le 1$.



$x^2 > y$
$y < x^2$

$$P(X > \sqrt{Y})$$
$$= \iint_A f_{X,Y}(x,y)\,dy\,dx$$
$$= \int_0^1 \int_0^{x^2} f_{X,Y}(x,y)\,dy\,dx = \int_0^1 \int_{\sqrt{y}}^1 f_{X,Y}(x,y)\,dx\,dy$$
$$= 7/20$$



$$P(X^2 < Y < X)$$
$$= \iint_A 2x\,dy\,dx$$
$$= \int_0^1 \int_{x^2}^{x} 2x\,dy\,dx = \int_0^1 \int_y^{\sqrt{y}} 2x\,dx\,dy$$
$$= 1/6$$