4* More on Correspondences

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1 More on Correspondences

Let us restrict attention to correspondences that are nonempty- and compact-valued. As before, we assume tthat $X \subseteq \mathbb{R}^d$ and $Y \subseteq \mathbb{R}^q$ are Euclidean spaces (with $d, q \in \mathbb{N}$).

1.1 Upper and lower hemi-continuity in terms of sequences

Definition 1. A correspondence $\Gamma : X \subseteq \mathbb{R}^d \rightrightarrows Y \subseteq \mathbb{R}^q$ is

- ▷ *upper hemi-continuous* at $x \in X$ if, for any open subset $O \subseteq Y$ such that $\Gamma(x) \subseteq O$, there exists $\epsilon > 0$ such that $\Gamma(B_\epsilon(x)) \subseteq O$.
- ▷ *lower hemi-continuous* at $x \in X$ if, for any open subset $O \subseteq Y$ such that $\Gamma(x) \cap O \neq \varnothing$, there exists $\epsilon > 0$ such that $\Gamma(z) \cap O \neq \varnothing$ for all $z \in B_\epsilon(x)$.

Proposition 1. *$\Gamma : X \subseteq \mathbb{R}^d \rightrightarrows Y \subseteq \mathbb{R}^q$ is upper hemi-continuous (resp. lower hemi-continuous) if and only if $\Gamma^{-1}(O) = \{x \in X : \Gamma(x) \subseteq O\}$ (resp. $\Gamma_{-1}(O) = \{x \in X : \Gamma(x) \cap S \neq \varnothing\}$) is open for every open $O \subseteq Y$.*

Lemma 1. *Suppose $\Gamma : X \subseteq \mathbb{R}^d \rightrightarrows Y \subseteq \mathbb{R}^q$ is a nonempty- and compact-valued upper hemi-continuous correspondence. Then, $\Gamma(S)$ is compact in Y for any compact subset S of X.*

Proof. Let $\mathcal{O}$ be an open cover of $\Gamma(S)$. We wish to find a finite subset of $\mathcal{O}$ that covers $\Gamma(S)$. Note that, for each $x \in S$, $\mathcal{O}$ is also an open cover of $\Gamma(x)$. Since $\Gamma(x)$ is compact, there exist a finite subcover $\{O_1(x), \ldots, O_{n_x}(x)\} \subseteq \mathcal{O}$ such that $\Gamma(x) \subseteq \bigcup_{i=1}^{n_x} O_i(x) =: O(x)$. By Proposition 1 in "4. Correspondences", $\Gamma^{-1}(O(x))$ is open in X for each $x \in X$. Moreover, $\Gamma(S) \subseteq \bigcup_{x \in S} O(x)$ so that $S \subseteq \bigcup_{x \in S} \Gamma^{-1}(O(x))$; i.e., $\{\Gamma^{-1}(O(x)) : x \in S\}$ is an open cover of S. By compactness of S, there exists $\{x_1, \ldots, x_n\} \subseteq S$ such that $\{\Gamma^{-1}(O(x_i)) : i \in \{1, \ldots, n\}\}$ is a finite subcover of S. But then, by definition of Γ^{-1}, $\{O(x_n), \ldots, O(x_n)\}$ must cover $\Gamma(S)$. Therefore, $\{O_j(x_i) : j \in \{1, \ldots, n_{x_i}\}, i \in \{1, \ldots, n\}\}$ is a finite subcover of $\Gamma(S)$. ■

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Proposition 2* Let $F : X \rightrightarrows Y$ be a nonempty- and compact-valued correspondence.

(i) F is upper hemi-continuous at $x \in X$ if and only if, for any sequence $(x_n)_n$ in X and any sequence $(y_n)_n$ in Y such that $x_n \to x$ and $y_n \in \Gamma(x_n)$ for all $n \in \mathbb{N}$, there exists a subsequence of $(y_n)_n$ that converges to a point in $F(x)$.

(ii) F is lower hemi-continuous at $x \in X$ if and only if, for any sequence $(x_n)_n$ in X with $x_n \to x$ and $y \in F(x)$, there exists a sequence $(y_n)_n$ in Y such that $y_n \to y$ and $y_n \in \Gamma(x_n)$ for all $n \in \mathbb{N}$.

Proof. (i) Fix $x \in X$. Suppose that, for any sequence $(x_n)_n$ in X and any sequence $(y_n)_n$ in Y such that $x_n \to x$ and $y_n \in F(x_n)$ for all $n \in \mathbb{N}$, there exists a subsequence of $(y_n)_n$ that converges to a point in $\Gamma(x)$. By way of contradiction, suppose Γ is not upper hemi-continuous at x. Then, there exists an open subset $O \subseteq Y$ with $\Gamma(x) \subseteq O$ and $\Gamma(B^X_{n^{-1}}(x)) \backslash O \neq \varnothing$ for all $n \in \mathbb{N}$. But then there exist sequences $(x'_n)_n$ in x and $(y'_n)_n$ in Y such that $x_n \to x$ and $y_n \in \Gamma(x_n) \backslash O$ for all $n \in \mathbb{N}$. Since $\Gamma(x) \subseteq O$ and each y_n belongs to the closed set $Y \backslash O$, no subsequence of (y'_n) can converge to a point in $\Gamma(x) \subseteq O$.

Conversely, suppose that Γ is upper hemi-continuous at $x \in X$. If $(x_n)_n$ is a sequence in X with $x_n \to x$, then $S \coloneqq \{x, x_1, x_2, \ldots\}$ is sequentially compact in X. Then, by lemma 1 above, $\Gamma(S)$ is sequentially compact in Y. Thus, if $y_n \in \Gamma(x_n)$ for all $n \in \mathbb{N}$, then $(y_n)_n$—being a sequence in $\Gamma(S)$—must possess a subsequence $(y_{n_k})_k$ that converges to some $y \in \Gamma(S)$. To show that $y \in \Gamma(x)$, we can use the same argument as in the proof of part (ii) of Proposition 5 in "4. Correspondences".

(ii) Suppose Γ is lower hemi-continuous at $x \in X$. Fix a sequence $(x_n)_n$ in X with $x_n \to x$ and $y \in \Gamma(x)$. By lower hemi-continuity, for every $k \in \mathbb{N}$, there exists a $\delta_k > 0$ such that $\Gamma(z) \cap B^Y_{k^{-1}}(y) \neq \varnothing$ for all $z \in B^X_{\delta_k}(x)$. Since $x_n \to x$, there exists an $n_1 \in \mathbb{N}$ such that $\|x_{n_1} - x\| < \delta_1$, and for any $k = 2, 3, \ldots$, there exists an $n_k \in \{n_{k-1}+1, n_{k-1}+2, \ldots\}$ such that $\|x_{n_k} - x\| < \delta_k$. This gives a subsequence $(x_{n_k})_k$ such that $\Gamma(x_{n_k}) \cap B^Y_{k^{-1}}(y) \neq \varnothing$ for each $k \in \mathbb{N}$. Now pick any $(y_n)_n$ in Y such that

$$y_n \in \Gamma\left(x_{n_k}\right) \cap B^Y_{k^{-1}}\left(y\right) \ \ \forall n \in \{n_k, \ldots, n_{k+1}-1\} \ \ \forall k \in \mathbb{N}.$$

Then, we have $y_n \to y$ and $y_n \in \Gamma(x_n)$ for each $n \in \mathbb{N}$.

Conversely, suppose Γ is not lower hemi-continuous at $x \in X$. Then, there exists an open subset O of Y such that $\Gamma(x) \cap O \neq \varnothing$ and, for every $n \in \mathbb{N}$, there exists an $x_n \in B^X_{n^{-1}}(x)$ with $\Gamma(x_n) \cap O = \varnothing$. Note that $x_n \to x$, and pick any $y \in \Gamma(x) \cap O$. By hypothesis, there must exist a sequence $(y_n)_n$ in Y such that $y_n \to y$ and $y_n \in \Gamma(x_n)$ for all $n \in \mathbb{N}$. But since $y \in O$ and O is open, $y_n \in O$ for n sufficiently large, contradicting that $\Gamma(x_n) \cap O = \varnothing$ for all $n \in \mathbb{N}$. ∎

Remark 1. In the class notes, we gave a sequential definition for the case in which, for some $x \in X$, $F(x)$ could be empty and/or not compact.

Exercise 1. Define correspondences $\Gamma_1, \Gamma_2, \Gamma_3 : \mathbb{R} \rightrightarrows \mathbb{R}$ as follows

$$\Gamma_1(\theta) \coloneqq \begin{cases} [0,1) & \text{if } \theta = 0 \\ \{0\} & \text{if } \theta \neq 0 \end{cases}, \ \Gamma_2(\theta) \coloneqq \begin{cases} [0,1] & \text{if } \theta = 0 \\ \{0\} & \text{if } \theta \neq 0 \end{cases}, \ \Gamma_3(\theta) \coloneqq \begin{cases} \{0\} & \text{if } \theta = 0 \\ [0,1] & \text{if } \theta \neq 0 \end{cases}. \tag{1}$$

Which of them are upper hemi-continuous and which of them are lower hemi-continuous. Which of them are compact-valued?

Solution 1. Γ_1: not compact-valued, not lower hemicontinuous, upper hemi-continuous. Γ_2: compact-valued, not lower hemi-continuous, upper hemi-continuous. Γ_3: compact-valued, lower hemi-continuous, not upper hemicontinuous.

Exercise 2. Let $\Gamma : \Theta := \mathbb{R}_+ \to X := \mathbb{R}_+$ be $\Gamma(\theta) := [0, \theta]$. Show that Γ is continuous.

Solution 2. Fix $\theta \in \Theta$. Observe that Γ is compact-valued. Take a sequence $(\theta_n)_n$ in Θ and a sequence $(x_n)_n$ in X such that $\theta_n \to \theta$ and $x_n \in \Gamma(\theta_n) = [0, \theta_n]$ for all $n \in \mathbb{N}$. Define $\epsilon := \sup_{n\in\mathbb{N}} \|\theta_n - \theta\|$ and let $\overline{B}_\epsilon(\theta)$ (which contains all θ_n's) denote the closure of the open ball centred at θ with radius ϵ. Since the set

$$\left[0, \max_{\theta' \in \overline{B}_\epsilon(\theta)} \theta'\right]$$

which contains all x_n's is compact, there exists a convergent subsequence $(x_{n_k})_k$ of $(x_n)_n$ with limit point x in the set above. By construction, we have $0 \leq x_n \leq \theta_n$ for all $n \in \mathbb{N}$ so that $0 \leq x \leq \theta$; i.e., $x \in \Gamma(\theta)$. Hence, Γ is upper hemi-continuous at θ.

Fix $\theta \in \Theta$. Take a sequence $(\theta_n)_n$ in Θ with $\theta_n \to \theta$ and $x \in F(\theta)$ (and such an x exists because $\Gamma(\theta)$ is nonempty). Let $\gamma := \frac{x}{\theta} \leq 1$ and $y_n = \gamma\theta_n$ for all $n \in \mathbb{N}$. Then, $y_n \in \Gamma(\theta_n)$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} y_n = \gamma \lim_{n\to\infty} \theta_n = \gamma\theta = x.$$

Hence, Γ is lower hemi-continuous at $\theta \in \Theta$.

Exercise 3. Let $f : \mathbb{R}^d_+ \to \mathbb{R}_+$ be a continuous function and define $\Gamma : \mathbb{R}^d_+ \to \mathbb{R}_+$ by $\Gamma(\theta) := [0, f(\theta)]$. Show that Γ is continuous. Hint: Modify the proof from the previous exercise by letting $f(\theta) = \theta$.

Solution 3. Fix $\theta \in \Theta$. Observe that Γ is compact-valued. Take a sequence $(\theta_n)_n$ in Θ and a sequence $(x_n)_n$ in X such that $\theta_n \to \theta$ and $x_n \in \Gamma(\theta_n) = [0, f(\theta_n)]$ for all $n \in \mathbb{N}$. Define $\epsilon := \sup_{n\in\mathbb{N}} \|\theta_n - \theta\|$ and let $\overline{B}_\epsilon(\theta)$ (which contains all θ_n's) denote the closure of the open ball centred at θ with radius ϵ. Since the set

$$\left[0, \max_{\theta' \in \overline{B}_\epsilon(\theta)} f(\theta')\right]$$

which contains all x_n's is compact, there exists a convergent subsequence $(x_{n_k})_k$ of $(x_n)_n$ with limit point x in the set above. By construction, we have $0 \leq x_n \leq f(\theta_n)$ for all $n \in \mathbb{N}$ so that, by continuity of f, $0 \leq x \leq f(\theta)$; i.e., $x \in \Gamma(\theta)$. Hence, Γ is upper hemi-continuous at θ.

Fix $\theta \in \Theta$. Take a sequence $(\theta_n)_n$ in Θ with $\theta_n \to \theta$ and $x \in F(\theta)$ (and such an x exists because $\Gamma(\theta)$ is nonempty). Let $\gamma := \frac{x}{f(\theta)} \leq 1$ and $y_n = \gamma f(\theta_n)$ for all $n \in \mathbb{N}$. Then, $y_n \in \Gamma(\theta_n)$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} y_n = \gamma \lim_{n\to\infty} f(\theta_n) = \gamma f(\theta) = x,$$

where we used that f is continuous. Hence, Γ is lower hemi-continuous at $\theta \in \Theta$.

1.2 A simpler theorem of the maximum

Let us state a simpler theorem of the maximum that requires Γ to be continuous everywhere and we omit the result that solution correspondence is closed at each point.

Theorem of the Maximum* Suppose $\Theta \subseteq \mathbb{R}^d$ and $X \subseteq \mathbb{R}^q$ and let $\Gamma : \Theta \rightrightarrows X$ be a nonempty-valued, compact-valued and continuous correspondence, and $f : X \times \Theta \to \mathbb{R}$ be a continuous function. Then, the solution correspondence $X^* : \Theta \rightrightarrows X$ defined as $X^*(\theta) := \arg\max_{x \in \Gamma(\theta)} f(x, \theta)$ is nonempty-valued, compact-valued and upper hemi-continuous.

Remark 2. To be precise, in class, we proved the version in which we only required Γ to be continuous at some $\theta_0 \in \Theta$ and obtained that: (i) X^* is nonempty-valued, compact-valued, upper hemi-continuous at θ_0, and closed at θ_0; and (ii) f^* is continuous at θ_0.

Remark 3. Let $X := \Theta := \mathbb{R}$ and $f(x, \theta) = x$. Consider the problem of maximising f with respect to x given each of the three constraint correspondences in 1. Let X^*_Γ denote the solution correspondence given constraint correspondence $\Gamma : \Theta \rightrightarrows X$:

$$X^*_{\Gamma_1}(\theta) = \begin{cases} \varnothing & \text{if } \theta = 0 \\ \{0\} & \text{if } \theta \neq 0 \end{cases}, \; X^*_{\Gamma_2}(\theta) = \begin{cases} \{1\} & \text{if } \theta = 0 \\ \{0\} & \text{if } \theta \neq 0 \end{cases}, \; X^*_{\Gamma_3}(\theta) = \begin{cases} \{0\} & \text{if } \theta = 0 \\ \{1\} & \text{if } \theta \neq 0 \end{cases}.$$

Observe that: $X^*_{\Gamma_1}(0)$ is empty (i.e., compactness of Γ is necessary); $X^*_{\Gamma_2}$ is not upper hemi-continuous at 0 (i.e., lower hemi-continuity of Γ is necessary); $X^*_{\Gamma_3}$ is not upper hemi-continuous at 0 (i.e., upper hemi-continuity of Γ is necessary).

Remark 4. The theorem does not tell us that X^* is lower hemi-continuous—and it cannot. To see this, suppose $X := \Theta := [0, 1]$, $\Gamma(\theta) := [0, 1]$ for all $\theta \in \Theta$, and $f(x, \theta) := x\theta$. Then,

$$X^*(\theta) = \begin{cases} [0, 1] & \text{if } \theta = 0 \\ \{1\} & \text{if } \theta \neq 0 \end{cases},$$

which is not lower hemi-continuous (but it is, nonempty-valued, compact-valued and upper hemi-continuous).

Proof of Theorem of Maximum. Fix $\theta \in \Theta$. Because the set $\Gamma(\theta)$ is nonempty and compact, and $f(x, \theta)$ is continuous in x, extreme value theorem tells us that $f(\cdot, \theta)$ attains a maximum at some $x^* \in \Gamma(\theta)$; i.e., $X^*(\theta)$ is nonempty. Moreover, since $X^*(\theta) \subseteq \Gamma(\theta)$ and $\Gamma(\theta)$ is compact, $X^*(\theta)$ is bounded. To show that $X^*(\theta)$ is closed, take any convergent sequence $(x^*_n)_n$ in $X^*(\theta)$. Because $\Gamma(\theta)$ is closed, x^*_n converges to some $x^* \in \Gamma(\theta)$. Because $f^*(\theta) = f(x^*_n, \theta)$ for all $n \in \mathbb{N}$ and f is continuous, we must have $f(x^*, \theta) = f^*(\theta)$. Hence, $x^* \in X^*(\theta)$; i.e., $X^*(\theta)$ is closed. Since we chose θ arbitrarily, it follows that X^* is nonempty-valued and compact-valued.

To show that X^* is upper hemicontinuous, fix $\theta \in \Theta$ and take any sequence $(\theta_n)_n$ in Θ that converges to θ. Choose some $x_n \in X^*(\theta_n)$ for all $n \in \mathbb{N}$. By upper hemi-continuity of Γ, there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that $x_{n_k} \to x \in \Gamma(\theta)$. Fix any $z \in \Gamma(\theta)$. Since Γ is lower hemi-continuous, there exists a sequence $(z_{n_k})_k$ such that $z_{n_k} \to z$ and $z_{n_k} \in \Gamma(\theta_{n_k})$ for all $k \in \mathbb{N}$. Since $f(x_{n_k}, \theta_{n_k}) \geq f(z_{n_k}, \theta_{n_k})$ for all $k \in \mathbb{N}$ and f is continuous, we must have $f(x, \theta) \geq f(z, \theta)$. Since this holds for any $z \in \Gamma(\theta)$, we must have $x \in X^*(\theta)$; i.e., X^* is upper hemi-continuous. ∎

Remark 5. In class, we proved the result by relying on the following lemmata:

Lemma 3 X^* is closed at θ_0.

Lemma 4 Suppose $Z \subseteq \mathbb{R}^d$ and $Y \subseteq \mathbb{R}^q$ and let $F_1, F_2 : Z \rightrightarrows Y$ with $F_1(z) \cap F_2(z) \neq \varnothing$ for all $z \in Z$. Define $F : Z \rightrightarrows Y$ by $F := F_1 \cap F_2$. If F_1 is compact-valued and upper hemi-continuous at $z_0 \in Z$, and if F_2 is closed at z_0, then F is upper hemi-continuous at z_0.

We used the fact that Γ is lower hemi-continuous (as well as the fact that f is continuous) to prove lemma 3. We then obtained the result by letting $F_1 = \Gamma$ and $F_2 = X^*$ in lemma 4. We did so to highlight why we require f to be continuous (i.e., not just upper semi-continuous), as well as to obtain the additional result that X^* has a closed graph.

Remark 6. Recall that for existence of a maximum, we only need the objective function to be upper semi-continuous. So what can we get if we only have that f is upper semi-continuous? Let $X := [0,1]$, $\Theta := (0,1]$, $\Gamma(\theta) := [0,\theta]$ for all $\theta \in \Theta$ and

$$f(x,\theta) := \begin{cases} 1 - 2x & \text{if } x \in [0, 0.5) \\ 3 - 2x & \text{if } x \in [0.5, 1] \end{cases}.$$

Observe that f is upper semi-continuous (and in particular, not lower semi-continuous), and

$$X^*(\theta) = \begin{cases} \{0\} & \text{if } \theta \in (0, 0.5) \\ \{0.5\} & \text{if } \theta \in (0, 0.5) \end{cases},$$

which is not upper hemi-continuous at 0.5. However, observe that

$$f^*(\theta) := \max_{x \in \Gamma(\theta)} f(x,\theta) = \begin{cases} 1 & \text{if } \theta \in (0, 0.5) \\ 2 & \text{if } \theta \in (0, 0.5) \end{cases}$$

is upper semi-continuous.

Proposition 2. *Suppose $\Theta \subseteq \mathbb{R}^d$ and $X \subseteq \mathbb{R}^q$ and let $\Gamma : \Theta \rightrightarrows X$ be a nonempty-valued, compact-valued and upper hemi-continuous correspondence, and $f : X \times \Theta \to \mathbb{R}$ be an upper semi-continuous function. Prove that $f^*(\theta) := \max_{x \in \Gamma(\theta)} f(x,\theta)$ is upper semi-continuous.*

Proof. Take any sequence $(\theta_n)_n$ in Θ with $\theta_n \to \theta$ for some $\theta \in \Theta$. Toward a contradiction, suppose that $f^*(\theta) < \limsup_{n\to\infty} f^*(\theta_n)$. For each $n \in \mathbb{N}$, take $x_n \in \Gamma(\theta_n)$ be such that $f(x_n, \theta_n) = f^*(\theta_n)$ which exists because φ is upper semi-continuous and $\Gamma(\theta_n)$ is compact (why?). Thus, there exists a subsequence $(f(x_{n_k}, \theta_{n_k}))_k$ of $(f(x_n, \theta_n))_n$ such that $f(x_{n_k}, \theta_{n_k}) \to \limsup_{n\to\infty} f^*(\theta_n)$ (why?). Hence, there exists an $\epsilon > 0$ and $K \in \mathbb{N}$ such that

$$f(x_{n_k}, \theta_{n_k}) - f^*(\theta) \geq \epsilon \; \forall k \geq K.$$

Then, by upper hemi-continuity of Γ, there exists a subsequence $(x_{n_{k_\ell}})_\ell$ of $(x_{n_k})_k$ that converges to a point $x \in \Gamma(\theta)$. Then,

$$\limsup_{\ell\to\infty} f\left(x_{n_{k_\ell}}, \theta_{n_{k_\ell}}\right) - f^*(\theta) \geq \epsilon \Leftrightarrow \limsup_{\ell\to\infty} f\left(x_{n_{k_\ell}}, \theta_{n_{k_\ell}}\right) > f(x,\theta) = f^*(\theta),$$

which contradicts the upper semicontinuity of φ (which requires the inequality above to hold with $\leq$). ■