ECON 6190

Problem Set 2

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Problem 1: X_i are i.d.i.d with $\mathbb{E}[X_i] = \mu_i$ and $var[X_i] = \sigma_i^2$.

1. We have that
$$\mathbb{E}[\hat{X}] = \mathbb{E}\left[\frac{\sum_{i=1}^n X_i}{n}\right] = \frac{\sum_{i=1}^n \mathbb{E}[X_i]}{n} = \frac{1}{n}\sum_{i=1}^n \mu_i$$

2. We have that
$$\begin{aligned} var[\hat{X}] &= var\left[\frac{\sum_{i=1}^n X_i}{n}\right] \\ &= \frac{1}{n^2}\sum_{i=1}^n var[X_i] \\ &= \frac{\sum_{i=1}^n \sigma_i^2}{n^2} \end{aligned}$$

Problem 2: From Bayes' Rule, we have that
$$P\left\{\mu = \frac{1}{2}\middle| X_1 < 0\right\} = \frac{P\left\{X_1 < 0 \mid \mu = \frac{1}{2}\right\} P\left\{\mu = \frac{1}{2}\right\}}{P\left\{X_1 < 0 \mid \mu = \frac{1}{2}\right\} P\left\{\mu = \frac{1}{2}\right\} + P\left\{X_1 < 0 \mid \mu = -\frac{1}{2}\right\} P\left\{\mu = -\frac{1}{2}\right\}}$$
which simplifies to
$$\frac{P\left\{X_1 < 0 \mid \mu = \frac{1}{2}\right\}}{P\left\{X_1 < 0 \mid \mu = \frac{1}{2}\right\} + P\left\{X_1 < 0 \mid \mu = -\frac{1}{2}\right\}}$$
Since we have that $X_1 \sim \mathcal{N}(\mu, \sigma^2)$, we have that $(X_1 - \mu)/\sigma \sim \mathcal{N}(0, 1)$. Thus, defining Φ as the cdf of the standard normal, we get that this equation is equivalent to
$$\frac{P\{(X_1 - 1/2)/\sigma < (0 - 1/2)/\sigma\}}{P\{(X_1 - 1/2)/\sigma < (0 - 1/2)/\sigma\} + P\{(X_1 + 1/2)/\sigma < (0 + 1/2)/\sigma\}} = \frac{\Phi(-\frac{1}{2\sigma})}{\Phi(-\frac{1}{2\sigma}) + \Phi(\frac{1}{2\sigma})} = \frac{1 - \Phi(\frac{1}{2\sigma})}{1 - \Phi(\frac{1}{2\sigma}) + \Phi(\frac{1}{2\sigma})}$$
which means that
$$P\left\{\mu = \frac{1}{2}\middle| X_1 < 0\right\} = 1 - \Phi\left(\frac{1}{2\sigma}\right)$$

Problem 3 We have that the standard normal density is given by
$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$
Thus,
$$\phi'(x) = -2x\frac{1}{2\sqrt{2\pi}} e^{-\frac{x^2}{2}} = -x\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = -x\phi(x)$$

Then to find $\mathbb{E}[Z^2]$, we can use the definition and get that

$$\mathbb{E}[Z^2] = \int z^2 \phi(z) dz$$

Setting $u = z$, $du = dz$, $dv = z\phi(z)dz$, and $v = -\phi(z) = \int dv$ we get that

$$\mathbb{E}[Z^2] = \int udv = uv - \int vdu = z \int z\phi(z)dz + \int \phi(z)dz = 1$$

where the last equality follows from the fact that the mean of a standard normal is 0, and the integral over $\mathbb{R}$ of any pdf is 1.

Problem 4

(a) We have from the definition that the marginal distribution of Y is

$$f_Y(y) = \int f(x,y)dx = \int \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - 2\frac{\rho xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2}\right)\right) dx$$

This simplifies as follows:

$$\begin{aligned}
f_Y(y) &= \frac{1}{\sqrt{2\pi}\sigma_Y} \exp\left(-\frac{1}{2}\frac{y^2}{\sigma_Y^2}\frac{1}{1-\rho^2}\right) \frac{1}{\sqrt{2\pi}\sigma_X\sqrt{1-\rho^2}} \int \exp\left[\left(-\frac{1}{2(1-\rho^2)}\right)\left(\frac{x^2}{\sigma_X^2} - \frac{2\rho xy}{\sigma_X\sigma_Y}\right)\right] dx \\
&= \frac{1}{\sqrt{2\pi}\sigma_Y} \exp\left(-\frac{1}{2}\frac{y^2}{\sigma_Y^2}\frac{1}{1-\rho^2}\right) \frac{1}{\sqrt{2\pi}\sigma_X\sqrt{1-\rho^2}} \int \exp\left(-\underbrace{\frac{1}{2\sigma_X^2(1-\rho^2)}}_{A} x^2 + \underbrace{\frac{\rho y}{\sigma_X\sigma_Y(1-\rho^2)}}_{B} x\right) dx \\
&= \frac{1}{\sqrt{2\pi}\sigma_Y} \exp\left(-\frac{1}{2}\frac{y^2}{\sigma_Y^2}\frac{1}{1-\rho^2}\right) \frac{1}{\sqrt{2\pi}\sigma_X\sqrt{1-\rho^2}} \sqrt{\frac{\pi}{A}} \exp\left(\frac{B^2}{4A}\right) \qquad \text{by the Gaussian Integral Rule} \\
&= \frac{1}{\sqrt{2\pi}\sigma_Y} \exp\left(-\frac{1}{2}\frac{y^2}{\sigma_Y^2}\frac{1}{1-\rho^2}\right) \frac{1}{\sqrt{2\pi}\sigma_X\sqrt{1-\rho^2}} \sqrt{2\pi\sigma_X^2(1-\rho^2)} \exp\left(\frac{\rho^2 y^2}{\sigma_X^2\sigma_Y^2(1-\rho^2)^2}\frac{1}{2}\sigma_X^2(1-\rho^2)\right) \\
&= \frac{1}{\sqrt{2\pi}\sigma_Y} \exp\left(-\frac{1}{2}\frac{y^2}{\sigma_Y^2}\frac{1}{1-\rho^2}\right) \exp\left(\frac{\rho^2 y^2}{2\sigma_Y^2(1-\rho^2)}\right) \\
&= \frac{1}{\sqrt{2\pi}\sigma_Y} \exp\left(-\frac{y^2}{2\sigma_Y^2}\left(\frac{1}{1-\rho^2} - \frac{\rho^2}{1-\rho^2}\right)\right) = \frac{1}{\sqrt{2\pi}\sigma_Y} \exp\left(-\frac{y^2}{2\sigma_Y^2}\right) \sim \mathcal{N}(0, \sigma_Y^2)
\end{aligned}$$

(b) Recall that the conditional density of a random variable is the joint density divided by the marginal density of the other random variable. We have that

$$f(x \mid Y = y) = \frac{f(x,y)}{f_Y(y)}$$

Thus,

$$
\begin{aligned}
f(x \mid Y = y) &= \left[\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - 2\frac{\rho x y}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2}\right)\right)\right]\left[\frac{1}{\sqrt{2\pi}\sigma_Y}\exp\left(-\frac{y^2}{2\sigma_Y^2}\right)\right]^{-1} \\
&= \frac{1}{\sqrt{2\pi}\sigma_X\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - 2\frac{\rho x y}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2}\right) + \left(\frac{y^2}{2\sigma_Y^2}\right)\right) \\
&= \frac{1}{\sqrt{2\pi}\sigma_X\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2}\left(\frac{x^2}{\sigma_X^2(1-\rho^2)} - \frac{2\rho x y}{\sigma_X\sigma_Y(1-\rho^2)} + \frac{y^2}{\sigma_Y^2(1-\rho^2)} - \frac{y^2}{\sigma_Y^2}\right)\right) \\
&= \frac{1}{\sqrt{2\pi}\sigma_X\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2\sigma_X^2(1-\rho^2)}\left(x^2 - 2\frac{\sigma_X}{\sigma_Y}\rho x y + \frac{\sigma_X^2}{\sigma_Y^2}\rho^2 y^2\right)\right) \\
&= \frac{1}{\sqrt{2\pi}\sigma_X\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2}\frac{(x - \frac{\sigma_X}{\sigma_Y}\rho y)^2}{(1-\rho)^2\sigma_X^2}\right) \sim \mathcal{N}\left(\frac{\sigma_X}{\sigma_Y}\rho y, (1-\rho)^2\sigma_X^2\right)
\end{aligned}
$$

(c) We have that Z is a linear combination of two jointly normal random variables, so it is jointly normal with Y. It thus suffices to show that $cov(Z, Y) = 0$, because with jointly normal random variables uncorrelatedness implies independence. Then

$$
\begin{aligned}
cov(Z, Y) &= \mathbb{E}[ZY] - \mathbb{E}[Z]\,\mathbb{E}[Y] \\
&= \mathbb{E}\left[\frac{XY}{\sigma_X} - \frac{\rho}{\sigma_Y}Y^2\right] - \frac{\mathbb{E}[X]\,\mathbb{E}[Y]}{\sigma_X} - \frac{\rho}{\sigma_Y}\,\mathbb{E}[Y]^2 \\
&= \frac{\mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]}{\sigma_X} - \frac{\rho}{\sigma_Y}\left(\mathbb{E}[Y^2] - \mathbb{E}[Y]^2\right) \\
&= \frac{cov(X, Y)}{\sigma_X} - \frac{\rho}{\sigma_Y}var(Y) \\
&= \sigma_Y\rho - \rho\sigma_Y = 0
\end{aligned}
$$

Thus, Z and Y are independent.

Problem 5

(a) We have that $\mathbb{E}[u] = \mathbb{E}[H'e] = H'\,\mathbb{E}[e] = H'0 = 0$. We also have that $var(u) = var(H'e) = H'var(e)H = H'I_n\sigma^2 H = I_n\sigma^2$. Thus, $u = H'e \sim \mathcal{N}(0, I_n\sigma^2)$.

(b) We have that $\mathbb{E}[u] = \mathbb{E}[A^{-1}e] = A^{-1}\,\mathbb{E}[e] = A^{-1}0 = 0$. We also have that $var(u) = var(A^{-1}e) = A^{-1}var(e)A^{-1'} = A^{-1}\Sigma A^{-1'} = A^{-1}AA'A^{-1'} = (A^{-1}A)(A'A^{-1'}) = I_nI_n = I_n$. Thus, $u = A^{-1}e \sim \mathcal{N}(0, I_n)$.

Problem 6 We have that

$$
\begin{aligned}
cov(\hat{\sigma}^2, \bar{X}) &= \mathbb{E}[\hat{\sigma}^2\bar{X}] - \mathbb{E}[\hat{\sigma}^2]\,\mathbb{E}[\bar{X}] \\
&= \mathbb{E}[\hat{\sigma}^2(\bar{X} - \mu)] + \mathbb{E}[\mu\hat{\sigma}^2] - \mu\,\mathbb{E}[\hat{\sigma}^2] \\
&= \mathbb{E}[\hat{\sigma}^2(\bar{X} - \mu)]
\end{aligned}
$$

There are a number of sufficient conditions. One would be if the sample mean $\bar{X}$ is equal to the population mean μ.