Econ 6190 Mid Term Exam

14 October 2021

Instructions

This exam consists of six questions, not of equal length or difficulty. Answer all questions. This exam counts toward 35% of your final grade. Remember to always explain your answer. Good luck!

1. **[10 pts]** Let X have a discrete distribution. In the note *Random Vector and Their Distribution* we defined the conditional distribution function of Y given $X = x$ as

$$F_{Y|X}(y|x) = P\{Y \leq y | X = x\}$$

for any x such that $P\{X = x\} > 0$. Verify that

$$\lim_{y \to -\infty} F_{Y|X}(y|x) = 0, \quad \lim_{y \to \infty} F_{Y|X}(y|x) = 1.$$

2. **[5 pts]** Let C and D be two events. Show that if $C \Rightarrow D$, then $P\{C\} \leq P\{D\}$.

3. **[25 pts]** Suppose $\theta > 0$ is a random variable with density

$$g(\theta) = \begin{cases} \theta e^{-\theta} & \text{if } \theta > 0 \\ 0 & \text{if } \theta \leq 0 \end{cases},$$

(notice here we use notation θ as both the random variable and the specific values it can take) and X is another random variable with conditional density

$$f(x|\theta) = \begin{cases} \frac{1}{\theta} & \text{if } 0 < x < \theta \\ 0 & \text{otherwise} \end{cases}.$$

(a) Find $f(x)$, the marginal density of X.

(b) Find $g(\theta|x)$, the conditional density of θ given $X = x$.

(c) Find $\mathbb{E}[(\theta - a)^2 | X = x]$ for some given constant a. (You are NOT required to work out the final integration.)

4. **[15 pts]** Let Y, X_1, X_2 be three continuous random variables and assume all their related joint, marginal and conditional densities are well defined. Show the following variation of law of iterated expectations holds:

$$\mathbb{E}[Y|X_1 = c] = \mathbb{E}[\mathbb{E}[Y|X_1 = c, X_2]|X_1 = c]$$

for any constant c.

5. **[15 pts]** Let $\{X_1, X_2 \dots X_n\}$ be a random sample of size n from the uniform distribution $U[0, \theta]$ for some unknown parameter $\theta > 0$.

   (a) Find the pdf of $T = \max\{X_1, X_2 \dots X_n\}$.

   (b) Derive the bias of T as an estimator for θ. Is T asymptotically unbiased?

6. **[30 pts]** Suppose $\{X_1, X_2 \dots X_n\}$ is a random sample from a population distribution F with mean $\mathbb{E}X = 0$ and variance $\text{var}(X) = \sigma^2 > 0$. Consider estimating σ^2 by $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$.

   (a) Find the distribution of $n\frac{\hat{\sigma}^2}{\sigma^2}$ when F is normal, that is, when $F \sim \text{N}(0, \sigma^2)$.

   (b) Suppose now the distribution F is unknown. Impose suitable assumptions to derive the stochastic order of magnitude of $\hat{\sigma}^2 - \sigma^2$. Carefully state your reasoning.

   (c) Thus find the asymptotic distribution of $\sqrt{n}(\hat{\sigma}^2 - \sigma^2)$. Is there any additional assumption you need to make other than what is stated in (b)?

   (d) Propose an estimator, say $\hat{m}$, for standard deviation $m = \sqrt{\sigma^2}$. Thus find the asymptotic distribution of $\sqrt{n}(\hat{m} - m)$.