Problem Set 4

Due: TA Discussion, 17 September 2024.

1 Exercises from class notes

All from "2. Euclidean Topology.pdf".

Exercise 6. A point x is an *isolated point* in $S \subseteq \mathbb{R}$ if $x \in S$ and there exists $\epsilon > 0$ such that $B_\epsilon(x) \cap S = \{x\}$. For example, $\{1\}$ is an isolated point in $S = \{1\} \cup [2,3]$. What real-valued functions $f : S \to \mathbb{R}$ is continuous at 1?

Exercise 7. Prove Proposition 7 using ϵ-δ definition of continuity.

Exercise 8. Let f and g be continuous at x_0. Prove or disprove: $\max(f, g)$ is continuous at x_0. **Hint:** $\max\{f, g\} = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$.

Exercise 9. Prove or disprove: $f : S \to \mathbb{R}$ is continuous at x_0 if and only if, for every *monotonic* sequence $(x_n)_n$ in S converging to x_0, $f(x_n) \to f(x_0)$. **Hint:** The following Lemma could be useful: (x_n) converges to x if and only if for every subsequence x_{n_k} there exists sub-subsequence $x_{n_{k_l}}$ that converges to x. Bonus points if you also prove this Lemma!

2 Additional Exercises

Definition 1. Let $A, S \subset \mathbb{R}$, and let $f : S \to R$ be a real valued function. The *preimage* of A under f is defined as

$$f^{-1}(A) := \{x \in S : f(x) \in A\}.$$

Exercise 1. Let $S \subset \mathbb{R}$ be open. Prove: A function $f : S \to \mathbb{R}^d$ is continuous if and only if for every open set $A \subset \mathbb{R}^d$, $f^{-1}(A)$ is open.

Note: We defined continuity for real valued functions, but the definition extends naturally to functions with values in $\mathbb{R}^d$: f is continuous at $x_0 \in \mathbb{R}^d$ if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that $||f(x) - f(x_0)|| < \epsilon$ for all x in $B_\delta(x_0)$,