ECON 6170 Problem Set 3 Solutions

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Problem 1. In class, we proved the Bolzano-Weierstrass theorem (i.e., that every bounded sequence has a convergent subsequence; see Module 1) for sequences in $\mathbb{R}$. Use this result to extend the statement to sequences in $\mathbb{R}^k$.

Proof. We use proof by induction. The base case $k = 1$ is taken care of by Bolzano-Weierstrass in $\mathbb{R}^1$, which we've shown in class. For the induction step, suppose the claim holds for $k = n$. Consider a bounded sequence $(x_r)_{r=1}^{\infty}$ in $\mathbb{R}^{n+1}$.

The remainder of our proof entails finding a subsequence (x_{r_s}) whose first n terms converge and a subsubsequence $(x_{r_{s_t}})$ whose final term also converges.

(x_r) induces a bounded sequence in $\mathbb{R}^n$, call it (x_r^*), whose terms consist of the first n entries of the corresponding terms in the original sequence. By the induction hypothesis, this induced sequence has a convergent subsequence, $(x_s^*) := (x_{r_s}^*)$. Let (x_s) be the corresponding subsequence of (x_r). The final entries of the terms in (x_s) form a bounded sequence in $\mathbb{R}$, which, by the base case, has a convergent subsequence. Let $(x_t) := (x_{s_t})$ be the corresponding subsequence of (x_s). Then this subsequence converges to a point in $\mathbb{R}^{n+1}$. □

Theorem (Bolzano-Weierstrass)**.** *A set $A \subseteq \mathbb{R}^d$ is sequentially compact if and only if it is closed and bounded.*

Proof. Suppose A is sequentially compact. Let (x_n) be a sequence converging to some $x \in \mathbb{R}^d$. We want to show $x \in A$. Sequential compactness implies some subsequence (x_{n_k}) converges to a point in A, call it y. But subsequences of a convergent sequence converge to the same limit, so $x = y \in A$. Therefore, A is closed. Suppose A is unbounded. Then, for all $M \in \mathbb{R}$, there exists $x \in A$ satisfying $\|x\| > M$. Define a sequence in A as follows: Take $\|x_n\| > n$ for all $n \in \mathbb{N}$. Clearly $\|x_n\| \to \infty$, so $\|x_{n_k}\| \to \infty$ for any subsequence $(x_{n_k})_k$. It follows that $\|x_{n_k} - x\| \geq \|x_{n_k}\| - \|x\| \geq 1$, for sufficiently large k. Therefore this arbitrary subsequence doesn't converge to x. Since x is also arbitrary, the subsequence doesn't converge to any limit in $\mathbb{R}^d$. Thus, A is not sequentially compact, a contradiction. Therefore, A must be bounded, and because it is closed, A must be compact.

Conversely, suppose A is closed and bounded. Because A is bounded, every sequence in A is also bounded, so by the original Bolzano-Weierstrass theorem, every sequence has a subsequence that converges. By closedness of A, that subsequence converges to a point in A. Therefore, A is sequentially compact. □