ECON 6130 — Prelim exam — Fall 2023

October 12, 2023

Name: ______________________________

This is a closed-book exam. Some questions are harder. Don't worry if you cannot answer them all and plan your time accordingly. Good luck!

Problem. Where does the return on stocks come from?

Historically, broad stock market indexes such as the S&P 500 have outperformed risk free assets such as short-term government bonds. Many people attribute this *equity premium* to vague factors like "the economy is doing great", etc. We will investigate what drives the equity premium in a simple model using the tools that we learned in class.

Consider an infinite-horizon endowment economy with a unit mass of identical agents. There is a unique consumption good that is tradable but non-storable. Each agent is endowed with a stream y_t of consumption goods that evolves according to

$$y_{t+1} = x_{t+1} y_t,$$

where x_{t+1} is a random variable that is equal to $x_h > 0$ with some probability q and equal to $x_l > 0$ with probability $1 - q$. The draws are independent across time and $x_h \geq x_l$. Each agent maximizes

$$\sum_{t=0}^{\infty} \beta^t u\left(c_t\right)$$

where as usual $0 < \beta < 1$, and u is strictly increasing, strictly concave, differentiable and satisfies the Inada condition.

1. (5 points) Define an Arrow-Debreu equilibrium in this environment. Use s_t to denote the draw of the stochastic event in t and $s^t = (s_0, \ldots, s_t)$ to denote the history of events up to time t.

   Solution: The problem of each agent is

$$\max_{\{c_t\}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u\left(c_t\right) \pi_t\left(s^t\right),$$

where $\pi_t\left(s^t\right)$ is the probability of history s^t, where the maximization is subject to

$$\sum_{t=0}^{\infty}\sum_{s^t} q_t^0\left(s^t\right) c_t\left(s^t\right) \leq \sum_{t=0}^{\infty}\sum_{s^t} q_t^0\left(s^t\right) y_t\left(s^t\right).$$

An Arrow-Debreu equilibrium is an allocation $\{c_t(s_t)\}$ and prices $\left\{q_t^0\left(s^t\right)\right\}$ such that given prices the allocation $\{c_t(s_t)\}$ solves the agent's problem, and the allocation is feasible, that is $c_t\left(s^t\right) \leq y_t\left(s^t\right)$ for all t and s^t.

2. (10 points) Show that for any asset with a stochastic dividend process $\left\{d_t\left(s^t\right)\right\}_{t\geq\tau+1}$ its price in period τ, in units of period τ goods, is given by

$$p_\tau^\tau\left(s^\tau\right) = \sum_{t\geq\tau+1}\sum_{s^t|s^\tau} \beta^{t-\tau}\frac{u'\left(y_t\left(s^t\right)\right)}{u'\left(y_\tau\left(s^\tau\right)\right)}\pi_t\left(s^t|s^\tau\right) d_t\left(s^t\right).$$

Solution: We have done this in class. You need to take the first-order conditions of the agent's optimization problem and impose the market clearing conditions.

3. (15 points) From now on, assume that preferences are CRRA such that $u'(c) = c^{-\alpha}$, where α is the coefficient of relative risk aversion. The price of the entire endowment process from $\tau+1$ onward (the stock market) is thus given by

$$p_\tau^e\left(s^\tau\right) = \sum_{t\geq\tau+1}\sum_{s^t|s^\tau} \beta^{t-\tau}\frac{y_\tau^\alpha\left(s^\tau\right)}{y_t^\alpha\left(s^t\right)} y_t\left(s^t\right)\pi_t\left(s^t|s^\tau\right).$$

Given the structure of the economy, this price is recursive and only depends on the endowment y_τ at τ. Denote by $P^e(y)$ the price of equity when the current endowment is y. Show that

$$P^e(y) = \beta y^\alpha\left[q\left(x_h y\right)^{-\alpha}\left(P^e\left(x_h y\right) + y x_h\right) + (1-q)\left(x_l y\right)^{-\alpha}\left(P^e\left(x_l y\right) + y x_l\right)\right].$$

Further show that we can write $P^e(y) = wy$ for some w that only depends on parameters. Find w.

Solution: We can write

$$\begin{aligned}
p_\tau^e\left(s^\tau\right) &= \sum_{t\geq\tau+1}\sum_{s^t|s^\tau} \beta^{t-\tau}\frac{y_\tau^\alpha\left(s^\tau\right)}{y_t^\alpha\left(s^t\right)} y_t\left(s^t\right)\pi_t\left(s^t|s^\tau\right) \\
&= \sum_{s^{\tau+1}|s^\tau} \beta\frac{y_\tau^\alpha\left(s^\tau\right)}{y_{\tau+1}^\alpha\left(s^{\tau+1}\right)} y_{\tau+1}\left(s^{\tau+1}\right)\pi_{\tau+1}\left(s^{\tau+1}|s^\tau\right) + E\left[\sum_{t\geq\tau+2}\beta^{t-\tau}\frac{y_\tau^\alpha\left(s^\tau\right)}{y_t^\alpha\left(s^t\right)} y_t\left(s^t\right)|s^\tau\right]
\end{aligned}$$

By the law total expectation,

$$
\begin{aligned}
p^e_\tau\left(s^\tau\right) &= \sum_{s^{\tau+1}|s^\tau} \beta \frac{y^\alpha_\tau\left(s^\tau\right)}{y^\alpha_{\tau+1}\left(s^{\tau+1}\right)} y_{\tau+1}\left(s^{\tau+1}\right)\pi_{\tau+1}\left(s^{\tau+1}|s^\tau\right) \\
&+ \sum_{s^{\tau+1}|s^\tau} E\left[\sum_{t\geq\tau+2} \beta^{t-\tau}\frac{y^\alpha_\tau\left(s^\tau\right)}{y^\alpha_t\left(s^t\right)} y_t\left(s^t\right)|s^{\tau+1}\right] P\left(s^{\tau+1}|s^\tau\right) \\
&= q\beta\frac{y^\alpha}{\left[yx_h\right]^\alpha} yx_h + (1-q)\,\beta\frac{y^\alpha}{\left[yx_l\right]^\alpha} yx_l \\
&+ \beta y^\alpha_\tau\left(s^\tau\right) \sum_{s^{\tau+1}|s^\tau} E\left[\sum_{t\geq\tau+2}\beta^{t-(\tau+1)}\frac{1}{y^\alpha_t\left(s^t\right)}\frac{y^\alpha_{\tau+1}\left(s^{\tau+1}\right)}{y^\alpha_{\tau+1}\left(s^{\tau+1}\right)} y_t\left(s^t\right)|s^{\tau+1}\right] P\left(s^{\tau+1}|s^\tau\right) \\
&= q\beta\frac{y^\alpha}{\left[yx_h\right]^\alpha} yx_h + (1-q)\,\beta\frac{y^\alpha}{\left[yx_l\right]^\alpha} yx_l \\
&+ \beta y^\alpha\left[q\left(yx_h\right)^{-\alpha}P^e\left(x_h y\right) + (1-q)\left(yx_l\right)^{-\alpha}P^e\left(x_l y\right)\right]
\end{aligned}
$$

which is the desired expression. Finally, note that

$$
\begin{aligned}
wy &= \beta y^\alpha\left[q\left(x_h y\right)^{-\alpha}\left(wx_h y + yx_h\right) + (1-q)\left(x_l y\right)^{-\alpha}\left(wx_l y + yx_l\right)\right] \\
w\left(1-\beta\left(q\left(x_h\right)^{-\alpha}x_h + (1-q)\left(x_l\right)^{-\alpha}x_l\right)\right) &= \beta\left[q\left(x_h\right)^{1-\alpha} + (1-q)\left(x_l\right)^{1-\alpha}\right] \\
w &= \frac{\beta\left(qx_h^{1-\alpha} + (1-q)\,x_l^{1-\alpha}\right)}{1-\beta\left(qx_h^{1-\alpha} + (1-q)\,x_l^{1-\alpha}\right)}
\end{aligned}
$$

or

$$
\frac{w}{w+1} = \beta\left[q\left(x_h\right)^{1-\alpha} + (1-q)\left(x_l\right)^{1-\alpha}\right].
$$

4. (5 points) Consider now a risk-free asset that pays one unit of consumption good in period $\tau+1$ only, regardless of the state of the world. What is the price $p^f_\tau\left(s^\tau\right)$ of that asset in period τ? Define its rate of return as

$$
r^f = \frac{1-p^f_\tau\left(s^\tau\right)}{p^f_\tau\left(s^\tau\right)}.
$$

   What is r^f? Does it depend on t or on s^τ?

   Solution: Using our earlier equation

$$
p^\tau_\tau\left(s^\tau\right) = \sum_{s^{\tau+1}|s^\tau}\beta\frac{u'\left(y_{\tau+1}\left(s^{\tau+1}\right)\right)}{u'\left(y_\tau\left(s^\tau\right)\right)}\pi_{\tau+1}\left(s^{\tau+1}|s^\tau\right) = \beta\left(qx_h^{-\alpha} + (1-q)\,x_l^{-\alpha}\right)
$$

   so

$$
r^f = \left(\beta\left(qx_h^{-\alpha} + (1-q)\,x_l^{-\alpha}\right)\right)^{-1} - 1.
$$

5. (5 points) We are interested in figuring out the return on equity and how it compares to the

return on risk-free assets. Define the *realized* return on equity in state x_i, for $i \in \{h, l\}$, as

$$r_i^e = \frac{P^e(yx_i) + yx_i - P^e(y)}{P^e(y)}.$$

The expected equity return is then $r^e = qr_h^e + (1-q)\, r_l^e$. Compute the equity premium $r^e - r^f$ as a function of parameters.

Solution: Combining

$$\begin{aligned} r^e - r^f &= qr_h^e + (1-q)\, r_l^e - r_f \\ &= q\frac{wx_h + x_h - w}{w} + (1-q)\,\frac{wx_l + x_l - w}{w} - \left(\beta\left(qx_h^{-\alpha} + (1-q)\, x_l^{-\alpha}\right)\right)^{-1} + 1 \\ &= q\frac{w+1}{w}x_h + (1-q)\,\frac{w+1}{w}x_l - \left(\beta\left(qx_h^{-\alpha} + (1-q)\, x_l^{-\alpha}\right)\right)^{-1} \end{aligned}$$

Note that

$$\frac{w+1}{w} = \frac{1}{\beta\left(qx_h^{1-\alpha} + (1-q)\, x_l^{1-\alpha}\right)}$$

and so

$$r^e - r^f = \frac{qx_h + (1-q)\, x_l}{\beta\left(qx_h^{1-\alpha} + (1-q)\, x_l^{1-\alpha}\right)} - \frac{1}{\beta\left(qx_h^{-\alpha} + (1-q)\, x_l^{-\alpha}\right)}.$$

6. (10 points) If the agent is risk neutral ($\alpha = 0$) is the equity premium positive or negative? If the agent has log preference ($\alpha = 1$) is the equity premium positive or negative? What happens to the equity premium if there is technological improvement, in the sense that both x_l and x_h are multiplied by $\lambda > 1$? Suppose that the economy is doing great, but that risk disappears ($x_h = x_l \gg 0$), what happens to the equity premium? What can you conclude about the relation between risk and the outperformance of equities compared to risk-free bonds?

   Solution: Under risk neutrality

$$r^e - r^f = 0.$$

   Under log preferences,

$$r^e - r^f = \frac{1}{\beta}\left(qx_h + (1-q)\, x_l - \frac{1}{qx_h^{-1} + (1-q)\, x_l^{-1}}\right).$$

   we will show that

$$\frac{qx_h + (1-q)\, x_l}{\beta\left(qx_h^{1-\alpha} + (1-q)\, x_l^{1-\alpha}\right)} > \frac{1}{\beta\left(qx_h^{-\alpha} + (1-q)\, x_l^{-\alpha}\right)}.$$

   Which is the same as showing that

$$E(x)\, E\left(x^{-\alpha}\right) > E\left(x^{1-\alpha}\right).$$

Now, since x^{-1} is a convex function Jensen's inequality implies that

$$E\left(x^{-1}\right) \geq \left(E\left(x\right)\right)^{-1}$$

or

$$\left(E\left(x^{-1}\right)\right)^{-1} \leq E\left(x\right)$$

and so

$$r^e \geq r^f.$$

If both x_l and x_h are multiplied by $\lambda > 1$, the equity premium is multiplied by λ^α.

If risks disappears,

$$r^e - r^f = \frac{x}{\beta x^{1-\alpha}} - \frac{1}{\beta x^{-\alpha}} = 0,$$

then the equity premium disappears as well.

This analysis, in a slightly more complicated setup, was initially done by Mehra and Prescott (1985). There, they showed that with realistic parameters the model was not able to capture the large magnitude of the empirical equity premium. This sparked a large literature that is still ongoing.