# Econ 6190 Final Exam



9:00 am - 11:30 am, 14 December 2021

*Instructions*

*This exam consists of three questions, not of equal length or difficulty. Answer all questions. Remember: (1) read the questions carefully; (2) always explain your answer; (3) relax and good luck!*

1. The density of a normal distribution $\mathrm{N}(\mu, \sigma^2)$ with mean $\mu$ and variance $\sigma^2 > 0$ is

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

Answer the following questions.

(a) [**30 pts**] Suppose we have a random sample $\{X_i, i = 1 \dots n\}$ drawn from $X \sim \mathrm{N}(\mu_0, \sigma^2)$ where $\mu_0$ is **known** but $\sigma^2$ is **unknown**.

i. Find the Fisher information $\mathcal{F}$ for estimating $\sigma^2$.

ii. Find the MLE for $\sigma^2$, say $\tilde{\sigma}^2$.

iii. Is $\tilde{\sigma}^2$ Cramer-Rao efficient? If yes, show the claim is true; if not, explain why [hint: the 4th central moment of $X$ is $3\sigma^4$].

iv. Based on $\tilde{\sigma}^2$, derive a **finite sample valid** two sided 95% confidence interval for $\sigma^2$.

v. Fisher information $\mathcal{F}$ depends on the unknown parameter $\sigma^2$ and is usually unknown. Propose an estimator for $\mathcal{F}$, say $\tilde{\mathcal{F}}$, and find the asymptotic distribution of $\sqrt{n}(\tilde{\mathcal{F}} - \mathcal{F})$.

(b) [**15 pts**] Suppose now $X \sim \mathrm{N}(\mu, \sigma^2)$ where both $\mu$ and $\sigma^2$ are **unknown**. We hope to use a random sample $\{X_i, i = 1 \dots n\}$ drawn from $X$ to test hypothesis: $\mathbb{H}_0 : \mu = \mu_0$ for some $\mu_0 \in \mathbb{R}$ against $\mathbb{H}_1 : \mu \neq \mu_0$.

i. Let $\beta = (\mu, \sigma^2)$. Write down the log likelihood of $\beta$ under $\mathbb{H}_0$.

ii. The unconstrained MLE of $\beta$ is $\hat{\beta} = (\bar{X}_n, \hat{\sigma}^2)$, where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$. Based on this and your answer to (b)-(i), derive the likelihood ratio statistic $LR_n$ for testing $\mathbb{H}_0 : \mu = \mu_0$ vs $\mathbb{H}_1 : \mu \neq \mu_0$. Simplify as much as you can.

iii. Show the likelihood ratio test based on $LR_n > c$ for some $c$ is equivalent to $|T| > b$ for some $b$, where $T = \frac{\bar{X}_n - \mu_0}{\sqrt{\frac{\hat{\sigma}^2}{n}}}$.

2. Let $\{X_i, i = 1 \dots n\}$ be a random sample from a distribution with pdf

$$f(x|\theta) = \frac{1}{\theta},\ 0 \le x \le \theta, \theta > 0.$$

Answer the following questions.

(a) **[20 pts]** First, we are interested in estimating of $\theta$.

i. Find a method of moment estimator of $\theta$, say $\hat{\theta}_{MME}$.

ii. Find the mean square error of $\hat{\theta}_{MME}$.

iii. The MLE estimator of $\theta$ is $\hat{\theta}_{MLE} = \max\{X_i, i = 1 \dots n\}$. The density of $\hat{\theta}_{MLE}$ is

$$f(t) = \begin{cases} \frac{n}{\theta^n} t^{n-1} & 0 \le t \le \theta \\ 0 & \text{otherwise} \end{cases}. \tag{1}$$

Based on (1), find the mean square error of $\hat{\theta}_{MLE}$.

iv. Thus find the stochastic order of magnitude of both $\hat{\theta}_{MME}$ and $\hat{\theta}_{MLE}$. Which estimator would you prefer and why?

(b) **[10 pts]** Second, we hope to test hypothesis $\mathbb{H}_0 : \theta = 1$ vs $\mathbb{H}_1 : \theta < 1$.

i. Consider a test that rejects $\mathbb{H}_0$ if $\hat{\theta}_{MLE} < b$ when sample size is 10. Find the value of $b$ such that size of the test is exactly 0.1. What is the power of this test when $\theta < 1$?

(c) **[10 pts]** We still hope to test hypothesis $\mathbb{H}_0 : \theta = 1$ vs $\mathbb{H}_1 : \theta < 1$. But now, instead of trying to control size exactly, we only hope to control size asymptotically using some statistic constructed from $\hat{\theta}_{MME}$.

i. As a first step, find the asymptotic distribution of $\sqrt{n}(\hat{\theta}_{MME} - \theta)$.

ii. Let the asymptotic variance of $\sqrt{n}(\hat{\theta}_{MME} - \theta)$ be $V$. Find an estimator of $V$ and show it is consistent.

iii. Based on (i) and (ii), propose a test that controls size asymptotically at 10%. Carefully state your reasoning.

3. Let $X$ be a random variable with finite second moment. Answer the following questions.

(a) **[5 pts]** Show the solution of $\min_{a \in \mathbb{R}} \mathbb{E}[X - a]^2$ is $\mathbb{E}X$.

(b) **[10 pts]** Let $\{X_i\}_{i=1}^n$ be a sample from $X$ that are identically distributed but **not** independent: specifically, $\text{var}(X_i) = \sigma^2, \text{cov}(X_i, X_j) = \sigma_{ij}^2 > 0$ for $i \ne j, i = 1 \dots n, j = 1 \dots n$. Derive a set of condition(s) under which we can show $\frac{1}{n}\sum_{i=1}^n X_i \xrightarrow{p} \mathbb{E}X$.