Econ 6190 Problem Set 5

Fall 2024

1. Consider a random variable Z_n with the probability distribution

$$Z_n = \begin{cases} -n & \text{with probability } \frac{1}{n} \\ 0 & \text{with probability } 1 - \frac{2}{n} \\ 2n & \text{with probability } \frac{1}{n} \end{cases}.$$

   (a) Does $Z_n \xrightarrow{p} 0$ as $n \to \infty$? Give your reasoning clearly.

   (b) Calculate $\mathbb{E}Z_n$. Does $\mathbb{E}Z_n \to 0$ as $n \to \infty$?

   (c) Calculate $\text{var}[Z_n]$.

2. Let X_n and Y_n be sequences of random variables, and let X be a random variable.

   (a) If $X_n \xrightarrow{p} c$ and $X_n - Y_n \xrightarrow{p} 0$, show $Y_n \xrightarrow{p} c$.

   (b) If $X_n \xrightarrow{p} X$ and a_n is a deterministic sequence such that $a_n \to a$, show that $a_n X_n \xrightarrow{p} aX$.

   (c) If $X_n \xrightarrow{p} 0$, show that $\frac{\sin X_n}{X_n} \xrightarrow{p} 1$.

3. Let X be a random variable and let A be a set in $\mathbb{R}$. Show that $\mathbb{E}[\mathbf{1}\{X \in A\}] = P\{X \in A\}$, where

$$\mathbf{1}\{X \in A\} = \begin{cases} 1 & \text{if } X \in A \\ 0 & \text{if } X \notin A \end{cases}.$$

4. Let $\{X_1 \dots X_n\}$ be random sample.

   (a) Suppose X_i has pdf $f(x) = e^{-x+\theta}\mathbf{1}\{x \geq \theta\}$ for some constant θ. Show that

$$\min(X_1, X_2, \dots X_n) \xrightarrow{p} \theta.$$

   (b) Suppose X_i is $U[0, \theta]$ for some constant $\theta > 0$. Show that

$$\max(X_1, X_2, \dots X_n) \xrightarrow{p} \theta.$$

5. [Hansen 7.6] Take a random sample $\{X_1, ..., X_n\}$. Which of the following statistics converge in probability by the weak law of large numbers and continuous mapping theorem? For each, which moments are needed to exist?

(a) $\frac{1}{n}\sum_{i=1}^{n} X_i^2$,

(b) $\frac{1}{n}\sum_{i=1}^{n} X_i^3$,

(c) $\max_{i\leq n} X_i$,

(d) $\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2$,

(e) $\frac{\sum_{i=1}^{n} X_i^2}{\sum_{i=1}^{n} X_i}$ (assuming $\mathbb{E}X > 0$),

(f) $\mathbf{1}\{\frac{1}{n}\sum_{i=1}^{n} X_i > 0\}$,

(g) $\frac{1}{n}\sum_{i=1}^{n} X_i Y_i$.

6. [Hansen 7.7] A weighted sample mean takes the form $\bar{X}_n^* = \frac{1}{n}\sum_{i=1}^{n} w_i X_i$ for some non negative constants w_i satisfying $\frac{1}{n}\sum_{i=1}^{n} w_i = 1$. Assume X_i is i.i.d.

   (a) Show that $\bar{X}_n^*$ is unbiased for $\mu = \mathbb{E}[X]$,

   (b) Calculate $\operatorname{var}\left(\bar{X}_n^*\right)$,

   (c) Show that a sufficient condition for $\bar{X}_n^* \xrightarrow{p} \mu$ is that $n^{-2}\sum_{i=1}^{n} w_i^2 \to 0$,

   (d) Show that a sufficient condition for the condition in part (c) is $\frac{\max_{i\leq n} w_i}{n} \to 0$ as $n \to \infty$.