Problem Set 11

Due: TA Discussion, 22 November 2023

1 Exercises from class notes

All from "7. Comparative Statics.pdf".

Exercise 1. Consider the problem of maximising an objective function $f : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$ subject to K equality constraints; i.e.,

$$\max_{\mathbf{x} \in X} f(\mathbf{x}, \boldsymbol{\theta}) \text{ s.t. } h_k(\mathbf{x}, \boldsymbol{\theta}) = 0 \; k \in \{1, \ldots, K\},$$

where $h_k : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$ for each $k \in \{1, \ldots, K\}$. Suppose that f and h_k's are all $\mathbf{C}^2$ and concave, constraint qualification (for equality constraints) is satisfied, and that there is a unique solution for all $\boldsymbol{\theta} \in \Theta$. Apply the implicit function theorem on the first-order condition of the Lagrangian to give an expression for how the solution varies with $\boldsymbol{\theta}$. What must be true to apply the same argument when there are inequality constraints?

Exercise 2. State and prove the Envelope Theorem for the equality constrained optimisation problem from Exercise 1. **Hint:** You may make the same endogenous assumption as in the theorem above and an additional endogenous assumption regarding the Lagrange multipliers. What further assumptions on the Lagrangian can you make to replace the endogenous assumptions?

Exercise 6. Suppose that X and Θ are open sublattices of $\mathbb{R}^d$ and $\mathbb{R}^m$ respectively. Prove that $f : X \times \Theta \to \mathbb{R}$ that is $\mathbf{C}^2$ has increasing differences in $(\mathbf{x}, \boldsymbol{\theta}) \in X \times \Theta$ if and only if

$$\frac{\partial^2 f}{\partial x_i \partial \theta_j}(\mathbf{x}, \boldsymbol{\theta}) \geq 0 \; \forall (i, j) \in \{1, \ldots, d\} \times \{1, \ldots, m\}.$$

Exercise 7. Suppose $(X, \geq)$ and $(\Theta, \geq)$ are partially ordered sets and that $f : X \times \Theta \to \mathbb{R}$ has single-crossing differences in (x, θ). Prove that single-crossing property is an ordinal property. **Hint:** Show that, for any $\phi : \mathbb{R} \times \Theta \to \mathbb{R}$ such that $\phi(\cdot, \theta)$ is strictly increasing for any $\theta \in \Theta$, the function $\tilde{f} : X \times \Theta \to \mathbb{R}$ defined by $\tilde{f}(x, \theta) := \phi(f(x, \theta), \theta)$ also has single-crossing differences.

Remark 1. Recall that increasing differences implies single-crossing differences. Thus, one way to show that a function has single-crossing differences is to show that its strictly increasing transformation has increasing differences.

2 Additional Exercises

Definition 1. Let $(X, \geq)$ be a lattice. $f : (X, \geq) \mapsto \mathbb{R}$ is *log-supermodular* if $\ln f$ is a supermodular function.

Exercise 2. Prove that if f is log-supermodular then f is quasi-supermodular.

Exercise 3. Let $f : \mathbb{R}^d \to \mathbb{R}$ denote the firm's production function and consider the firm's profit maximisation problem:

$$\max_{\mathbf{y} \in \mathbb{R}^d_+} pf(\mathbf{y}) - \mathbf{q} \cdot \mathbf{y},$$

where p and $\mathbf{q}$ are output and input prices respectively. Suppose that f is nondecreasing and supermodular. Prove that if the price of the firm's output increases and/or the price of any of its inputs decreases, then the firm increases the use of all of its inputs.

Exercise 4. Let $f : \mathbb{R}^d \to \mathbb{R}$, and define $c : \mathbb{R} \to \mathbb{R}$ as

$$c(x) := \min_{\mathbf{y} \in \mathbb{R}^d_+ : f(\mathbf{y}) \geq x} \mathbf{q} \cdot \mathbf{y}.$$

Show that

$$X^*(p) \in \arg\max_{x \in \mathbb{R}} px - c(x).$$

is increasing in the strong set order.

Exercise 5. (LeChatelier's Principle) Let Θ be a poset. The function $F : \mathbb{R}^{d_1}_{++} \times \mathbb{R}^{d_2}_{++} \times \Theta \mapsto \mathbb{R}$ has increasing differences in $((x, y); \theta)$ and, for each $\theta \in \Theta$, $F(\cdot, \theta)$ is a supermodular function. Suppose that

$$(x', y') \in \arg\max_{(x,y) \in \mathbb{R}^{d_1}_{++} \times \mathbb{R}^{d_2}_{++}} F(x, y, \theta').$$

Let $\theta'' > \theta'$ and assume that the sets

$$Z^*(y', \theta'') := \arg\max_{x \in \mathbb{R}^{d_1}_{++}} F(x, y', \theta''),$$
$$Z^{**}(\theta'') := \arg\max_{(x,y) \in \mathbb{R}^{d_1}_{++} \times \mathbb{R}^{d_2}_{++}} F(x, y, \theta'')$$

are nonempty and compact. Show that there is x^* and (x^{**}, y^{**}) with the following properties:

(i) $x^* \in Z^*(y', \theta'')$,

(ii) $(x^{**}, y^{**}) \in Z^{**}((\theta''))$,

(iii) $x' \leq x^* \leq x^{**}$,

(iv) $y' \leq y^{**}$.

Hint: To prove this, you should choose (x^{**}, y^{**}) to be the largest element in $Z^{**}(\theta'')$. Show first that $y^{**} \geq y'$. Proving $x' \leq x^* \leq x^{**}$ is a little more delicate.

Remark 2. LeChatelier's Principle is idea that firms react more to input price changes in the long-run than the short run: Let $x = L$ (labour input) and $y = K$ (capital input); in the short run following the increase in a parameter θ (e.g., wages), the firm cannot change its capital level K', but is free to vary L, so it chooses L^* instead of L'; in the long run, the firm is free to vary K as well, and the choice of K increases from K' to K^{**}, with the choice of L increasing again from L^* to L^{**}. The exercise tells us that for LeChatelier's Principle to hold, we need K and L to be complements (because $F(\cdot, \theta)$ is supermodular) and for the MRS between K and L to be monotone with θ.