ECON 6170 Problem Set 10 Answers

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Exercise 1.

(i) u is continuous and Γ is compact, so we can apply the extreme-value theorem.

(ii) Suppose $x \in \operatorname{int}\Gamma$. Then $px_1 + x_2 < m$. In particular, $px_1 + x_2 + \varepsilon \leq m$ for sufficiently small ε. But $x_1^{\frac{1}{2}} + (x_2 + \varepsilon)^{\frac{1}{2}} > x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}}$, so x is not a solution.

(iii) The Lagrangian ignoring the nonnegativity constraints is

$$x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}} + \mu(m - px_1 - x_2)$$

Setting the gradient equal to 0, we obtain

$$\frac{1}{2x_1^{\frac{1}{2}}} - \mu p = 0 = \frac{1}{2x_2^{\frac{1}{2}}} - \mu$$

and the budget constraint at equality. Rearranging, we get

$$px_1^{\frac{1}{2}} = x_2^{\frac{1}{2}}$$

or

$$x_2 = p^2 x_1$$

Plugging this into the budget constraint at equality, we get

$$(p + p^2)x_1 = m$$

or

$$x_1^* = \frac{m}{p + p^2}$$

and

$$x_2^* = \frac{pm}{1 + p}$$

The constraint qualification holds and a maximum does exist, so by Additional Exercise 1 on Problem Set 9, maximising u over the set of (x_1, x_2) that satisfy the FOCs for some λ, will give a solution to the problem with a budget constraint at equality and no nonnegativity constraints. In this case, only one pair (x_1, x_2) satisfies the FOCs, so this is the solution to this problem.

*Based on solutions provided by Professor Takuma Habu

We still need to show that this problem has the same solution as the inequality-constrained problem with nonnegativity constraints. It is WLOG to assume $p \geq 1$.[1] It then follows that it is weakly better to expend all income on good 2 than to expend all income on good 1. Expending all income on good 2 will give $u(0, m) = \sqrt{m}$. Note that

$$u(x_1^*, x_2^*) = \sqrt{m}\left(\sqrt{\frac{1}{p(1+p)}} + \sqrt{\frac{p}{1+p}}\right)$$

To show that $x_1 = 0$ or $x_2 = 0$ is not optimal, it suffices to show that the sum inside the parentheses exceeds 1. This follows from

$$\frac{1}{p(1+p)} + 2\sqrt{\frac{1}{(1+p)^2}} + \frac{p}{1+p} = \frac{2+p}{1+p} + \frac{1}{p(1+p)} > \frac{2+p}{1+p} > 1$$

Therefore, the nonnegativity constraints will be non-binding in the original problem, so we can safely ignore them. From the second part of this question, we also know that we can ignore the interior of the budget set. From the first part of this question we know that a maximiser lies somewhere in the budget set. Therefore, the maximiser of the equality-constrained problem ignoring the nonnegativity constraints will solve the original problem.

Exercise 2. Theorem 3 in Static Optimisation tells us that there exists $\mu^* \in \mathbb{R}^K$ and $\lambda^* \in \mathbb{R}^J$ such that

$$\lambda_j^* \geq 0 \text{ for all } j \tag{6}$$

$$\lambda_j^* g_j(x^*) = 0 \text{ for all } j \tag{7}$$

$$\nabla f(x^*) + \sum_{k=1}^{K} \mu_k^* \nabla h_k(x^*) + \sum_{j=1}^{J} \lambda_j^* \nabla g_j(x^*) = 0^\mathsf{T} \tag{8}$$

Note that (8) is just $\nabla_x \mathcal{L}(x, \mu, \lambda) = 0^\mathsf{T}$, which is just condition (i) in the question. Condition (ii) is just $h_k(x^*) = 0$ for all k, which we are given in the question. The first part of condition (iii) is satisfied by (6); the second part is just $g_j(x^*) \geq 0$ for all j, which we are given in the question; and the third part is satisfied by (7). Therefore, (x^*, μ^*, λ^*) is a critical point of $\mathcal{L}$ and thus $x^* \in S_X$. It follows that $f(x^\circ) \geq f(x^*)$. Moreover (ii) and the second part of (iii) imply that x° is in the constraint set. Therefore, x° is also a solution to (1).

Exercise 3.

(i) The constrained optimisation problem is

$$\max\ x_1 + x_2 \text{ st } m - p_1 x_1 - p_2 x_2, x_1, x_2 \geq 0$$

The associated Lagrangian is

$$\mathcal{L}(x, \lambda) = x_1 + x_2 + \lambda_1(m - p_1 x_1 - p_2 x_2) + \lambda_2 x_1 + \lambda_3 x_2$$

[1]Otherwise just define $p' := \frac{1}{p}$, $m' := \frac{m}{p}$, switch the labels of x_1 and x_2, and write $p' x_1 + x_2 = m'$.

(ii) If $m > p_1x_1 + p_2x_2$ then the consumer's utility can be increased by increasing either of x_1 or x_2 without violating any of the constraints. Therefore, at the optimal consumption bundle the budget constraint must bind.

$$Dg(x^*) = \begin{bmatrix} -p_1 & -p_2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

It cannot be the case that all of the constraints bind, as $m > 0$. Therefore, the binding constraints are either (a) the budget constraint alone, (b) the budget constraint and $x_1 \geq 0$, or (c) the budget constraint and $x_2 \geq 0$. The associated $Dg_E(x^*)$ has rank $|E|$ in all of these cases, given $p_1, p_2 > 0$. That is,

$$\text{rank} \begin{bmatrix} -p_1 & -p_2 \\ 1 & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} -p_1 & -p_2 \\ 0 & 1 \end{bmatrix} = 2$$

and

$$\text{rank} \begin{bmatrix} -p_1 & -p_2 \end{bmatrix} = 1$$

(iii) By the previous exercise, the critical points of the Lagrangian that maximise $f(x)$ over S_X solve this consumer's problem.

Exercise 4.

(i) The firm's profit maximisation problem is

$$\max\ py - w_1x_1 - w_2x_2 - w_3x_3 \text{ st } x_1(x_2 + x_3) - y \geq 0, y \geq 0, x_1 \geq 0, x_2 \geq 0, \text{ and } x_3 \geq 0$$

Because $p > 0$, the constraint $f(x_1, x_2, x_3) \geq y$ will bind, so we can replace y with $f(x_1, x_2, x_3)$:

$$\max\ px_1(x_2 + x_3) - w_1x_1 - w_2x_2 - w_3x_3 \text{ st } x_1 \geq 0, x_2 \geq 0, \text{ and } x_3 \geq 0$$

The associated Lagrangian is

$$\mathcal{L}(x, \lambda) = px_1(x_2 + x_3) - w_1x_1 - w_2x_2 - w_3x_3 + \lambda_1x_1 + \lambda_2x_2 + \lambda_3x_3$$

The equations defining the critical points are

(1)

$$\begin{aligned} 0^\mathsf{T} &= \nabla_x \mathcal{L}(x, \lambda) \\ &= \begin{bmatrix} p(x_2 + x_3) - w_1 + \lambda_1 & px_1 - w_2 + \lambda_2 & px_1 - w_3 + \lambda_3 \end{bmatrix} \end{aligned}$$

(2) $\lambda \geq 0$, $g(y, x) \geq 0$, and $\lambda^\mathsf{T} g(y, x) \geq 0$, where $g(y, x)$ is the vector of constraints.

(ii) Suppose, WLOG, that $w_2 \geq w_3$. Then (1) implies that $\lambda_2 \geq \lambda_3$, with strict inequality iff $w_2 > w_3$.

(a) First consider the case $w_2 > w_3$. Then $\lambda_2 > \lambda_3 \geq 0$, so by complementary slackness, $x_2 = 0$.

Suppose $x_3 > 0$. Then $\lambda_3 = 0$ by complementary slackness $\implies x_1 = w_3/p$ by FOC 3 $\implies \lambda_1 = 0$ by complementary slackness and $\lambda_2 = w_3 - w_2$ by FOC 2 $\implies x_3 = w_1/p$. Therefore,

$$(x_1, x_2, x_3, \lambda_1, \lambda_2, \lambda_3) = (w_3/p, 0, w_1/p, 0, w_3 - w_2, 0)$$

is a critical point of $\mathcal{L}$.

Suppose $x_3 = 0$. Then $\lambda_1 = w_1 > 0$ by FOC 1 $\implies x_1 = 0$ by complementary slackness $\implies \lambda_2 = w_2$ and $\lambda_3 = w_3$. Therefore,

$$(0, 0, 0, w_1, w_2, w_3)$$

is also a critical point of $\mathcal{L}$.

(b) Now consider the alternative case $w_2 = w_3$. Then $\lambda_2 = \lambda_3$.

Suppose $x_3 > 0$. Then $\lambda_2 = \lambda_3 = 0 \implies x_1 = w_2/p > 0$ by FOC 2 $\implies \lambda_1 = 0$ by complementary slackness $\implies x_2 + x_3 = w_1/p$. It follows that

$$(w_2/p, z, w_1/p - z, 0, 0, 0)$$

is a critical point for any $0 \leq z < w_1/p$. By the symmetry between x_2 and x_3, we can also include the case $z = w_1/p$.

Suppose $x_2 = x_3 = 0$. Then $\lambda_1 = w_1 > 0$ by FOC 1 $\implies x_1 = 0 \implies \lambda_2 = \lambda_3 = w_2$. It follows that

$$(0, 0, 0, w_1, w_2, w_2)$$

is also a critical point.

(iii) The profit maximisation problem does not have a solution, so none of the above critical points could be a solution. To see that the PMP does not have a solution, let $x_1 = x_2 = x_3 \to \infty$. Then profit is

$$\begin{aligned}
\lim_{x_1 \to \infty} 2px_1^2 - (w_1 + w_2 + w_3)x_1 &= \lim_{x_1 \to \infty} x_1 \cdot \lim_{x_1 \to \infty} (2px_1 - (w_1 + w_2 + w_3)) \\
&= \infty \cdot \infty \\
&= \infty
\end{aligned}$$