Econ 6190 Problem Set 10

Fall 2024

1. Take the model $X \sim \mathrm{N}(\mu, 4)$. A sample of $n = 16$ independent realizations of X was collected, and the sample mean $\bar{X} = 20.5$. A sample of $n = 16$ independent realizations of X was collected, and the sample mean $\bar{X} = 20.5$. Find a 98% confidence interval for μ.

2. [Hansen 14.4] You have the point estimate $\hat{\theta} = 0.45$ and standard errors $s(\hat{\theta}) = 0.28$. You are interested in $\beta = \exp(\theta)$.

   (a) Find $\hat{\beta}$.

   (b) Use the delta method to find a standard error $s(\hat{\beta})$.

   (c) Use the above to calculate a 95% asymptotic confidence interval for $\hat{\beta}$.

   (d) Calculate a 95% asymptotic confidence interval $[L, U]$ for the original parameter θ. Calculate a 95% asymptotic confidence interval for β as $[\exp(L), \exp(U)]$. Can you explain why this is a valid choice? Compare this interval with your answer in (c).

3. [Hansen] Answer the following questions.

   (a) A confidence interval for the mean of a variable X is $[L, U]$. You decided to rescale your data, so set $Y = \frac{X}{1000}$. Find the confidence interval for the mean of Y.

   (b) In general, let $C = [L, U]$ be a $1-\alpha$ confidence interval for θ. Consider $\beta = h(\theta)$ where $h(\theta)$ is mono tonically increasing. Set $C_\beta = [h(L), h(U)]$. Evaluate the converage probability of C_β for β. Is C_β a $1 - \alpha$ confidence interval?

4. Let the random variable X be normally distributed with mean μ and variance 1. You are given a random sample of 16 observations.

   (a) Construct a one sided 95% confidence interval for μ that has form $[\hat{L}, \infty)$ for some statistic $\hat{L}$.

   (b) Construct a two sided 95% confidence interval for μ.

   (c) Show that the rejection of the null $\mathbb{H}_0 : \mu = 0$ against $\mathbb{H}_1 : \mu \neq 0$ with size 5% based on t test corresponds to the rejection of $\mathbb{H}_0 : \mu = 0$ when zero does not lie in the 95% confidence interval for μ constructed in part (b).

   (d) How would your answers be affected when you would not have known the variance of the random variable?

(e) How would your answers be affected when you would not have known the variance of the random variable but the sample size is 100?

5. [Hansen] Answer the following questions.

   (a) A confidence interval for the mean of a variable X is $[L, U]$. You decided to rescale your data, so set $Y = \frac{X}{1000}$. Find the confidence interval for the mean of Y.

   (b) In general, let $C = [L, U]$ be a $1-\alpha$ confidence interval for θ. Consider $\beta = h(\theta)$ where $h(\theta)$ is monotonically increasing. Set $C_\beta = [h(L), h(U)]$. Evaluate the converge probability of C_β for β. Is C_β a $1-\alpha$ confidence interval?

6. [Hansen 14.7] A friend suggests the following confidence interval for θ: they draw a random number $U \sim U[0,1]$ and set

$$C = \begin{cases} \mathbb{R} & \text{if } U \leq 0.95 \\ \emptyset & \text{if } U > 0.95 \end{cases}.$$

   (a) What is the coverage probability of C?

   (b) Is C a good choice for a confidence interval? Explain.

Q1

Variance of $\bar{X}$ is $\text{var}(\bar{X}) = \frac{\sigma^2}{n} = \frac{4}{16} = \frac{1}{4}$. Hence $\text{stdev}(\bar{X}) = \frac{1}{2}$, and the $z_{1-1\%} = 2.33$. Hence a 98% confidence interval is

$$\begin{aligned} &\left[\bar{X} - z_{1-1\%}\text{stdev}(\bar{X}), \bar{X} + z_{1-1\%}\text{stdev}(\bar{X})\right] \\ =&[19.335, 21.665] \end{aligned}$$

Q4

(a) We know that $\frac{\bar{X}-\mu}{\sqrt{\frac{1}{16}}} \sim N(0,1)$. And since $z_{1-5\%} = 1.65$,

$$P\left\{\frac{\bar{X}-\mu}{\sqrt{\frac{1}{16}}} \leq 1.65\right\} = 0.95$$

i.e.,

$$P\left\{\mu \geq \bar{X} - \sqrt{\frac{1}{16}}1.65\right\} = 0.95$$

so that a one sided 95% confidence interval is given by $[\bar{X} - \frac{1.65}{4}, \infty)$.

(b) Similarly, since $z_{1-2.5\%} = 1.96$,

$$P\left\{-1.96 \leq \frac{\bar{X}-\mu}{\sqrt{\frac{1}{16}}} \leq 1.96\right\} = 0.95$$

i.e.,

$$P\left\{\bar{X} - 1.96\sqrt{\frac{1}{16}} \leq \mu \leq \bar{X} + 1.96\sqrt{\frac{1}{16}}\right\} = 0.95$$

so that the 95% confidence interval is given by $[\bar{X} - \frac{1.96}{4}, \bar{X} + \frac{1.96}{4}]$.

(c) If zero does not lie in the 95% confidence interval for μ constructed in part (b), that means

$$0 < \bar{X} - 1.96\sqrt{\frac{1}{16}}, \text{or } 0 > \bar{X} + 1.96\sqrt{\frac{1}{16}},$$

i.e.,

$$\frac{\bar{X}}{\sqrt{\frac{1}{16}}} > 1.96, \text{or } \frac{\bar{X}}{\sqrt{\frac{1}{16}}} < -1.96 \tag{1}$$

On the other hand, When we conduct the test of $\mu = 0$ at the 5% level of significance, we could use the statistic

$$T = \frac{\bar{X} - 0}{\sqrt{\frac{1}{16}}} \sim N(0,1) \text{ under } \mathbb{H}_0.$$

So our decision is to reject if

$$\frac{\bar{X}}{\sqrt{\frac{1}{16}}} > 1.96 \text{ or } \frac{\bar{X}}{\sqrt{\frac{1}{16}}} < -1.96,$$

which is the same as (1).

(d) We can use the exact distribution result

$$\frac{\bar{X} - \mu}{\sqrt{\frac{s^2}{16}}} \sim t_{n-1},$$

where $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$. The 95% confidence interval would then be given by $[\bar{X} - t_{15,1-2.5\%}\frac{s}{4}, \bar{X} + t_{15,1-2.5\%}\frac{s}{4}]$, where $t_{15,1-2.5\%}$ is the $(1-2.5\%)$-th quantile of t_{15} distribution.

(e) Since sample size 100 is large, even if we do not know σ^2, we could still use standard normal as an approximation for the distribution of $\frac{\bar{X}-\mu}{\sqrt{\frac{s^2}{100}}}$. The confidence interval $[\bar{X} - 1.96\frac{s}{10}, \bar{X} + 1.96\frac{s}{10}]$ is asymptotically valid.

Q 2

(a) By plugging in, we have $\hat{\beta} = \exp(\hat{\theta}) = \exp(0.45) \approx 1.57$

(b) By delta method,

$$\begin{aligned}\sqrt{n}(\hat{\beta} - \beta) &= \sqrt{n}(\exp(\hat{\theta}) - \exp(\theta)) \\ &\approx \sqrt{n}\exp(\tilde{\theta})(\hat{\theta} - \theta) \\ &\xrightarrow{d} \mathrm{N}(0, \exp(2\theta)V_\theta),\end{aligned}$$

where $\tilde{\theta}$ lie between $\hat{\theta}$ and θ, and V_θ is asymptotic variance of $\sqrt{n}(\hat{\theta} - \theta)$. Thus the standard error of $\hat{\beta}$ is given by

$$\begin{aligned}\sqrt{\frac{\exp(2\hat{\theta})\hat{V}_\theta}{n}} &= \exp(\hat{\theta})\sqrt{\frac{\hat{V}_\theta}{n}} = \exp(\hat{\theta})s(\hat{\theta}) \\ &= \exp(0.45) \cdot 0.28 \approx 0.44.\end{aligned}$$

(c) Based on (b), an 95% asymptotic confidence interval for β is given by

$$[\hat{\beta} - z_{1-2.5\%} \cdot 0.44, \hat{\beta} + z_{1-2.5\%} \cdot 0.44],$$

or approximately $[0.71, 2.43]$.

(d) 95% asymptotic confidence interval for θ is given by

$$[\hat{\theta} - z_{1-2.5\%} \cdot s\left(\hat{\theta}\right), \hat{\beta} + z_{1-2.5\%} \cdot s\left(\hat{\theta}\right)],$$

which is approximately $[-0.099, 0.999]$. Based on this, alternative confidence interval for β is given by

$$[\exp(L), \exp(U)] \approx [0.91, 2.71].$$

This is also asymptotically valid confidence interval because

$$\begin{aligned}
&P\{\exp(L) \le \beta \le \exp(U)\} \\
=&P\{\exp(L) \le \exp(\theta) \le \exp(U)\} \\
=&P\{L \le \theta \le U\} \\
\to&1 - \alpha.
\end{aligned}$$

But note $[\exp(L), \exp(U)]$ is longer than the length of the confidence interval we derived in (b) using delta method.

Q3 / Q5

(a) Confidence interval for $\mathbb{E}[Y]$ is given by $[\frac{L}{1000}, \frac{U}{1000}]$. Note

$$\begin{aligned}
&P\{\frac{L}{1000} \le \mathbb{E}[Y] \le \frac{U}{1000}\} \\
=&P\{\frac{L}{1000} \le \frac{\mathbb{E}[X]}{1000} \le \frac{U}{1000}\} \\
=&P\{L \le \mathbb{E}[X] \le U\}
\end{aligned}$$

so the coverage probability of $[\frac{L}{1000}, \frac{U}{1000}]$ for $\mathbb{E}[Y]$ is the same as the coverage probability of $[L, U]$ for $\mathbb{E}[X]$.

(b) Since h is monotonically increasing,

$$\begin{aligned}
1 - \alpha =&P\{\theta \in C\} \\
=&P\{L \le \theta \le U\} \\
=&P\{h(L) \le h(\theta) \le h(U)\} \\
=&P\{h(L) \le \beta \le h(U)\} \\
=&P\{\beta \in C_\beta\}.
\end{aligned}$$

Thus C_β is a valid confidence interval for β.

Q6

(a) Note that

$$\begin{aligned}
P\{\theta \in C\} &= 0.95 \cdot P\{\theta \in \mathbb{R}\} + 0.05 \cdot P\{\theta \in \emptyset\} \\
&= 0.95 \cdot 1 + 0.05 \cdot 0 \\
&= 0.95 \cdot 1.
\end{aligned}$$

So the coverage probability of C is 0.95.

(b) Although C is a valid 95% confidence interval, it does not use the data at all. So it is an odd construction. One way to criticize this confidence interval is to look at its expected length. Given the coverage, shorter is better. However, expected length of C is ∞, (because $0.95 \cdot \infty + 0.05 \cdot 0 = \infty$), i.e., infinity.