3 Production

3.1 Firm

Assumptions 3.1.

(i) L commodities

(ii) Production plan $y \in \mathbb{R}^L$

 (1) Net input: good i such that $y_i < 0$

 (2) Net output: good j such that $y_j > 0$

(iii) Production possibility set, $Y \subseteq \mathbb{R}^L$ of feasible production plans

(iv) Prices, $p \geq 0$, are unaffected by the activity of the firm.

We will also often assume:

Assumptions 3.2.

(i) Y is nonempty, closed and (strictly) convex.[11]

(ii) Free disposal: If $y \in Y$ and $y' \leq y$ then $y' \in Y$.

Definition 3.3. A production plan, $y \in Y$ is **efficient** if there does not exist $y' \in Y$ such that $y' \geq y$ and $y'_i > y_i$ for some i.

In the case of a single output, we partition y into output $q \in \mathbb{R}_+$ and inputs $z \in \mathbb{R}_+^{L-1}$. This allows us to define the following:

Definition 3.4. The **production function** $f\colon \mathbb{R}_+^{L-1} \to \mathbb{R}_+$ is defined by

$$\begin{aligned} f(z) &= \max q \\ &\quad \text{st } (q, -z) \in Y \end{aligned}$$

Definition 3.5. The **input requirement set**

$$V(q) := \{z \in \mathbb{R}_+^{L-1} \mid (q, -z) \in Y\}$$

gives all the input vectors that can be used to produce the output q.

[11]These properties are required for the general existence and/or uniqueness of the maximizers and minimizers defined in this section. In particular, strict convexity allows us to speak exclusively of demand and supply *functions*, rather than correspondences. An additional more technical property, the recession-cone property, is also required: see Kreps Proposition 9.7.

Definition 3.6. The **isoquant**

$$Q(q) := \{z \in \mathbb{R}_+^{L-1} \mid z \in V(q) \text{ and } z \notin V(q') \text{ for any } q' > q\}$$

gives all the input vectors that can be used to produce at most q units of output.

3.2 Cost minimization

Assumptions 3.7.

(i) $L-1$ inputs z

(ii) One output $q = f(z)$

(iii) $f \in C^2$

(iv) Input price $w \in \mathbb{R}_+^{L-1}$

Remark 3.8. Inputs with zero prices will not affect the decision-making of the firm and can thus be ignored.

The firm's **cost minimization problem** is

$$\begin{aligned} &\min_{z \in \mathbb{R}_+^{L-1}} w \cdot z \\ &\text{st } f(z) = q \end{aligned}$$

Definition 3.9. The associated value function is called the **cost function**:

$$\begin{aligned} C(w,q) := &\min_{z \in \mathbb{R}_+^{L-1}} w \cdot z \\ &\text{st } f(z) = q \end{aligned}$$

Proposition 3.10 (Properties of the cost function)**.**

(i) C is homogeneous of degree 1 in w.

(ii) C is concave in w.

(iii) If we assume free disposal, then C is nondecreasing in q.

(iv) If f is homogeneous of degree k in z, then C is homogeneous of degree $\frac{1}{k}$ in q.

**Proof.*

(i) Increasing w by a factor of α is a positive monotonic transformation and therefore does not affect the optimal choice of z, but does increase $w \cdot z$ by that factor.

(ii) Let $w, w' \in \mathbb{R}_+^{L-1}$. Suppose $C(w,q) = w \cdot z$ and $C(w',q) = w' \cdot z'$. Let $w'' = \alpha w + (1-\alpha)w'$ for some $\alpha \in [0,1]$. Then, for z'' a cost minimizer at w'',

$$\begin{aligned} C(w'',q) &= w'' \cdot z'' \\ &= (\alpha w + (1-\alpha)w') \cdot z'' \\ &= \alpha w \cdot z'' + (1-\alpha)w' \cdot z'' \end{aligned}$$

We know $w \cdot z'' \geq C(w,q)$ and $w' \cdot z'' \geq C(w',q)$. So $C(w'',q) \geq \alpha C(w,q) + (1-\alpha)C(w',q)$.

(iii) Suppose $q' > q$. By free disposal, q can be produced from the same input vector used to produce q'.

(iv) Homogeneity of degree k of f implies

$$f(z) = q \iff \frac{1}{q} f(z) = 1 \iff f\left(\frac{z}{q^{1/k}}\right) = 1$$

Therefore,

$$\begin{aligned} C(w,q) &= \min_z w \cdot z \text{ st } f(z) = q \\ &= \min_z w \cdot z \text{ st } f\left(\frac{z}{q^{1/k}}\right) = 1 \\ &= q^{1/k} \min_z w \cdot \frac{z}{q^{1/k}} \text{ st } f\left(\frac{z}{q^{1/k}}\right) = 1 \\ &= q^{1/k} C(w,1) \end{aligned}$$

□

3.3 Homogeneous functions

Definition 3.11. $f \colon X \subseteq \mathbb{R}^n \to \mathbb{R}$ is **homogeneous of degree k** if

$$f(\alpha x) = \alpha^k f(x)$$

where k is a nonnegative integer, for all $\alpha > 0, x \in X$

Proposition 3.12. *If f is homogeneous of degree k, then for $i = 1, 2, \ldots, n$, $\frac{\partial f}{\partial x_i}$ is homogeneous of degree $k-1$.*

**Proof.* Let $f_i := \frac{\partial f}{\partial x_i}$.

$$\begin{aligned}
f(\alpha x) &= \alpha^k f(x) && \text{hod k} \\
\alpha f_i(\alpha x) &= \alpha^k f_i(x) && \text{differentiating wrt } x_i \\
f_i(\alpha x) &= \alpha^{k-1} f_i(x) && \text{dividing by } \alpha \\
\implies \quad f_i(\alpha x) &\text{ is homogenous of degree } k-1
\end{aligned}$$

□

Proposition 3.13 (**Euler's formula**)**.** *If f is homogeneous of degree k and differentiable, then at any x*

$$\sum_{i=1}^{n} \frac{\partial f(x)}{\partial x_i} x_i = k f(x)$$

**Proof.*

$$\begin{aligned}
f(\alpha x) &= \alpha^k f(x) && \text{hod k} \\
\sum_{i=1}^{n} f_i(\alpha x) x_i &= k\alpha^{k-1} f_i(x) && \text{differentiating wrt } \alpha \\
\sum_{i=1}^{n} f_i(x) x_i &= k f_i(x) && \text{evaluating at } \alpha = 1
\end{aligned}$$

□

Proposition 3.14. *If the production function f is homogeneous of degree k, then*

$$\mathrm{MRTS}_{ij}(z) := \frac{\frac{\partial f(z)}{\partial z_i}}{\frac{\partial f(z)}{\partial z_j}} = \frac{\frac{\partial f(\alpha z)}{\partial z_i}}{\frac{\partial f(\alpha z)}{\partial z_j}} = \mathrm{MRTS}_{ij}(\alpha z)$$

Proof.

$$\frac{f_i(\alpha z)}{f_j(\alpha z)} = \frac{\alpha^{k-1} f_i(z)}{\alpha^{k-1} f_j(z)} = \frac{f_i(z)}{f_j(z)}$$

□

3.4 Profit maximization

The firm's **profit maximization problem** is

$$\max_y p \cdot y$$
$$\text{st } y \in Y$$

Definition 3.15. The associated value function is called the **profit function**:

$$\pi(p) := \max_y p \cdot y$$
$$\text{st } y \in Y$$

In the single-output case, this becomes

$$\pi(p, w) := \max_{z \in \mathbb{R}_+^{L-1}} pf(z) - w \cdot z$$

Henceforth, we consider only the single-output case.

Proposition 3.16 (Properties of the profit function)**.**

(i) *Homogeneous of degree* 1*.*

(ii) *Nondecreasing in output price* p*.*

(iii) *Nonincreasing in input prices* w*.*

(iv) *Convex in* (p, w)*.*

(v) *Continuous.*

**Proof.*

(i) $\max_z \alpha(pf(z) - w \cdot z) = \alpha \max_z pf(z) - w \cdot z$.

(ii) $p' \geq p \implies p'f(z) \geq pf(z)$ for all z.

(iii) $w' \geq w \implies w' \cdot z \geq w \cdot z$.

(iv) Let $(p'', w'') := \alpha(p, w) + (1 - \alpha)(p', w')$ and z, z', z'' be the solution to the profit maximization problem with the corresponding output prices and input price vectors. Then by definition of z and z',

$$\pi(p, w) = pf(z) - w \cdot z \geq pf(z'') - w \cdot z''$$
$$\pi(p', w') = p'f(z') - w' \cdot z' \geq p'f(z'') - w' \cdot z''$$

implying

$$\begin{aligned}\alpha\pi(p,w)+(1-\alpha)\pi(p',w') &\geq \alpha(pf(z'')-w\cdot z'')\\ &\quad+(1-\alpha)(p'f(z'')-w'\cdot z'')\\ &=(\alpha p+(1-\alpha)p')f(z'')\\ &\quad-(\alpha w+(1-\alpha)w')\cdot z''\\ &=\pi(p'',z'')\end{aligned}$$

(v) See Kreps Proposition 9.9. □

Remark 3.17. Note that π being convex in (p,w) implies that π is convex in p and w individually.

Definition 3.18. The **unconditional input demand function**

$$x(p,w) := \arg\max_{z\in\mathbb{R}_+^{L-1}} pf(z)-w\cdot z$$

is the solution to the profit maximization problem. The **output supply function**

$$q(p,w) := f(x(p,w))$$

is the output level when the profit is maximized.

Proposition 3.19 (**Hotelling's lemma**). *If π is differentiable,*[12] *then for* $(p,w)\in\mathbb{R}^L_{++}$,

$$\begin{aligned}q(p,w) &= \frac{\partial\pi(p,w)}{\partial p}\\ x_j(p,w) &= -\frac{\partial\pi(p,w)}{\partial w_j}\end{aligned}$$

Proof. Apply the Envelope Theorem and note that $x(p,w)$ is the profit maximizer and $q(p,w)$ is the production function evaluated at the maximizer. □

[12] In fact, if the output supply and unconditional demand functions are well-defined – or equivalently, the associated correspondences are singleton-valued – then π is necessarily differentiable. See Kreps Proposition 9.22. An analogous result holds for the cost function: see Kreps Proposition 9.24j.

Definition 3.20. The **conditional input demand function**

$$z(w,q) := \arg\min_{z\in\mathbb{R}_+^{L-1}} w\cdot z$$
$$\text{st } f(z) = q$$

is the solution to the cost minimization problem.

Proposition 3.21 (**Shepard's lemma**)**.** *If C is differentiable, then for $w \in \mathbb{R}_{++}^{L-1}$,*

$$z_i(w,q) = \frac{\partial C(w,q)}{\partial w_i}$$

Proof. Similarly, apply the Envelope Theorem to the cost function (the value function of the cost minimization problem). Note that the Envelope Theorem also holds true for minimization problems. Equivalently, we can rewrite

$$-C(w,q) := \max_{z\in\mathbb{R}_+^{L-1}} -w\cdot z \text{ st } f(z) = q.$$

and apply the regular Envelope Theorem. $\square$

Proposition 3.22. *Suppose that the profit function is twice continuously differentiable. Then*

(i)

$$\frac{\partial q(p,w)}{\partial p} \geq 0$$

(ii)

$$\frac{\partial x_j(p,w)}{\partial w_j} \leq 0$$

(iii)

$$\frac{\partial x_j(p,w)}{\partial w_i} = \frac{\partial x_i(p,w)}{\partial w_j}$$

Proof. By applying Hotelling's lemma, note that

$$D^2\pi(p,w) = \begin{bmatrix} \frac{\partial q(p,w)}{\partial p} & \frac{\partial q(p,w)}{\partial w_1} & \cdots & \frac{\partial q(p,w)}{\partial w_n} \\ -\frac{\partial x_1(p,w)}{\partial p} & -\frac{\partial x_1(p,w)}{\partial w_1} & \cdots & -\frac{\partial x_1(p,w)}{\partial w_n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\partial x_n(p,w)}{\partial p} & -\frac{\partial x_n(p,w)}{\partial w_1} & \cdots & -\frac{\partial x_n(p,w)}{\partial w_n} \end{bmatrix}$$

is symmetric and positive semidefinite because the profit function π is twice continuously differentiable and convex. Then, (i) and (ii) follows from the fact that a positive semidefinite matrix has nonnegative diagonal entries, and (iii) follows from symmetry. $\square$

Proposition 3.23. *Suppose that the cost function is twice continuously differentiable. Then*

(i)

$$\frac{\partial z_i(w,q)}{\partial w_i} \leq 0$$

(ii)

$$\frac{\partial z_i(w,q)}{\partial w_j} = \frac{\partial z_j(w,q)}{\partial w_i}$$

(iii)

$$\frac{\partial MC(w,q)}{\partial w_i} = \frac{\partial z_i(w,q)}{\partial q} \implies \begin{cases} > 0 & \textit{Normal Input} \\ < 0 & \textit{Inferior Input} \end{cases}$$

where $MC(w,q) = \frac{\partial C(w,q)}{\partial q}$.

Proof. Using Shepard's lemma, write the Hessian of C as

$$D^2C(w,q) = \begin{bmatrix} \frac{\partial MC(w,q)}{\partial q} & \frac{\partial MC(w,q)}{\partial w_1} & \cdots & \frac{\partial MC(w,q)}{\partial w_n} \\ \frac{\partial z_1(w,q)}{\partial q} & \frac{\partial z_1(w,q)}{\partial w_1} & \cdots & \frac{\partial z_1(w,q)}{\partial w_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial z_n(w,q)}{\partial q} & \frac{\partial z_n(w,q)}{\partial w_1} & \cdots & \frac{\partial z_n(w,q)}{\partial w_n} \end{bmatrix}$$

Then, (ii) and (iii) follow from the symmetry of the second derivatives. Since C is concave in w, the sub-matrix

$$\begin{bmatrix} \frac{\partial z_1(w,q)}{\partial w_1} & \cdots & \frac{\partial z_1(w,q)}{\partial w_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial z_n(w,q)}{\partial w_1} & \cdots & \frac{\partial z_n(w,q)}{\partial w_n} \end{bmatrix}$$

is negative semidefinite and its diagonal entries must be nonpositive. This proves (i). $\square$

3.5 Comparative Statics

Assumptions 3.24.

(i) Two inputs (x_1, x_2)

(ii) One output $q = f(x)$

(iii) $f \in C^2$ and the Hessian, H_f, is negative definite

(iv) $f(0, x_2) = f(x_1, 0) = 0$, *i.e.*, both inputs necessary

(v) Inada conditions on x_1, x_2

(vi) Output price $p > 0$

(vii) Input price $w \gg 0$

Consider the profit maximization problem:

$$\max_{x \in \mathbb{R}^2_{++}} pf(x) - w \cdot x$$

The first order conditions are

$$\begin{aligned} pf_1(x) - w_1 &= 0 \\ pf_2(x) - w_2 &= 0 \end{aligned}$$

Since the Lagrangian is strictly concave, the first order conditions are sufficient. To determine the sign of $\frac{\partial x_1(p,w)}{\partial w_1}$, we apply the Implicit Function Theorem. Since H_f is negative definite,

$$H(x) = \begin{bmatrix} pf_{11}(x) & pf_{12}(x) \\ pf_{21}(x) & pf_{22}(x) \end{bmatrix}$$

has strictly positive determinant. This satisfies the condition for the IFT, so there exists an implicit function

$$x(p, w) = (x_1(p, w), x_2(p, w))$$

which is C^1 near (x, p, w). Writing x as an implicit function of (p, w), we have

$$\begin{aligned} pf_1(x(p, w)) - w_1 &= 0 \\ pf_2(x(p, w)) - w_2 &= 0 \end{aligned}$$

Taking the derivative with respect to w_1 gives

$$\begin{aligned} pf_{11}\frac{\partial x_1}{\partial w_1} + pf_{12}\frac{\partial x_2}{\partial w_1} - 1 &= 0 \\ pf_{21}\frac{\partial x_1}{\partial w_1} + pf_{22}\frac{\partial x_2}{\partial w_1} &= 0 \end{aligned}$$

Writing it in matrix form,

$$\begin{bmatrix} pf_{11} & pf_{12} \\ pf_{21} & pf_{22} \end{bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial w_1} \\ \frac{\partial x_2}{\partial w_1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Since the first matrix is non-singular, invert it and find that

$$\frac{\partial x_1}{\partial w_1} = \frac{pf_{22}}{|H(x)|} < 0$$
$$\frac{\partial x_2}{\partial w_1} = \frac{-pf_{12}}{|H(x)|}$$

where the inequality is from the negative definiteness of H_f. Thus we have shown that the demand for an input always decreases with its price.

To determine the effect of a price change on output, *i.e.*, the sign of $\frac{\partial q}{\partial w_1}$, we write

$$q(p, w) = f(x(p, w))$$

and take the derivative with respect to w_1:

$$\begin{aligned} \frac{\partial q}{\partial w_1} &= f_1 \frac{\partial x_1}{\partial w_1} + f_2 \frac{\partial x_2}{\partial w_1} \\ &= \frac{p(f_1 f_{22} - f_2 f_{12})}{|H(x)|} \end{aligned}$$

where the sign depends on the term $f_1 f_{22} - f_2 f_{12}$. To find this, we consider the cost minimization problem:

$$\min_{x \in \mathbb{R}^2_{++}} \; w \cdot x$$
$$\text{st } f(x) = q$$

The first order conditions are

$$\begin{aligned} -w_1 + \lambda f_1(x) &= 0 \\ -w_2 + \lambda f_2(x) &= 0 \\ q - f(x) &= 0 \end{aligned}$$

where $\lambda(w, q)$ is the Lagrangian multiplier. Taking the derivative with respect to q gives:

$$\begin{aligned} \frac{\partial \lambda}{\partial q} f_1 + \lambda \frac{\partial f_1}{\partial x_1} \frac{\partial x_1}{\partial q} + \lambda \frac{\partial f_1}{\partial x_2} \frac{\partial x_2}{\partial q} &= 0 \\ \frac{\partial \lambda}{\partial q} f_2 + \lambda \frac{\partial f_2}{\partial x_1} \frac{\partial x_1}{\partial q} + \lambda \frac{\partial f_2}{\partial x_2} \frac{\partial x_2}{\partial q} &= 0 \\ 1 - \frac{\partial f_1}{\partial x_1} \frac{\partial x_1}{\partial q} - \frac{\partial f_1}{\partial x_2} \frac{\partial x_2}{\partial q} &= 0 \end{aligned}$$

Writing it in matrix form,

$$\begin{bmatrix} \lambda f_{11} & \lambda f_{12} & f_1 \\ \lambda f_{21} & \lambda f_{22} & f_2 \\ f_1 & f_2 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial q} \\ \frac{\partial x_2}{\partial q} \\ \frac{\partial \lambda}{\partial q} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

where the first matrix is the Hessian of the Lagrangian, $H_c(x)$, and is thus invertible. By Cramer's Rule,

$$\begin{aligned} \frac{\partial x_1}{\partial q} &= \frac{\begin{vmatrix} 0 & \lambda f_{12} & f_1 \\ 0 & \lambda f_{22} & f_2 \\ 1 & f_2 & 0 \end{vmatrix}}{|H_c(x)|} \\ &= \frac{\lambda(f_{12}f_2 - f_{22}f_1)}{|H_c(x)|} \quad \begin{cases} > 0 & \text{Normal Input} \\ < 0 & \text{Inferior Input} \end{cases} \end{aligned}$$

where λ and $|H_c(x)|$ are strictly positive. Combined with the result from the profit maximization problem, we conclude:

(i) If input 1 is normal, $\frac{\partial x_1}{\partial q} > 0$, then $f_{12}f_2 - f_{22}f_1 > 0$ and $\frac{\partial q}{\partial w_1} < 0$.

$$w_1 \uparrow \implies q \downarrow \implies x_1 \downarrow$$

(ii) If input 1 is inferior, $\frac{\partial x_1}{\partial q} < 0$, then $f_{12}f_2 - f_{22}f_1 < 0$ and $\frac{\partial q}{\partial w_1} > 0$.

$$w_1 \uparrow \implies q \uparrow \implies x_1 \downarrow$$

In either case, this reinforces the substitution effect where x_1 necessarily decreases when w_1 increases, keeping output level q fixed.

3.6 Duality

Fix an output level q and suppose we observe $C(w, q)$ for all $w \gg 0$. We can recover an "outer bound" of the (unobserved) input requirement set,

$$V^*(q) := \{x \in \mathbb{R}_+^{L-1} \mid w \cdot x \geq C(w, q) \text{ for all } w \in \mathbb{R}_{++}^{L-1}\}$$

Proposition 3.25. *$V^*(q)$ is convex.*

Proof. Suppose $x, x' \in V^*(q)$. Let $\alpha \in [0,1]$ and $x'' := \alpha x + (1-\alpha)x'$. We want to show that $x'' \in V^*(q)$. Since $x \in V^*(q)$, $w \cdot x \geq C(w,q)$ for all $w \in \mathbb{R}^{L-1}_{++}$ and similarly $x' \in V^*(q)$ implies $w \cdot x' \geq C(w,q)$ for all $w \in \mathbb{R}^{L-1}_{++}$. Then

$$w \cdot x'' = \alpha w \cdot x + (1-\alpha) w \cdot x' \geq C(w,q)$$

Thus $x'' \in V^*(q)$. □

Remark 3.26. This doesn't imply that the true input requirement set $V(q)$ is convex, but it does imply that the non-convex part of $V(q)$ is not economically relevant since a cost-minimizing firm would never choose something in that region of $V(q)$.

Proposition 3.27 (Relationship between $V(q)$ and $V^*(q)$)**.**

(i) *$V(q) \subseteq V^*(q)$.*

(ii) *If $V(q)$ is closed, convex and comprehensive upward,*[13] *then $V(q) = V^*(q)$.*

**Proof.*

(i) Suppose $x \notin V^*(q)$. We want to show $x \notin V(q)$. If $x \notin V^*(q)$ then there exists some $w \in \mathbb{R}^{L-1}_{++}$ such that $w \cdot x < C(w,q)$. If $x \in V(q)$ then $C(w,q)$ is not the minimum, contradicting the definition of C.

(ii) Suppose not. In particular, suppose $x \in V^*(q)$ and $x \notin V(q)$. $V(q)$ and $\{x\}$ are both closed, convex, disjoint, nonempty subsets of $\mathbb{R}^{L-1}$ and $\{x\}$ is compact. Applying a version of the separating hyperplane theorem,[14] we obtain $w^* \neq 0$ such that $w^* \cdot x < w^* \cdot x'$ for all $x' \in V(q)$. In particular, $w^* \cdot x < C(w^*, q)$, which contradicts the definition of $V^*(q)$. We also want to show that $w^* \geq 0$. Suppose instead that for some i, $w^*_i < 0$. Because $V(q)$ is comprehensive upward, this implies we can choose $x' \in V(q)$ with x_i sufficiently large that $w^* \cdot x' < w^* \cdot x$, contradicting our choice of w^*.

□

Now, let

$$C^*(w,q) := \min_{x \in V^*(q)} w \cdot x$$

[13] $V(q)$ is comprehensive upward if $x \in V(q)$ and $x' \geq x$ imply $x' \in V(q)$. That is, the same output can always be produced using more input. If Y has the free disposal property, then for all q, $V(q)$ is comprehensive upward. The converse is not true. See Kreps Proposition 9.23c.

[14] Covered in the math class.

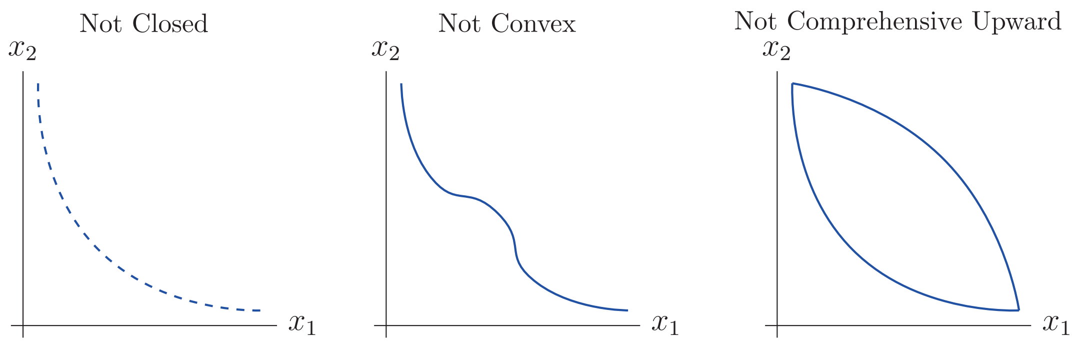

Figure 1: Three classes of $V(q)$ for which $V(q) \neq V^*(q)$

Proposition 3.28.

$$C^*(w,q) = C(w,q)$$

Proof. $V(q) \subseteq V^*(q)$ implies $C(w,q) \geq C^*(w,q)$. Suppose that for some $\bar{w} \in \mathbb{R}^{L-1}_{++}$, we have $C^*(\bar{w},q) = \bar{w} \cdot \bar{x} < C(\bar{w},q)$. Then $\bar{x} \notin V^*(q)$ which contradicts the definition of C^*. This implies $C^*(w,q) \geq C(w,q)$ for all $w \in \mathbb{R}^{L-1}_{++}$. Combining both inequalities, we have $C^*(w,q) = C(w,q)$. $\square$