ECON 6170

Problem Set 10

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Worked with Wanxi Zhou and Fenglin Ye on Exercise 4.

Exercise 1. Consider the problem of maximizing $u : \mathbb{R}^2 \to \mathbb{R}$ given by $u(x_1, x_2) \coloneqq x_1^{0.5} + x_2^{0.5}$ subject to the budget constraint, *i.e.*

$$\Gamma \coloneqq \left\{(x_1, x_2) \in \mathbb{R}^2 : px_1 + x_2 \leq m; x_1, x_2 \geq 0\right\}$$

where $p, m > 0$

(i) Prove that a solution to the utility maximization problem exists.

Proof. Note that the partials of u are

$$\frac{\partial u}{\partial x_1} = \frac{0.5}{x_1^{0.5}} \qquad \text{and} \qquad \frac{\partial u}{\partial x_2} = \frac{0.5}{x_2^{0.5}}$$

and the Hessian is

$$H_f = \begin{bmatrix} -\frac{0.25}{x_1^{1.5}} & 0 \\ & \\ 0 & -\frac{0.25}{x_2^{1.5}} \end{bmatrix}$$

Thus, since this is a diagonal matrix, the eigenvalues are negative and it is negative definite. Additionally, the constraint functions are all affine and therefore concave. It remains to show that $x^\star$ and $\lambda^\star$ exist that satisfy the KKT conditions. From the KKT Theorem, it suffices to show that the constraint qualification holds. Since u is strictly increasing in x_1 and x_2, the non-negativity constraints will not bind. Thus, the only binding constraint is $g(x) = px_1 + x_2 \leq m \equiv m - px_1 - x_2 \geq 0$. We have that $Dg(x) = \begin{bmatrix} -p & -1 \end{bmatrix}$, and since this is a 2×1 matrix, it has rank 1. Thus, the constraint qualification holds.

Thus, by the sufficiency of concavity to KKT, there exists $x^\star$ that satisfies the KKT conditions and $x^\star$ is a global maximum. $\square$

(ii) Prove that a solution must lie on the boundary of the set Γ.

Proof. FSOC, assume that the global maximum $x^\star$ is such that $x^\star \in \operatorname{int}\Gamma$. Since interiors of sets are open, $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(x^\star) \subseteq \operatorname{int}\Gamma$. However, there exists $x' \in B_\varepsilon(x^\star)$ such that $x_1' > x_1^\star$ and $x_2' > x_2^\star$. Since u is strictly increasing in both inputs, $u(x') > u(x^\star)$, which contradicts the fact that $x^\star$ is a global maximum. Thus, since the global maximum $x^\star \in \Gamma$ and $x^\star \notin \operatorname{int}\Gamma$, it must be that $x^\star \in \partial\Gamma$. $\square$

(iii) Solve the Lagrangian as an equality-constrained one while ignoring the nonnegativity constraints. Does the solution to the Lagrangian identify a solution to the original problem? Why or why not?
Solution. We have that the new Lagrangian is

$$\mathcal{L} = x_1^{0.5} + x_2^{0.5} + \lambda(m - px_1 - x_2)$$

The first order conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= \frac{0.5}{x_1^{0.5}} - p\lambda = 0 && \implies \lambda = \frac{0.5}{px_1^{0.5}} \\
\frac{\partial \mathcal{L}}{\partial x_2} &= \frac{0.5}{x_2^{0.5}} - \lambda = 0 && \implies \lambda = \frac{0.5}{x_2^{0.5}} \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= m - px_1 - x_2 = 0 && \implies m = px_1 + x_2
\end{aligned}$$

Combining, we get that

$$x_2 = p^2 x_1 \implies x_1^\star = \frac{m}{p + p^2}$$

and thus,

$$x_2^\star = \frac{pm}{1+p}$$

Note that the first order conditions are not zero at $x^\star$, since $\lambda^\star = -\frac{0.5(1+p)}{\sqrt{pm}} \neq 0$. However, this is still a solution of the primal problem. Notice that

$$px_1^\star + x_2^\star = \frac{m + pm}{1+p} = m$$

meaning that $x^\star \in \partial\Gamma$. This is a corner solution to the original problem, and does maximize it.

Exercise 2. Show that if the primal problem attains a global maximum at some $x^\star \in \mathbb{R}^d$ such that $h_k(x^\star) = 0$ for all k, $g_j(x^\star) \geq 0$ for all j, and the constraint qualification holds at $x^\star$, then an $x^\circ \in S_X$ that solves the other problem is also a global maximum.

Proof. We have that f, h_k, g_j are $\mathbf{C}^1$, and we have that $x^\star$ solves the problem

$$\max_{x \in \mathbb{R}^d} f(x) \text{ s.t. } h_k(x) = 0 \text{ and } g_j(x) \geq 0 \; \forall \; k, j$$

Since $x^\star$ is a local maximum given that it is also a global maximum, we have that, from KKT with equality and inequality constraints (Theorem 3 in the notes), and the fact that the constraint qualification is met, that there exist $\mu^\star \in \mathbb{R}^K$ and $\lambda^\star \in \mathbb{R}^J$ such that

$$\lambda_j^\star \geq 0 \; \forall \; j \tag{1}$$

$$\lambda_j^\star g_j(x^\star) = 0 \; \forall \; j \tag{2}$$

$$\nabla f(x^\star) + \sum_{k=1}^{K} \mu_k^\star \nabla h_k(x^\star) + \sum_{j=1}^{J} \lambda_j^\star \nabla g_j(x^\star) = 0 \tag{3}$$

We will show that $(x^\star, \mu^\star, \lambda^\star)$ is a critical point of the Lagrangian. Note that (i) is met immediately by (3). For (ii), note that $\frac{\partial \mathcal{L}}{\partial \mu_k} = h_k(x^\star)$ for all k, and since $x^\star$ solves the problem, $h_k(x^\star) = 0 \; \forall \; k$. Finally, for (iii), note that $\frac{\partial \mathcal{L}}{\partial \lambda_j} = g_j(x^\star)$. From the conditions of the primal problem, $\frac{\partial \mathcal{L}}{\partial \lambda_j} \geq 0$ for all j, and from (1) we have that $\lambda_j^\star \geq 0$ for all j. Proof that $(x^\star, \mu^\star, \lambda^\star)$ is a critical point of the Lagrangian follows from (2).

Since $(x^\star, \mu^\star, \lambda^\star) \in S$, we have that $x^\star \in S_X$. Thus, for any x° that is a global maximum of

$$\max_{x \in S_X} f(x)$$

we will have that $f(x^\circ) \geq f(x^\star)$. It remains only to show that $h_k(x^\circ) = 0 \; \forall \; k$ and that $g_j(x^\circ) \geq 0 \; \forall \; j$. Both conclusions follow from the above observations that $\frac{\partial \mathcal{L}}{\partial \mu_k} = h_k(x^\circ)$ for all k and that $\frac{\partial \mathcal{L}}{\partial \lambda_j} = g_j(x^\circ)$ for all j. Since x° is a critical point by definition, the quantities are zero and non-negative respectively. Thus, x° is feasible in the primal problem, and since $f(x^\circ) \geq f(x^\star)$, x° is a global maximum of the primal problem. $\square$

Exercise 3. Consider the consumer's problem of maximizing $u : \mathbb{R}^2 \to \mathbb{R}$ given by $u(x_1, x_2) := x_1 + x_2$ subject to the budget set

$$B(p_1, p_2, m) := \left\{(x_1, x_2) \in \mathbb{R}^2_+ : m - p_1x_1 - p_2x_2 \geq 0\right\}$$

where $p_1, p_2, m > 0$.

(i) The constrained optimization problem is

$$\max_{x \in \mathbb{R}^2_+} x_1 + x_2 \text{ s.t. } m - p_1x_1 - p_2x_2 \geq 0$$

and the Lagrangian is

$$\mathcal{L}(x, \lambda) = x_1 + x_2 + \lambda(m - p_1x_1 - p_2x_2)$$

(ii) First, note that the optimum must be on the border of the budget set. To see why, consider FSOC an optimal $x^\star \in \operatorname{int} B$. Then it must be the case that $\exists\ \varepsilon > 0$ s.t. $B_\varepsilon(x^\star) \subseteq B$, since interiors are open. However, $\exists\ x' \in B_\varepsilon(x^\star)$ where $x_1' > x_1^\star$ and $x_2' > x_2^\star$. It would be the case that $u(x') > u(x^\star)$ by definition, which is a contradiction of the fact that $x^\star$ is optimal. Thus, $x^\star \in \partial B$. Finally, it's clear that if the budget isn't entirely exhausted, then utility could be improved by spending more budget on at least one good. Thus, $x^\star$ is such that $m - p_1x_1^\star - p_2x_2^\star = 0$

Since $p_1, p_2, m > 0$, this means that at least one element of $x^\star$ is strictly positive. WLOG, say that $x_1^\star > 0$. Then, at least one non-negativity constraint does not hold with equality, and we have that

$$\operatorname{rank}(Dg_k(x)) = \operatorname{rank}\left(\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}\right) = 2 = |E|$$

(iii) By inspection, note that the first order conditions of the Lagrangian eliminate any x terms, so it would be impossible to isolate optimal x using them. In fact, there are no critical points – since the utility function is locally non-satiated, it has no critical points on the entire domain, let alone the feasible set.

Exercise 4. Suppose a firm's production function is given by $f : \mathbb{R}^3 \to \mathbb{R}$, where

$$f(x_1, x_2, x_3) := x_1(x_2 + x_3)$$

The unit price of firm's output is $p > 0$ and the input prices are $w_i > 0$ for $i \in \{1, 2, 3\}$.

(i) The firm's profit maximization problem is

$$\max_{q, x \in \mathbb{R}^1_+ \times \mathbb{R}^3_+} p \cdot q - w \cdot x \text{ s.t. } q = x_1(x_2 + x_3)$$

which simplifies to the problem

$$\max_{x \in \mathbb{R}^3} p \cdot (x_1(x_2 + x_3)) - w \cdot x \text{ s.t. } x \geq 0$$

The Lagrangian is

$$\mathcal{L}(x, \lambda) = p \cdot (x_1(x_2 + x_3)) - w \cdot x - \lambda \cdot x$$

for $\lambda \in \mathbb{R}^3_+$. Since the firm needs positive production, we need that $x_1 > 0$, and at least one of x_2, x_3 must be positive. WLOG, assume that $x_2 > 0$ as well, so those constraints don't bind. The Lagrangian becomes

$$\mathcal{L}(x, \lambda_3) = p \cdot (x_1(x_2 + x_3)) - w \cdot x - \lambda_3 x_3$$

and the first order conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= p(x_2 + x_3) - w_1 = 0 && \Longrightarrow x_2^\star + x_3^\star = \frac{w_1}{p} \\
\frac{\partial \mathcal{L}}{\partial x_2} &= px_1 - w_2 = 0 && \Longrightarrow x_1^\star = \frac{w_2}{p} \\
\frac{\partial \mathcal{L}}{\partial x_3} &= px_1 - w_3 - \lambda_3 = 0 && \Longrightarrow x_1^\star = \frac{w_3}{p} + \lambda_3 \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= x_3^\star = 0
\end{aligned}$$

(ii) We have that $x_1^\star = \frac{w_2}{p}$, that $x_2^\star + x_3^\star = \frac{w_1}{p}$, and that $x_3^\star = 0$. Thus, for any choice of (p, w), we have that there is a critical point of the form

$$x^\star = \left(\frac{w_2}{p}, \frac{w_1}{p}, 0\right)$$

However, our assumption earlier that $x_2^\star > 0$ was WLOG, so we can change it to an assumption that $x_3^\star > 0$, and get another critical point of the form

$$x^\star = \left(\frac{w_2}{p}, 0, \frac{w_1}{p}\right)$$

(iii) Fix some $(p, w) \in \mathbb{R}^4_{++}$, and consider a point $x^\star$ of the two identified above. Let's say that $x_2^\star = \frac{w_1}{p}$ and $x_3^\star = 0$. We have that the attained profit is

$$\pi(p, w) = p \cdot \frac{w_1 w_2}{p^2} - \frac{w_1 w_2}{p} - \frac{w_2 w_1}{p} - 0 = -\frac{w_1 w_2}{p} < 0$$

This is negative, but by choosing to produce $f(x) = 0$, the firm could attain zero profit, which would be a strict improvement. Note that this also holds for the other critical point, so neither are optimal.