Optional Problem Set 12

Due: N/A

1 Exercises from class notes

All from "8. Fixed Point Theorems.pdf".

Exercise 1. Complete the proof of Theorem 1; i.e., show that there is a smallest fixed point and any nonempty subset of fixed points has a supremum in the set of all fixed points.

Solution 1. Define $Z' := \{x \in X : x \geq f(x)\}$. Since X is complete, $\sup X \in X$ and because f is a self-map on X, $\sup X \geq f(x)$ for all $x \in X$. In particular, $\sup X \geq f(\sup X)$ so that $\sup X \in Z'$; i.e., Z' is nonempty. Since $Z' \subseteq X$, by completeness of X, $\inf Z' \in X$ and by definition, $z' \geq \inf Z'$ for all $z' \in Z$. Since f is increasing and by definition of Z', we must have

$$f\left(z'\right) \geq f\left(\inf Z'\right) \geq z\ \forall z' \in Z'.$$

Therefore, $f(\inf Z')$ is a lower bound of Z'. By definition, $\inf Z'$ is the greatest lower bound of Z' and so $\inf Z' \geq f(\inf Z')$. Since f is increasing, we also have $f(\inf Z') \geq f(f(\inf Z'))$; i.e., $f(\inf Z') \in Z'$. By definition $\inf Z'$, it follows that $f(\inf Z') \geq \inf Z'$. Hence, $\inf Z'$ is a fixed point. This must also be the smallest fixed point because any fixed point must be contained in Z'.

Solution 1. (iv) Let $\mathcal{E} \subseteq X$ be the set of fixed points of f (which we already showed is nonempty) and fix any nonempty subset $E \subseteq \mathcal{E}$. Define $Y' := \{x \in X : x \geq \sup E\}$ (set of upper bounds of E). We proceed as follows: (1) show that Y' is a complete lattice; (2) f restricted to Y', denoted $f|_{Y'}$, is a self-map on Y'; (3) conclude from part (iii) that $f|_{Y'}$ has a smallest fixed point $\underline{e} \in \mathcal{E}$ that that equals $\sup E$ so that $\sup E \in \mathcal{E}$.

(1) We wish to show that for any nonempty subset $S' \subseteq Y'$, $\sup S' \in Y'$ and $\inf S' \in Y'$. Fix a nonempty $S' \subseteq Y'$. Since $S' \subseteq X$ and X is a complete lattice, $\sup S' \in X$ and $\inf S' \in X$. By definition of Y', $y' \geq \sup E$ for all $y' \in Y'$ so that $\sup E$ is a lower bound of Y'. Because $\inf Y'$ is the greatest lower bound, we must have $\inf Y' \geq \sup E$ and so $\inf Y' \in Y'$. Because $S' \subseteq Y$, we must have $\inf S' \geq \inf Y'$ so that $\inf S' \geq \sup E$; i.e., $\inf S' \in Y'$. Since $\sup S' \geq \inf S'$, we must also have $\sup S' \in Y'$.

(2) For any $e \in E$, we have $\sup E \geq e$ so that $f(\sup E) \geq f(e) = e$; i.e., $f(\sup E)$ is an upper bound of E. Since $\sup E$ is the least upper bound of E, we must have $f(\sup E) \geq \sup E$ so that $f(\sup E) \in Y'$. Moreover, for all $y' \in Y'$, $y' \geq \sup E$ so that $f(y') \geq f(\sup E) \geq \sup E$. Hence, $f|_{Y'} : Y' \to Y'$; i.e., $f|_{Y'}$ is a self-map on Y'.

(3) Since $f|_{Y'}$ is an increasing self-map on a complete lattice Y', by (iii), it has a smallest fixed point $\underline{e} \in Y$. Since $\underline{e}$ must be fixed point of f, we have $\underline{e} \in \mathcal{E}$. Moreover, if $e' \in \mathcal{E}$ is an upper bound on E, $e' \geq \sup E$ so that $e' \in Y'$. Then, e' is a fixed point of $f|_{Y'}$ and we must have $e' \geq \underline{e}$. Hence, $\underline{e}$ is the least upper bound of E in $\mathcal{E}$; i.e., $\underline{e} = \sup E \in \mathcal{E}$.

Exercise 2. Show that the smallest fixed point is also increasing in θ in Proposition 1.

Solution 2. Fix $\theta'' > \theta'$. Since $f(x,\theta)$ is increasing in θ for any $x \in X$, $f(x,\theta'') \geq f(x,\theta')$, which, in turn, implies that

$$Y'' \equiv \left\{x \in X : x \geq f\left(x,\theta''\right)\right\} \subseteq \left\{x \in X : x \geq f\left(x,\theta'\right)\right\} \equiv Y'.$$

By Tarski's fixed point theorem, the smallest fixed points in Y' and Y'' exist and, in fact, are given by $\underline{x}(\theta') := \inf Y'$ and $\underline{x}(\theta'') := \inf Z''$. Since $Y'' \subseteq Y'$, we must have $\overline{x}(\theta'') \geq \overline{x}(\theta')$.

Exercise 3. Prove that the set of stable matching is a sublattice of $(V, \leq)$ and that, for any two stable matchings μ and μ': (i) $(\mu \vee \mu')(m)$ is preferred with respect to $\succsim_m$ over $\mu(m)$ and $\mu'(m)$; (ii) $(\mu \wedge \mu')(m)$ is the worse with respect to $\succsim_m$ than $\mu(m)$ and $\mu(m')$.

Solution 3. Let ν be the fantasy defined by giving each men and the best partner out of μ and μ', and each woman the worst. Then, ν is in fact a matching: $w = \nu(m)$ and $\nu(w) \neq m$ would imply that m and w would agree as to which is the better matching, μ or μ'. Then, the other matching could not be stable because (m,w) would be a blocking pair (e.g., if $w = \nu(m) = \mu(m)$ say and $\nu(w) \neq m$, then $w \succ_m \mu'(m)$—as $\mu(m) \neq \nu'(m)$) because otherwise we could not have $\nu(w) \neq \mu(w)$. Also $\nu(w) \neq \mu(w)$ implies that $m \succ_w \mu'(w)$. Hence, (m,w) is a blocking pair for μ'.)

2 Additional Exercises

2.1 Existence of a Walrasian equilibrium

Consider an economy with $I \in \mathbb{N}$ consumers and $N \in \mathbb{N}$ goods. Each consumer $i \in \{1, 2 \ldots, I\}$ is associated with a utility function $u^i : \mathbb{R}^N_+ \to \mathbb{R}$ and an endowment $\mathbf{e}^i = (e^i_1, e^i_2, \ldots, e^i_N) \in \mathbb{R}^N_{++}$. You may assume that u^i is continuous, strictly increasing and strictly quasiconcave.

Part (i) Given a price vector $\mathbf{p} = (p_1, p_2, \ldots, p_N) \in \mathbb{R}^N_{++}$, write down the consumer's maximisation problem and prove that a unique solution exists (you may cite well-known mathematical results/theorems covered in class). Let $x^i_n(\mathbf{p})$ denote consumer i's demand function for good $n \in \{1, 2, \ldots, N\}$ given price $\mathbf{p} \in \mathbb{R}^N_{++}$. What can you say about $\mathbf{x}^i(\mathbf{p})$?

Part (ii) Define an excess demand function as $\mathbf{z} : \mathbb{R}^N_{++} \to \mathbb{R}^N$, where the nth coordinate of $\mathbf{z}(\mathbf{p})$ is given by

$$\mathbf{z}_n\left(\mathbf{p}\right) = \sum_{i=1}^{I} x^i_n\left(\mathbf{p}\right) - \sum_{i=1}^{I} e^i_n.$$

Prove that $\mathbf{z}$: (a) is continuous, (b) is homogeneous of degree zero (i.e., $\mathbf{z}(\lambda\mathbf{p}) = \mathbf{z}(\mathbf{p})$ for all $\lambda > 0$ and all $\mathbf{p} \in \mathbb{R}^N_{++}$), and (c) satisfies Walras' law (i.e., $\mathbf{p} \cdot \mathbf{z}(\mathbf{p}) = 0$ for all $\mathbf{p} \in \mathbb{R}^N_{++}$).

(d) Interpret the fact that $\mathbf{z}$ satisfies homogeneity of degree zero. What does property Walras' law imply about the good-N market when goods-$1, 2, \ldots, N-1$ markets are in equilibrium (i.e., supply equals demand)? If $\mathbf{p}^* \in \mathbb{R}^N_{++}$ is a competitive equilibrium, what must be true about the excess demand function at $\mathbf{p}^*$?

Part (iii) If $z_n(\mathbf{p}) > 0$ for some $n \in \{1, 2, \ldots, N\}$, then there is excess demand for good n at price $\mathbf{p}$. Intuition tells us that p_n should be higher to clear the market and so one idea is to consider the price of good n to be

$$\tilde{f}_n(\mathbf{p}) = p_n + \mathbf{z}_n(\mathbf{p}).$$

Letting $\tilde{f}(\cdot) = (\tilde{f}_1(\cdot), \tilde{f}_2(\cdot), \ldots, \tilde{f}_N(\cdot))$, finding a competitive equilibrium is equivalent to fining a fixed point of $\tilde{f}$. Instead of $\tilde{f}$, consider, for each $n \in \{1, 2 \ldots, N\}$ and any $\epsilon \in (0, 1)$,

$$f_n^\epsilon(\mathbf{p}) := \frac{\epsilon + p_n + \max\{\bar{z}_n(\mathbf{p}), 0\}}{N\epsilon + 1 + \sum_{k=1}^N \max\{\bar{z}_k(\mathbf{p}), 0\}},$$

where $\bar{z}_n(\mathbf{p}) := \min\{z_n(\mathbf{p}), 1\}$. (a) Show that $f^\epsilon(\cdot) = (f_1^\epsilon(\cdot), f_2^\epsilon(\cdot), \ldots, f_N^\epsilon(\cdot))$ is a self-map on

$$S_\epsilon := \left\{ \mathbf{p} \in \mathbb{R}^N_{++} : \sum_{n=1}^N p_n = 1 \text{ and } p_n \geq \frac{\epsilon}{1+2N} \ \forall n \in \{1, 2, \ldots, N\} \right\}.$$

(b) Argue that a fixed point of f^ϵ, denoted $\mathbf{p}^\epsilon$, exists. (c) Take a sequence $(\epsilon^k)_k$ such that $\epsilon^k \to 0$ and a corresponding sequence of fixed points $(\mathbf{p}^k)_k$ such that $\mathbf{p}^k$ is a fixed point of f^{ϵ^k} for all $k \in \mathbb{N}$. Does $(\mathbf{p}^k)_k$ necessarily converge? If not, would it still have a subsequence that converges to some $\mathbf{p}^* \in S_0$? (d) Can you see why we use f^ϵ instead of $\tilde{f}$?

Part (iv) Under certain conditions, $\mathbf{p}^*$ from the previous part can be guaranteed to be strictly positive in every component (i.e., $\mathbf{p}^* \in \mathbb{R}^N_{++}$). Assuming this to be the case; i.e., you found a sequence $(\mathbf{p}^k)_k$ that converges to $\mathbf{p}^* \in S_0$ and $\mathbf{p}^* \in \mathbb{R}^N_{++}$, prove that a Walrasian equilibrium exists.

Hint: Write out the condition that each p_n^* must satisfy by expanding the definition of f_n^0. Multiply this condition by the excess demand function, sum across all goods, and use the Walras' law to get the following condition:

$$\sum_{n=1}^N z_n(\mathbf{p}^*) \max\{\bar{z}_n(\mathbf{p}^*), 0\} = 0.$$

Finally, use the fact that $\mathbf{p}^* \in \mathbb{R}^N_{++}$ and Walras' law to conclude that above implies $z_n(\mathbf{p}^*) = 0$ for all $n \in \{1, 2, \ldots, N\}$.

Solutions

Part (i) The consumer's problem is

$$\max_{\mathbf{x}^i \in \mathbb{R}^N_+} u_i\left(\mathbf{x}^i\right) \text{ s.t. } \mathbf{p}\cdot\mathbf{x}^i \leq \mathbf{p}\cdot\mathbf{e}^i = \max_{\mathbf{x}^i \in \Gamma^i(\mathbf{p})} u_i\left(\mathbf{x}^i\right),$$

where

$$\Gamma^i\left(\mathbf{p}\right) := \left\{\mathbf{x}\in\mathbb{R}^N_+ : \mathbf{p}\cdot\mathbf{x} \leq \mathbf{p}\cdot\mathbf{e}^i\right\}.$$

That a solution exists follows from Weierstrass theorem because u^i is continuous and $\Gamma^i(\mathbf{p})$ is compact (i.e., closed and bounded) given $\mathbf{p}\in\mathbb{R}^N_{++}$. That the solution is unique follows from strict quasiconcavity of u^i. To see why, toward a contradiction, suppose $\mathbf{x}^i, \mathbf{y}^i$ are distinct solutions to the consumer's problem, then

$$u^i\left(\lambda\mathbf{x}^i + (1-\lambda)\,\mathbf{y}'\right) > \min\left\{u^i\left(\mathbf{x}^i\right), u^i\left(\mathbf{y}^i\right)\right\},$$

which contradicts that $\mathbf{x}^i, \mathbf{y}^i$ are optimal. It follows that each $x^i_n(\mathbf{p})$ is single-valued. Finally, theorem of the maximum tells us that $\mathbf{x}^i = (x^i_n)^N_{n=1}$ is continuous.

Part (ii)

(a) That $\mathbf{z}$ is continuous follows from the fact that each x^i is continuous in $\mathbf{p}$.

(b) Homogeneity of degree zero follows from the fact that $\Gamma(\lambda\mathbf{p}) = \Gamma(\mathbf{p})$ for any $\lambda > 0$. This condition implies that what matter is relative price and not absolute price between goods.

(c) The property follows from the fact that the budget constraint must bind at any optimal—note that this requires both u^i to be strictly increasing and strictly quasiconcave (because the two together imply that u^i is strongly increasing; i.e., if $\mathbf{x} \geq \mathbf{x}'$ and $\mathbf{x} \neq \mathbf{x}'$, then $u^i(\mathbf{x}) > u^i(\mathbf{y})$). Since $\mathbf{p}\cdot\mathbf{x}^i = \mathbf{p}\cdot\mathbf{e}^i$ for all $i \in \{1, 2\ldots, I\}$,

$$\sum_{i=1}^I \mathbf{p}\cdot\mathbf{x}^i = \sum_{i=1}^I \mathbf{p}\cdot\mathbf{e}^i \Leftrightarrow \mathbf{p}\cdot\sum_{i=1}^I \mathbf{x}^i = \mathbf{p}\cdot\sum_{i=1}^I \mathbf{e}^i \Leftrightarrow \mathbf{p}\cdot\sum_{i=1}^I\left(\mathbf{x}^i - \mathbf{e}^i\right) = 0 \Leftrightarrow \mathbf{p}\cdot\mathbf{z}\left(\mathbf{p}\right) = 0.$$

(d) Walras' law says that if $N-1$ markets are in equilibrium, then the Nth market must be in equilibrium. At any competitive equilibrium $\mathbf{p}^* \in \mathbb{R}^N_{++}$, $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.

Part (iii)

(a) Given any $\mathbf{p}\in S_\epsilon$, we need to show that $f^\epsilon(\mathbf{p}) = (f^\epsilon_1(\mathbf{p}), f^\epsilon_2(\mathbf{p}), \ldots, f^\epsilon_N(\mathbf{p})) \in S_\epsilon$. Observe first that

$$\sum_{n=1}^N f^\epsilon_n\left(\mathbf{p}\right) = \sum_{n=1}^N \frac{\epsilon + p_n + \max\left\{\bar{z}_n\left(\mathbf{p}\right), 0\right\}}{N\epsilon + 1 + \sum_{k=1}^n \max\left\{\bar{z}_k\left(\mathbf{p}\right), 0\right\}} = 1.$$

To prove the other condition, note that

$$f^\epsilon_n\left(\mathbf{p}\right) \geq \frac{\epsilon + 0 + 0}{N\epsilon + 1 + N} \geq \frac{\epsilon}{1 + 2N},$$

where the second inequality uses that $\epsilon \in (0,1)$. Hence, f^ϵ is a self-map on S_ϵ. It remains to show that f_n^ϵ is continuous to be able to use the Brouwer's fixed point theorem to conclude that a fixed point exists. But continuity of f_n^ϵ follows from the fact that z_n is continuous (note that denominator is bounded away from zero).

(b) Note that while $\epsilon \to 0$ is convergent, the corresponding sequence of fixed points $(\mathbf{p}^\epsilon)$ need not be convergent because we do not know if $\mathbf{p}^\epsilon$ is continuous in ϵ. Nevertheless, since $\mathbf{p}^\epsilon$ is bounded between zero and one, it must have a subsequence that converges—to say $\mathbf{p}^*$. Since $\mathbf{p}^\epsilon \in S_\epsilon$ for every ϵ and S_ϵ converges to S_0, it follows that $\mathbf{p}^* \in S_0$.

(c) Domain of $\tilde{f}$ is $\mathbb{R}^N_{++}$ but the image could be strictly negative (because $z_n(\mathbf{p})$ can be negative). So $\tilde{f}$ may not be a self-map. The domain is also not compact (since it is unbounded).

Part (iv) Observe that $\bar{z}$ inherits continuity from z and so

$$p_n^* \sum_{k=1}^N \max\left\{\bar{z}_k\left(\mathbf{p}^*\right), 0\right\} = \max\left\{\bar{z}_n\left(\mathbf{p}^*\right), 0\right\} \;\forall n \in \{1,2,\ldots,N\}.$$

Multiplying both sides by $z_n(\mathbf{p}^*)$ and summing across n gives

$$\sum_{n=1}^N z_n\left(\mathbf{p}^*\right) \max\left\{\bar{z}_n\left(\mathbf{p}^*\right), 0\right\} = \underbrace{\sum_{n=1}^N p_n^* z_n\left(\mathbf{p}^*\right)}_{=\mathbf{p}^*\cdot\mathbf{z}(\mathbf{p}^*)=0} \left(\sum_{k=1}^N \max\left\{\bar{z}_k\left(\mathbf{p}^*\right), 0\right\}\right) = 0,$$

where we used Walras' law. We now argue that $z_n^*(\mathbf{p}^*) \leq 0$ for all $n \in \{1,2,\ldots,N\}$. Toward a contradiction, suppose $z_n(\mathbf{p}^*) > 0$ for some $n \in \{1,2,\ldots,N\}$. Then, $\bar{z}_n(\mathbf{p}^*) > 0$ so that $z_n(\mathbf{p}^*)\max\{\bar{z}_n(\mathbf{p}^*), 0\} > 0$. Suppose now $z_n(\mathbf{p}^*) < 0$, then $z_n(\mathbf{p}^*)\max\{\bar{z}_n(\mathbf{p}^*), 0\} = 0$. Thus, for the left-hand side of display equation above to equal zero, we must have $z_n(\mathbf{p}^*) \leq 0$ for all $n \in \{1,2,\ldots,N\}$. Moreover, since Walras' law requires

$$\sum_{n=1}^N p_n^* z_n\left(\mathbf{p}^*\right) = 0,$$

and $\mathbf{p}^* \in \mathbb{R}^N_{++}$ $z_n^*(\mathbf{p}^*)$ cannot be negative; i.e., we must have $z_n(\mathbf{p}^*) = 0$ for all $n \in \{1,2,\ldots,N\}$.

2.2 Cournot oligopoly as a supermodular game

Consider $n \in \mathbb{N}$ with $n \geq 2$ firms operating as Cournot duopoly. Let $P : \mathbb{R}^n_+ \to \mathbb{R}_{++}$ denote the inverse demand function so that $P(Q)$ is the market price when Q is the aggregate quantity of goods produced. Let $C_i : \mathbb{R}_+ \to \mathbb{R}_+$ denote each firm $i \in \{1,2,\ldots,n\}$'s cost function. You may assume that P and Q are twice continuously differentiable, P is strictly decreasing, and C is is strictly increasing, and that all firm faces a common capacity constraint of $\bar{q} < \infty$.

Part (i) Suppose $n = 2$. What additional conditions, if any, on P and C are needed to guarantee that the game is supermodular? Show how each firm $i \in \{1,2\}$'s optimal output changes with firm $j \in \{1,2\}\backslash\{i\}$'s output?

Hint: A game is supermodular if (i) each player's set of strategies is a subcomplete sublattice, (ii) fixing other players' actions, each player $i \in \{1, 2 \ldots, n\}$'s payoff function is supermodular in own action, and (iii) each player's payoff function satisfies increasing differences in (own action; others actions).

Part (ii) Suppose $n = 2$ and that the game is supermodular. Let $Q_i^* : \mathcal{Q} \rightrightarrows \mathcal{Q}$ denote firm $i \in \{1, 2\}$'s best response correspondence and let $q_i^* : \mathcal{Q} \to \mathcal{Q}$ be defined via $q_i^*(q_{-i}) := \max Q_i^*(q_{-i})$. Consider the following sequence $(\mathbf{q}^k)_k = (\mathbf{q}^1, \mathbf{q}^2, \ldots)$ defined as

$$\begin{aligned}\mathbf{q}^1 &:= \overline{\mathbf{q}} = (\overline{q}, \overline{q}, \ldots, \overline{q}), \\ \mathbf{q}^2 &:= \left(q_1^*\left(\mathbf{q}^1\right), q_2^*\left(\mathbf{q}^1\right)\right) \\ \mathbf{q}^{k+1} &:= \left(q_1^*\left(\mathbf{q}^k\right), q_2^*\left(\mathbf{q}^k\right)\right) \ \forall k \in \{2, 3, \ldots\}.\end{aligned}$$

(a) Argue that q_i^* is well-defined. (b) Show that the sequence $(\mathbf{q}^k)_k$ is decreasing. (c) Argue that $(\mathbf{q}^k)_k$ converges to some point $\mathbf{e}^*$ and that $\mathbf{e}^*$ is a (pure-strategy) Nash equilibrium. (d) Show that $\mathbf{e}^*$ is the "largest" Nash equilibrium of the game (i.e., a Nash equilibrium $\overline{\mathbf{e}}$ is the largest equilibrium if (i) $\overline{\mathbf{e}}$ is a Nash equilibrium and (ii)

$$\overline{\mathbf{e}} = \sup \left\{ \mathbf{q} \in [0, \overline{q}]^2 : \mathbf{q}^*(\mathbf{q}) \geq \mathbf{q} \right\}.$$

Hint: For part (c), use the fact each firm i's payoff is continuous.

Part (iii) Suppose now that $n > 2$ and that firms are all identical. Suppose firms $2, 3, \ldots, n$ are each producing y units of output. Then, firm 1's profit from choosing q_1 of output can be thought of as firm 1 choosing aggregate output Q.

(a) Write down firm 1's profit as a function of (Q, y).

(b) What additional conditions, if any, on P and C are needed to guarantee firm 1's profit from part (a) has increasing differences in (Q, y)?

(c) How can you use this fact to establish the existence of a symmetric Cournot equilibrium using Tarski's fixed point theorem?

Solutions

Part (i) Fix $i, j \in \{1, 2\}$ with $i \neq j$. Firm i's profit function is given by $\pi_i : \mathbb{R}_+^2 \to \mathbb{R}$ such that

$$\pi_i(q_i, q_j) := P(q_i + q_j) q_i - C(q_i).$$

Note that π_i is trivially supermodular in q_i since q_i is one-dimensional. To ensure that π_i satisfies increasing differences (in $(q_i; -q_{-i})$), it suffices that the cross derivative of π_i is nonpositive; i.e.

$$\frac{\mathrm{d}^2 \pi_i(q_i, q_j)}{\mathrm{d}q_j \mathrm{d}q_i} = \frac{\mathrm{d}}{\mathrm{d}q_i}\left[P_j(q_i + q_j) q_i\right] = P''(q_i + q_j) q_i + P'(q_i, q_j) \leq 0.$$

Hence, a sufficient condition is that demand is concave which ensures that firm i's marginal revenue is decreasing in the output of the other firm j. In particular, we do not require conditions on

C.

Given the other firm's output $q_{-i} \in \mathcal{Q}$, firm i's problem is

$$Q_i^*\left(q_{-i}\right) = \max_{q_i \in [0,\overline{Q}]} P\left(q_i, q_j\right) q_i - C\left(q_i\right),$$

The objective is continuous and we're maximising over a compact set and hence a solution exists. Then, the monotone comparative static theorem tells us that $\max Q_i^*$ is strictly decreasing in q_{-i}.

Part (ii)

(a) By theorem of the maximum Q_i^* is a compact-valued correspondence and hence $\max Q_i^*$ is well-defined.

(b) Milgrom and Shannon gives us that

$$Q_i^*\left(q'_{-i}\right) \geq_S Q_i^*\left(q_{-i}\right) \ \forall q'_{-i} \geq q_{-i}$$

for each $i \in \{1,2\}$ and so

$$q_i^*\left(q'_{-i}\right) \geq q_i^*\left(q_{-i}\right) \ \forall q'_{-i} \geq q_{-i}.$$

This implies that

$$\mathbf{q}^*\left(\mathbf{q}'\right) = \left(q_1^*\left(q'_2\right), q_2^*\left(q'_1\right)\right) \geq \left(q_1^*\left(q_2\right), q_2^*\left(q_1\right)\right) = \mathbf{q}^*\left(\mathbf{q}\right) \ \forall \mathbf{q}' \geq \mathbf{q}.$$

Given that $\overline{\mathbf{q}} \geq \mathbf{q}$ for any feasible $\mathbf{q}$ and $\mathbf{q}^*(\cdot) \in [0,\overline{q}]^2$,

$$\overline{\mathbf{q}} \geq \mathbf{q}^*\left(\overline{\mathbf{q}}\right) \geq \mathbf{q}^*\left(\mathbf{q}\right) \ \forall \mathbf{q} \in [0,\overline{q}]^2.$$

In particular,

$$\overline{\mathbf{q}} \geq \mathbf{q}^*\left(\overline{\mathbf{q}}\right) \geq \mathbf{q}^*\left[\mathbf{q}^*\left(\mathbf{q}\right)\right]$$

and so on.

(b) Any decreasing sequence in a compact set has a limit; call this limit $\mathbf{e}^*$. Suppose that $\mathbf{e}^*$ is not a Nash equilibrium. Then, there exists an $i \in \{1,2\}$ and $q_i \in [0,\overline{q}]$ such that

$$\pi_i\left(q_i, e^*_{-i}\right) - \pi_i\left(e_i^*, e^*_{-i}\right) > 0.$$

By continuity of π_i, for sufficiently large k,

$$\pi_i\left(q_i, q^k_{-i}\right) - \pi_i\left(q_i^k, q^k_{-i}\right) > 0.$$

But this is a contradiction since q_i^k is a best response to q^k_{-i} by construction.

(c) We know that the largest Nash equilibrium of the game is given by

$$\overline{\mathbf{e}} = \sup\left\{\mathbf{q} \in [0,\overline{q}]^2 : \overline{q}^*\left(\mathbf{q}\right) \geq \mathbf{q}\right\}$$

Since $\overline{\mathbf{q}}$ is the maximum element, we have that

$$\begin{aligned}\mathbf{q}^1 &= \overline{\mathbf{q}} \geq \overline{\mathbf{e}} \\ \mathbf{q}^2 = \mathbf{q}^*\left(\mathbf{q}^1\right) &\geq \mathbf{q}^*\left(\overline{\mathbf{e}}\right) = \overline{\mathbf{e}} \\ \vdots &\geq \vdots \\ \mathbf{e}^* &\geq \overline{\mathbf{e}}.\end{aligned}$$

We proved in the previous part that $\mathbf{e}^*$ is a Nash equilibrium. Since $\overline{\mathbf{e}}$, by definition, is the largest Nash equilibrium it follows that $\mathbf{e}^* = \overline{\mathbf{e}}$.

Part (iii)

$\pi_1(Q, y) = P(Q)(Q - (n-1)y) - C(Q - (n-1)y)$.

(b)

It suffices that C is convex.

$$\begin{aligned}\frac{\mathrm{d}^2\pi_1(Q,y)}{\mathrm{d}Q\mathrm{d}y} &= (n-1)\frac{\mathrm{d}\pi_1(Q,y)}{\mathrm{d}Q}\left[-P(Q) + C'(Q-(n-1)y)\right] \\ &= (n-1)\left[-P'(Q) + C''(Q-(n-1)y)\right].\end{aligned}$$

From monotone comparative static theorem, we can conclude that the set

$$\overline{Q}(y) := \max \arg\max_{Q} \ \pi_1(Q, y)$$

increases with y. Define $q(y) = \frac{\overline{Q}(y)}{n}$. Since $\overline{Q}$ is increasing, q is also increasing. By Tarski's fixed point theorem, there exists y^* such that $q(y^*) = y^*$; i.e., a symmetric Cournot equilibrium exists.