Optional Problem Set 12

Due: N/A

1 Exercises from class notes

All from "8. Fixed Point Theorems.pdf".

Exercise 1. Complete the proof of Theorem 1; i.e., show that there is a smallest fixed point and any nonempty subset of fixed points has a supremum in the set of all fixed points.

Exercise 2. Show that the smallest fixed point is also increasing in θ in Proposition 1.

Exercise 3. Prove that the set of stable matching is a sublattice of $(V, \leq)$ and that, for any two stable matchings μ and μ': (i) $(\mu \vee \mu')(m)$ is preferred with respect to $\succsim_m$ over $\mu(m)$ and $\mu'(m)$; (ii) $(\mu \wedge \mu')(m)$ is the worse with respect to $\succsim_m$ than $\mu(m)$ and $\mu(m')$.

2 Additional Exercises

2.1 Existence of a Walrasian equilibrium

Consider an economy with $I \in \mathbb{N}$ consumers and $N \in \mathbb{N}$ goods. Each consumer $i \in \{1, 2 \ldots, I\}$ is associated with a utility function $u^i : \mathbb{R}^N_+ \to \mathbb{R}$ and an endowment $\mathbf{e}^i = (e^i_1, e^i_2, \ldots, e^i_N) \in \mathbb{R}^N_{++}$. You may assume that u^i is continuous, strictly increasing and strictly quasiconcave.

Part (i) Given a price vector $\mathbf{p} = (p_1, p_2, \ldots, p_N) \in \mathbb{R}^N_{++}$, write down the consumer's maximisation problem and prove that a unique solution exists (you may cite well-known mathematical results/theorems covered in class). Let $x^i_n(\mathbf{p})$ denote consumer i's demand function for good $n \in \{1, 2, \ldots, N\}$ given price $\mathbf{p} \in \mathbb{R}^N_{++}$. What can you say about $\mathbf{x}^i(\mathbf{p})$?

Part (ii) Define an excess demand function as $\mathbf{z} : \mathbb{R}^N_{++} \to \mathbb{R}^N$, where the nth coordinate of $\mathbf{z}(\mathbf{p})$ is given by

$$\mathbf{z}_n(\mathbf{p}) = \sum_{i=1}^{I} x^i_n(\mathbf{p}) - \sum_{i=1}^{I} e^i_n.$$

Prove that $\mathbf{z}$: (a) is continuous, (b) is homogeneous of degree zero (i.e., $\mathbf{z}(\lambda \mathbf{p}) = \mathbf{z}(\mathbf{p})$ for all $\lambda > 0$ and all $\mathbf{p} \in \mathbb{R}^N_{++}$), and (c) satisfies Walras' law (i.e., $\mathbf{p} \cdot \mathbf{z}(\mathbf{p}) = 0$ for all $\mathbf{p} \in \mathbb{R}^N_{++}$).

(d) Interpret the fact that $\mathbf{z}$ satisfies homogeneity of degree zero. What does property Walras' law imply about the good-N market when goods-$1, 2, \ldots, N-1$ markets are in equilibrium (i.e., supply equals demand)? If $\mathbf{p}^* \in \mathbb{R}^N_{++}$ is a competitive equilibrium, what must be true about the excess demand function at $\mathbf{p}^*$?

Part (iii) If $z_n(\mathbf{p}) > 0$ for some $n \in \{1, 2, \ldots, N\}$, then there is excess demand for good n at price $\mathbf{p}$. Intuition tells us that p_n should be higher to clear the market and so one idea is to consider the price of good n to be

$$\tilde{f}_n(\mathbf{p}) = p_n + \mathbf{z}_n(\mathbf{p}).$$

Letting $\tilde{f}(\cdot) = (\tilde{f}_1(\cdot), \tilde{f}_2(\cdot), \ldots, \tilde{f}_N(\cdot))$, finding a competitive equilibrium is equivalent to fining a fixed point of $\tilde{f}$. Instead of $\tilde{f}$, consider, for each $n \in \{1, 2 \ldots, N\}$ and any $\epsilon \in (0, 1)$,

$$f_n^\epsilon(\mathbf{p}) := \frac{\epsilon + p_n + \max\{\bar{z}_n(\mathbf{p}), 0\}}{N\epsilon + 1 + \sum_{k=1}^N \max\{\bar{z}_k(\mathbf{p}), 0\}},$$

where $\bar{z}_n(\mathbf{p}) := \min\{z_n(\mathbf{p}), 1\}$. (a) Show that $f^\epsilon(\cdot) = (f_1^\epsilon(\cdot), f_2^\epsilon(\cdot), \ldots, f_N^\epsilon(\cdot))$ is a self-map on

$$S_\epsilon := \left\{\mathbf{p} \in \mathbb{R}^N_{++} : \sum_{n=1}^N p_n = 1 \text{ and } p_n \geq \frac{\epsilon}{1+2N}\ \forall n \in \{1, 2, \ldots, N\}\right\}.$$

(b) Argue that a fixed point of f^ϵ, denoted $\mathbf{p}^\epsilon$, exists. (c) Take a sequence $(\epsilon^k)_k$ such that $\epsilon^k \to 0$ and a corresponding sequence of fixed points $(\mathbf{p}^k)_k$ such that $\mathbf{p}^k$ is a fixed point of f^{ϵ^k} for all $k \in \mathbb{N}$. Does $(\mathbf{p}^k)_k$ necessarily converge? If not, would it still have a subsequence that converges to some $\mathbf{p}^* \in S_0$? (d) Can you see why we use f^ϵ instead of $\tilde{f}$?

Part (iv) Under certain conditions, $\mathbf{p}^*$ from the previous part can be guaranteed to be strictly positive in every component (i.e., $\mathbf{p}^* \in \mathbb{R}^N_{++}$). Assuming this to be the case; i.e., you found a sequence $(\mathbf{p}^k)_k$ that converges to $\mathbf{p}^* \in S_0$ and $\mathbf{p}^* \in \mathbb{R}^N_{++}$, prove that a Walrasian equilibrium exists.

Hint: Write out the condition that each p_n^* must satisfy by expanding the definition of f_n^0. Multiply this condition by the excess demand function, sum across all goods, and use the Walras' law to get the following condition:

$$\sum_{n=1}^N z_n(\mathbf{p}^*) \max\{\bar{z}_n(\mathbf{p}^*), 0\} = 0.$$

Finally, use the fact that $\mathbf{p}^* \in \mathbb{R}^N_{++}$ and Walras' law to conclude that above implies $z_n(\mathbf{p}^*) = 0$ for all $n \in \{1, 2, \ldots, N\}$.

2.2 Cournot oligopoly as a supermodular game

Consider $n \in \mathbb{N}$ with $n \geq 2$ firms operating as Cournot duopoly. Let $P : \mathbb{R}^n_+ \to \mathbb{R}_{++}$ denote the inverse demand function so that $P(Q)$ is the market price when Q is the aggregate quantity of goods produced. Let $C_i : \mathbb{R}_+ \to \mathbb{R}_+$ denote each firm $i \in \{1, 2, \ldots, n\}$'s cost function. You may assume that P and Q are twice continuously differentiable, P is strictly decreasing, and C is is strictly increasing, and that all firm faces a common capacity constraint of $\overline{q} < \infty$.

Part (i) Suppose $n = 2$. What additional conditions, if any, on P and C are needed to guarantee that the game is supermodular? Show how each firm $i \in \{1,2\}$'s optimal output changes with firm $j \in \{1,2\} \backslash \{i\}$'s output?

Hint: A game is supermodular if (i) each player's set of strategies is a subcomplete sublattice, (ii) fixing other players' actions, each player $i \in \{1, 2 \ldots, n\}$'s payoff function is supermodular in own action, and (iii) each player's payoff function satisfies increasing differences in (own action; others actions).

Part (ii) Suppose $n = 2$ and that the game is supermodular. Let $Q_i^* : \mathcal{Q} \rightrightarrows \mathcal{Q}$ denote firm $i \in \{1,2\}$'s best response correspondence and let $q_i^* : \mathcal{Q} \to \mathcal{Q}$ be defined via $q_i^*(q_{-i}) := \max Q_i^*(q_{-i})$. Consider the following sequence $(\mathbf{q}^k)_k = (\mathbf{q}^1, \mathbf{q}^2, \ldots)$ defined as

$$
\begin{aligned}
\mathbf{q}^1 &:= \overline{\mathbf{q}} = (\overline{q}, \overline{q}, \ldots, \overline{q}) \,, \\
\mathbf{q}^2 &:= \left(q_1^* \left(\mathbf{q}^1 \right), q_2^* \left(\mathbf{q}^1 \right) \right) \\
\mathbf{q}^{k+1} &:= \left(q_1^* \left(\mathbf{q}^k \right), q_2^* \left(\mathbf{q}^k \right) \right) \; \forall k \in \{2, 3, \ldots\}.
\end{aligned}
$$

(a) Argue that q_i^* is well-defined. (b) Show that the sequence $(\mathbf{q}^k)_k$ is decreasing. (c) Argue that $(\mathbf{q}^k)_k$ converges to some point $\mathbf{e}^*$ and that $\mathbf{e}^*$ is a (pure-strategy) Nash equilibrium. (d) Show that $\mathbf{e}^*$ is the "largest" Nash equilibrium of the game (i.e., a Nash equilibrium $\overline{\mathbf{e}}$ is the largest equilibrium if (i) $\overline{\mathbf{e}}$ is a Nash equilibrium and (ii)

$$
\overline{\mathbf{e}} = \sup \left\{ \mathbf{q} \in [0, \overline{q}]^2 : \mathbf{q}^* (\mathbf{q}) \geq \mathbf{q} \right\}.
$$

Hint: For part (c), use the fact each firm i's payoff is continuous.

Part (iii) Suppose now that $n > 2$ and that firms are all identical. Suppose firms $2, 3, \ldots, n$ are each producing y units of output. Then, firm 1's profit from choosing q_1 of output can be thought of as firm 1 choosing aggregate output Q.

(a) Write down firm 1's profit as a function of (Q, y).

(b) What additional conditions, if any, on P and C are needed to guarantee firm 1's profit from part (a) has increasing differences in (Q, y)?

(c) How can you use this fact to establish the existence of a symmetric Cournot equilibrium using Tarski's fixed point theorem?