ECON 6170

Problem Set 2

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Exercise 14 Assume that $x_n \to x$ and $y_n \to y$. TF:

(i) $\{x_n + y_n\}_n \to x + y$: True.

Proof. Fix some $\varepsilon > 0$. Then choose $\delta = \varepsilon/2$. From the definition of convergence, $\exists\ N_x, N_y \in \mathbb{N}$ s.t. $\forall\ n > N_x, |x_n - x| < \delta$, and $\forall\ n > N_y, |y_n - y| < \delta$. Then taking $N = \max\{N_x, N_y\}$, we have that for all $n > N$, $|x_n + y_n - x - y| \leq |x_n - x| + |y_n - y| < 2\delta = \varepsilon$. Thus, $\{x_n + y_n\}_n \to x + y$ $\square$

(ii) $x_n y_n \to xy$: True.

Proof. Fix $\varepsilon > 0$. Note first that since convergent sequences are bounded, $|x_n|, |y_n|, |x|, |y| < M$ for some $M > 0$. Choose $\delta = \frac{\varepsilon}{2M}$. From the definition of convergence, $\exists\ N_x, N_y \in \mathbb{N}$ s.t. $\forall\ n > N_x, |x_n - x| < \delta$ and $\forall\ n > N_y, |y_n - y| < \delta$. Then choosing $N = \max\{N_x, N_y\}$, we get that $\forall\ n > N$,

$$|x_n y_n - xy| = |x_n y_n - x y_n - xy + x y_n| = |(x_n - x)y_n + x(y_n - y)| \leq |(x_n - x)y_n| + |(y_n - y)x|$$

and since convergent sequences are bounded by M, we have that

$$|x_n y_n - xy| < |x_n - x||M| + |y_n - y||M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus, $x_n y_n \to xy$. $\square$

(iii) $x_n - y_n \to x - y$: True.

Proof. Note that if $y_n \to y$, $-y_n \to -y$. Call $z_n = -y_n\ \forall\ n$. Then $z_n \to z = -y$, and this problem becomes $x_n + z_n \to x + z$, which was proved in part (i). $\square$

(iv) $\frac{1}{x_n} \to \frac{1}{x}$: False. Consider $x_n = \frac{1}{n}$. $x_n \to 0$, but $\frac{1}{x_n}$ diverges, and does not converge.

(v) $\frac{x_n}{y_n} \to \frac{x}{y}$: False. Choose $x_n = 1\ \forall\ n$, and this problem is identical to part (iv), which was shown to be false by counterexample.

Exercise 18 False! Consider the sequence $\{x_n\}_n$, where:

$$x_n = \begin{cases} -1 & n \text{ is odd} \\ n & n \text{ is even} \end{cases}$$

-1 is the unique real number where $(-1 - \varepsilon, -1 + \varepsilon)$ contains an infinite number of elements of $\{x_n\}$, but $\{x_n\}$ diverges by inspection.

Exercise 23 True!

Proof. We have that $x_n \to x \in \mathbb{R}$. Take $\varepsilon = 1$. From the definition of convergence, $\exists\ N \in \mathbb{N}$ s.t. $\forall\ n > N, |x_n - x| < 1$. Thus, we have that $\forall\ n > N$, $|x_n|$ is bounded by $|x| + 1$. Since there are only $N < \infty$ terms of x_n where $n \leq N$, we can define

$$b = \max\{|x| + 1, |x_1|, \ldots, |x_N|\}$$

and $x_n \in [-b, b]\ \forall\ n$, so $\{x_n\}$ is bounded. $\square$

Exercise 27 True!

Proof. We have that $\{x_n\}_n$ is bounded, which means that $\exists\ b$ s.t. $x_n \in [-b, b]\ \forall\ n$. Take a subsequence $\{x_{n_k}\}$. For $x_{n_k} \in \{x_{n_k}\}$, we have that $x_{n_k} \in \{x_n\}_n \subseteq [-b, b]$. Thus, $\{x_{n_k}\}$ is bounded. $\square$

Exercise 28 False! Consider the sequence $\{x_n\}_n$, where:

$$x_n = \begin{cases} -1 & n \text{ is odd} \\ n \cdot (-1)^{n/2} & n \text{ is even} \end{cases}$$

The subsequence $\{x_n \mid n \text{ is odd}\}$ is bounded by inspection, but the sequence is neither bounded above nor below.

Exercise 29 False! Consider the sequence $x_n = n$. It is unbounded, and any infinite subsequence of it is unbounded, as there is no infinite subset of the natural numbers that is bounded.

Exercise 31 You can not. Consider the following example. $\{x_n\}_n$ is such that $x_n = 1\ \forall\ n$. Then $\max S_n$ exists and is attained at every element of $\{x_n\}_n$. However, $\max\{m \in \mathbb{N} \mid x_m = \max S_1\}$ does not exist, because $\sup\{m \in \mathbb{N} \mid x_m = \max S_1\} = \infty \notin \mathbb{N}$.

Exercise 32 False! Consider the sequence $x_n = \frac{1}{n}$. By Proposition 9, $\limsup x_n = \lim x_n = 0$. However, given any $M \in \mathbb{N}$, $\sup\{x_n \mid n \geq M\} = \frac{1}{M} > 0$, so $\sup\{x_n \mid n \geq M\} > \limsup x_n\ \forall\ M \in \mathbb{N}$.

Exercise 33 First, the disproof:

Disproof. (Disproof by counterexample) Take $x_n = \frac{1}{n}$ and $y_n = n$. Then $\limsup x_n y_n = \limsup 1 = 1$. However, $x_n \to 0$ and y_n diverges, so $0 \limsup y_n = 0 \cdot \infty$ which is undefined. $\square$

A condition that would make this statement true is forcing x_n to converge to $x > 0$.

Exercise 1 This set of sequences includes only sequences that meet the following criterium: they must be constant after some $N < \infty$ non-constant terms. Formally, $\{x_n\}_n$ is $\star$-convergent to $x \in \mathbb{R}$ if

$$\exists\ N \in \mathbb{N} \text{ s.t. } \forall\ n > N, |x_n - x| = 0$$

Proof. FSOC, assume that there exists some $\{y_n\}_n$ that fails to meet this criterium but is still $\star$-convergent to some $y \in \mathbb{R}$. Since y_n fails to meet the criterium, that means that $\forall\ N \in \mathbb{N}, \exists\ n > N$ where $|y_n - y| > 0$. However, fixing any N, we can choose $\varepsilon < |y_n - y|$. Thus, $\{y_n\}$ is not $\star$-convergent by definition. $\square$