Econ 6190 Problem Set 9

Fall 2024

1. Let $X \sim \text{binomial}(5, \theta)$ with θ unknown. Consider testing $\mathbb{H}_0 : \theta = \frac{1}{2}$ versus $\mathbb{H}_1 : \theta > \frac{1}{2}$.

    (a) Consider test alpha that rejects $\mathbb{H}_0$ if and only if all "successes" are observed. Derive the power function of this test. Calculate its type I error. Express its type II error as a function of θ where $\theta > \frac{1}{2}$.

    (b) Consider an alternative test beta that rejects $\mathbb{H}_0$ if we observe $X = 3, 4$, or 5. Write down the power function of this test. Calculate its type I error. Express its type II error as a function of θ where $\theta > \frac{1}{2}$.

    (c) Between tests alpha and beta, which test has a smaller type I error? Which test has a smaller type II error? Which test would you prefer?

2. Take the model $X \sim \mathrm{N}(\mu, \sigma^2)$ with σ^2 unknown. A sample of size $n = 4$ yields $\sum_{i=1}^{4} X_i = 40$, $\sum_{i=1}^{4} \left(X_i - \bar{X}\right)^2 = 48$, where $\bar{X}$ is the sample average.

    (a) Propose a test for testing $\mathbb{H}_0 : \mu = 9$ and $\mathbb{H}_1 : \mu \neq 9$ given significance value $\alpha = 5\%$. What is the critical value? Can you reject the null? Draw a graph of the distribution of your statistic if the null hypothesis is correct and indicate the rejection region.

    (b) Do the same for $\mathbb{H}_0 : \mu = 7$ and $\mathbb{H}_1 : \mu > 7$ given significance value $\alpha = 5\%$.

3. Take the model $X \sim \mathrm{N}(\mu, 4)$. We want to test the null hypothesis $\mathbb{H}_0 : \mu = 20$ against $\mathbb{H}_1 : \mu > 20$. A sample of $n = 16$ independent realizations of X was collected, and the sample mean $\bar{X} = 20.5$.

    (a) Propose a test with size α equal to 1%. What is the condition for rejecting $\mathbb{H}_0$ for this test?

    (b) What is the p value of this test?

    (c) What is the condition for rejecting $\mathbb{H}_0$ with $\alpha = 1\%$ if we increase the size of the sample to $n = 25$?

    (d) We want a test with power 90% if $\mu = 21$. What is the size of the sample n needed for that? Explain briefly how n affects the power of the test.

    (e) Now consider the two-sided test $\mathbb{H}_0 : \mu = 20$ against $\mathbb{H}_1 : \mu \neq 20$. Write down the power function of the test if $\mu = 21$. Compare with (d). Do you need a larger or smaller n in order to achieve 90% power?

4. [Hansen 13.11] You have two samples (Madison and Ann Arbor) of monthly rents paid by n individuals in each sample. You want to test the hypothesis that the average rent in the two cities is the same. Construct an appropriate test.

5. [Hansen 13.13] You design a statistical test of some hypothesis $\mathbb{H}_0$ which has asymptotic size 5% but you are unsure of the approximation in finite samples. You run a simulation experiment on your computer to check if the asymptotic distribution is a good approximation. You generate data which satisfies $\mathbb{H}_0$. On each simulated sample, you compute the test. Out of $B = 50$ independent trials you find 5 rejections and 45 acceptances.

   (a) Based on the $B = 50$ simulation trials, what is your estimate $\hat{p}$ of p, the probability of rejection?

   (b) Find the asymptotic distribution for $\sqrt{B}(\hat{p} - p)$.

   (c) Test the hypothesis that $p = 0.05$ against $p \neq 0.05$. Does the simulation evidence support or reject the hypothesis that the size is 5%?

6. One very striking abuse of hypothesis testing is to choose size α **after** seeing the data and to choose them in such a way as to force rejection (or acceptance) of a null hypothesis. To see what the **true** Type I and Type II error probabilities of such a procedure are, calculate size and power of the following two trivial tests:

   (a) Always reject $\mathbb{H}_0$, no matter what data are obtained. (equivalent to the practice of choosing the α level to force rejection of $\mathbb{H}_0$)

   (b) Always accept $\mathbb{H}_0$, no matter what data are obtained. (equivalent to the practice of choosing the α level to force acceptance of $\mathbb{H}_0$)

7. [Final exam, 2021 fall] Suppose $X \sim \mathrm{N}(\mu, \sigma^2)$ where both μ and σ^2 are unknown. We hope to use a random sample $\{X_i, i = 1 \ldots n\}$ drawn from X to test hypothesis: $\mathbb{H}_0 : \mu = \mu_0$ for some $\mu_0 \in \mathbb{R}$ against $\mathbb{H}_1 : \mu \neq \mu_0$.

   (a) Let $\beta = (\mu, \sigma^2)$. Write down the log likelihood of β under $\mathbb{H}_0$.

   (b) The unconstrained MLE of β is $\hat{\beta} = (\bar{X}_n, \hat{\sigma}^2)$, where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$. Derive the likelihood ratio statistic LR_n for testing $\mathbb{H}_0 : \mu = \mu_0$ vs $\mathbb{H}_1 : \mu \neq \mu_0$. Simplify as much as you can.

   (c) Show the likelihood ratio test based on $LR_n > c$ for some c is equivalent to $|T| > b$ for some b, where $T = \frac{\bar{X}_n - \mu_0}{\sqrt{\frac{\hat{\sigma}^2}{n}}}$.