ECON 6170

Problem Set 8

Gabe Sekeres

October 25, 2024

1 Exercises from class notes

Exercise 8. Prove the following: Suppose $f : X \subseteq \mathbb{R}^d \to \mathbb{R}^m$ is differentiable at $x_0 \in \text{int}(X)$. Then $\frac{\partial f_i}{\partial x_j}(x_0)$ exists for any (i, j), and

$$Df(x_0) = \left[\frac{\partial f_i}{\partial x_j}(x_0)\right]_{ij} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_0) & \cdots & \frac{\partial f_1}{\partial x_d}(x_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x_0) & \cdots & \frac{\partial f_m}{\partial x_d}(x_0) \end{bmatrix}$$

Proof. We have that f is differentiable, meaning that there exists a linear transformation $D : \mathbb{R}^d \to \mathbb{R}^m$ such that

$$\lim_{h \to 0} \frac{\|f(x_0 + h) - (f(x_0) + Dh)\|_m}{\|h\|_d} = 0$$

Fix some $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, d\}$. Take $h = \eta e_j$ for some $\eta \in \mathbb{R}$ and e_j the standard jth basis vector in $\mathbb{R}^m$. Then we have

$$\begin{aligned} \lim_{\eta \to 0} \frac{\|f(x_0 + \eta e_j) - (f(x_0) + D\eta e_j)\|_m}{\|\eta e_j\|_d} &= \lim_{\eta \to 0} \frac{\|f(x_0 + \eta e_j) - (f(x_0) + D\eta e_j)\|_m}{|\eta|} \\ &= \lim_{\eta \to 0} \frac{\|f(x_0 + \eta e_j) - f(x_0) - \eta d_j\|_m}{|\eta|} \end{aligned}$$

where d_j is the jth column of D. This implies that, expanding the norm, we have that

$$\lim_{\eta \to 0} \frac{\|f(x_0 + \eta e_j) - (f(x_0) + D\eta e_j)\|_m}{\|\eta e_j\|_d} = \lim_{\eta \to 0} \frac{\sqrt{\sum_{i=1}^m \left(f_i(x_0 + \eta e_j) - f_i(x_0) - \eta d_{ij}\right)^2}}{|\eta|} = 0$$

which implies that

$$\lim_{\eta \to 0} \frac{f_i(x_0 + \eta e_j) - f_i(x_0) - \eta d_{ij}}{\eta} = 0 \Longrightarrow \lim_{\eta \to 0} \frac{f_i(x_0 + \eta e_j) - f_i(x_0)}{\eta} = d_{ij}$$

Thus, by definition $\frac{\partial f_j}{\partial x_j}(x_0)$ exists, and $Df(x_0) = \left[\frac{\partial f_i}{\partial x_j}(x_0)\right]_{ij}$. □

Exercise 9. Let $f(x, y) = \frac{xy}{x^2 + y^2}$, if $(x, y) \neq (0, 0)$, and let $f(0, 0) = 0$. Show that the partial derivatives of f exist at $(0, 0)$, but that f is not differentiable at $(0, 0)$.

Proof. Consider first $\frac{\partial f}{\partial x}(0, 0)$. From the definition of the partial derivative, we have that

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0$$

Similarly, we have that

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} 0 = 0$$

So the two partial derivatives do exist. However, f is not differentiable at $(0,0)$. To see why, note that the limit from two directions is:

$$\lim_{h\to 0} f(h,h) = \lim_{h\to 0} \frac{h^2}{2h^2} = \frac{1}{2}$$

and

$$\lim_{h\to 0} f(h,0) = \lim_{h\to 0} \frac{0}{h^2} = 0$$

So f is not continuous at $(0,0)$ and thus is not differentiable. □

Exercise 10. Let $f : (a,b) \subseteq \mathbb{R} \to Y \subseteq \mathbb{R}^d$ be differentiable, and let $g : Y \to \mathbb{R}$ be differentiable at $f(x_0)$ for $x_0 \in (a,b)$. Express $D(g \circ f)$ as a function of the partial derivatives of f and g.

Proof. We have that from the Chain rule:

$$D(g \circ f)(x_0) = Dg(f(x))Df(x_0)$$

From Exercise 8, we have that

$$Dg(f(x)) = \left[\tfrac{\partial g}{\partial f_j(x)} f(x)\right]_{1\times d} \quad \text{and} \quad Df(x_0) = \left[\tfrac{\partial f_j}{\partial x_0}(x_0)\right]_{d\times 1}$$

for $j = \{1, \ldots, d\}$. Thus, we have that

$$D(g \circ f)(x_0) = \left[\tfrac{\partial g}{\partial f_j(x)} f(x)\right]_{1\times d} \cdot \left[\tfrac{\partial f_j}{\partial x_0}(x_0)\right]_{d\times 1} = \sum_{i=1}^{d} \left(\frac{\partial g}{\partial f_i(x)} f(x)\right)\left(\frac{\partial f_i}{\partial x_0}(x_0)\right)$$

□

Exercise 11. Prove the following:
Theorem 1. (Young's Theorem with $d = 2$) *Suppose $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}^m$ and $f \in C^2$ at $x_0 \in \operatorname{int}(X)$. Then, when they both exist,*

$$\frac{\partial^2 f}{\partial x_1 \partial x_2}(x_0) = \frac{\partial^2 f}{\partial x_2 \partial x_1}(x_0)$$

Proof. We have that f is twice continuously differentiable. Consider the rectangle formed by $x_0 + h$, where the points are x_0, $(x_{0,1} + h_1, x_{0,2})$, $(x_{0,1}, x_{0,2} + h_2)$, and $(x_{0,1} + h_1, x_{0,2} + h_2)$. Define the distance functions

$$r(h) = f(x_{0,1} + h_1, x_{0,2} + h_2) - f(x_{0,1} + h_1, x_{0,2})$$

and

$$t(h) = f(x_{0,1} + h_1, x_{0,2} + h_2) - f(x_{0,1}, x_{0,2} + h_2)$$

Then we define

$$d(h) = f(x_{0,1} + h_1, x_{0,2} + h_2) - f(x_{0,1} + h_1, x_{0,2}) - f(x_{0,1}, x_{0,2} + h_2) + f(x_0)$$

and note that

$$d(h) = r(h_1, h_2) - r(0, h_2) = t(h_1, h_2) - t(h_1, 0)$$

Since these are all additive functions of f, which is twice continuously differentiable, all of these functions are continuous and differentiable on their domains, so the Mean Value Theorem applies. We have that there exists $y \in (0, h_1), y' \in (0, h_2)$ such that

$$d(h) = r(h_1, h_2) - r(0, h_2) = r'(y, h_2) \cdot (h_1, 0)$$

and

$$d(h) = t(h_1, h_2) - t(h_1, 0) = t'(h_1, y') \cdot (0, h_2)$$

so

$$r'(y, h_2) \cdot (h_1, 0) = t'(h_1, y') \cdot (0, h_2)$$

Thus, we have that

$$\frac{\partial}{\partial h}\left[f(x_{0,1} + y, x_{0,2} + h_2) - f(x_{0,1} + y, x_{0,2})\right](h_1, 0) = \frac{\partial}{\partial h}[f(x_{0,1} + h_1, x_{0,2} + y') - f(x_{0,1}, x_{0,2} + y')](0, h_2)$$

which implies that

$$h_1\left(\frac{\partial f}{\partial x_1}(x_{0,1} + y, x_{0,2} + h_2) - \frac{\partial f}{\partial x_1}(x_{0,1} + y, x_{0,2})\right) = h_2\left(\frac{\partial f}{\partial x_2}(x_{0,1} + h_1, x_{0,2} + y') - \frac{\partial f}{\partial x_2}(x_{0,1}, x_{0,2} + y')\right)$$

Since $f \in C^2$, we have that each of the parts inside the parentheses are continuous and differentiable. Thus, using the Mean Value Theorem again, we get that there exists $z \in (0, h_1), z' \in (0, h_2)$ such that this becomes

$$h_1\left(\frac{\partial}{\partial z}\frac{\partial f}{\partial x_1}(x_0 + z) \cdot (0, h_2)\right) = h_2\left(\frac{\partial}{\partial z'}\frac{\partial f}{\partial x_2}(x_0 + z') \cdot (h_1, 0)\right)$$

Recalling that $y, y', z, z' \in (0, h)$, we have that as $h \to 0$, $y, y', z, z' \to h$, and this becomes

$$h_1\left(\frac{\partial}{\partial h}\frac{\partial f}{\partial x_1}(x_0 + h) \cdot (0, h_2)\right) = h_2\left(\frac{\partial}{\partial h}\frac{\partial f}{\partial x_2}(x_0 + h) \cdot (h_1, 0)\right)$$

Simplifying the partial derivatives, we get that this is

$$h_1\left(h_2\frac{\partial^2 f}{\partial x_2 \partial x_1}(x_0 + h)\right) = h_2\left(h_1\frac{\partial^2 f}{\partial x_1 \partial x_2}(x_0 + h)\right)$$

So we have that

$$\frac{\partial^2 f}{\partial x_2 \partial x_1}(x_0 + h) = \frac{\partial^2 f}{\partial x_1 \partial x_2}(x_0 + h)$$

As $h \to 0$, since $f \in C^2$, we can conclude that

$$\frac{\partial^2 f}{\partial x_2 \partial x_1}(x_0) = \frac{\partial^2 f}{\partial x_1 \partial x_2}(x_0)$$

$\square$

Exercise 14. Let $f : X \subseteq \mathbb{R}^d \to \mathbb{R}$, where X is nonempty, open, and convex. For any $x, v \in \mathbb{R}^d$, let $S_{x,v} := \{t \in \mathbb{R} : x + tv \in X\}$ and define $g_{x,v} : S_{x,v} \to \mathbb{R}$ as $g_{x,v}(t) := f(x + tv)$. Then f is (resp. strictly) concave on X if and only if $g_{x,v}$ is (resp. strictly) concave for all $x, v \in \mathbb{R}^d$ with $v \neq 0$.

Proof. ($\Rightarrow$): We have that f is concave on X, meaning that $f''(x) \leq 0$ for all $x \in X$. We also have that from the chain rule,

$$g'_{x,v}(t) = f'(x + tv) \cdot v \Longrightarrow g''_{x,v}(t) = f''(x + tv) \cdot v^2$$

Thus, when $v \neq 0$, $g''_{x,v}(t) \leq 0$. A similar proof holds when f is strictly concave, replacing $\leq$ with $<$.

($\Leftarrow$): We have that g is concave for all $x, v \in \mathbb{R}^d$ where $v \neq 0$. Again from the chain rule, we have that

$$g''_{x,v}(t) = f''(x+tv)v^2 \implies f''(x+tv) = \frac{g''_{x,v}(t)}{v^2}$$

and since $v \neq 0$ and $x + tv \in X$ by definition, we have that $f''(x+tv)$ is concave whenever the argument is in X. A similar proof holds when f is strictly concave. $\square$

Exercise 17. Let $f : \mathbb{R}^2_{++} \to \mathbb{R}$ be defined by $f(x,y) \coloneqq x^\alpha y^\beta$ for some $\alpha, \beta > 0$. Compute the Hessian of f at $(x,y) \in \mathbb{R}^2_{++}$. Find conditions on α and β such that f is (i) strictly concave, (ii) concave but not strictly concave, and (iii) neither concave nor convex. How do your answers change if the domain of f was $\mathbb{R}^2_+$?

Solution. We have that

$$H_f = \begin{bmatrix} \frac{\partial^2 f}{(\partial x)^2}(x,y) & \frac{\partial^2 f}{\partial x \partial y}(x,y) \\ \frac{\partial^2 f}{\partial y \partial x}(x,y) & \frac{\partial^2 f}{(\partial y)^2}(x,y) \end{bmatrix} = \begin{bmatrix} \alpha(\alpha-1)x^{\alpha-2}y^\beta & \alpha\beta x^{\alpha-1}y^{\beta-1} \\ \alpha\beta x^{\alpha-1}y^{\beta-1} & \beta(\beta-1)x^\alpha y^{\beta-2} \end{bmatrix}$$

From Proposition 15, we have that H_f being negative definite implies that f is strictly concave. We have that the determinant of H_f is

$$\det(H_f) = (\alpha(\alpha-1)x^{\alpha-2}y^\beta)(\beta(\beta-1)x^\alpha y^{\beta-2}) - (\alpha\beta x^{\alpha-1}y^{\beta-1})^2$$

so simplifying, we get that

$$\det(H_f) = \alpha\beta x^{2\alpha-2}y^{2\beta-2}(1-\alpha-\beta)$$

Additionally, the trace of H_f is

$$\operatorname{tr}(H_f) = \alpha(\alpha-1)x^{\alpha-2}y^\beta + \beta(\beta-1)x^\alpha y^{\beta-2} = x^\alpha y^\beta \left(\frac{\alpha^2-\alpha}{x^2} + \frac{\beta^2-\beta}{y^2}\right)$$

A matrix is negative definite if its Eigenvalues are all negative. Equivalently, since this is a 2×2 matrix, it is negative definite if the determinant is positive and the trace is negative. This condition is satisfied when $1-\alpha-\beta > 0$ and when $\alpha^2-\alpha$ and $\beta^2-\beta$ are both negative. This implies that $\alpha, \beta \in (0,1)$ and $\alpha+\beta < 1$.

Similarly, this function is concave but not strictly concave if the Hessian is negative semi-definite but not negative definite. This happens when the determinant is non-negative and the trace is non-positive, which happens when $1-\alpha-\beta \leq 0$ and $\alpha^2-\alpha, \beta^2-\beta \leq 0$. Since we also need that the function not be strictly concave, this implies that $\alpha, \beta \in \{0,1\}$, and $\alpha \neq \beta$.

Finally, this function is neither concave nor convex when the determinant is negative, which implies that $1-\alpha-\beta < 0$, with the condition that $\alpha+\beta > 1$.

If the domain of f were instead $\mathbb{R}^2_+$, none of these conditions would be sufficient. Specifically, since we can have that $(x,y) = (0,0)$, it is possible that the Hessian takes indeterminate values depending on the values of α and β.

2 Additional Exercises

Theorem 2. Euler's Theorem *If $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is differentiable at $x \in \operatorname{int}(X)$ and homogenous of degree k, then*

$$\nabla f(x)x = kf(x)$$

Proof. We have that f is homogeneous of degree k, which means that $f(\lambda x) = \lambda^k f(x)$ for all $\lambda \in \mathbb{R}_{++}$. We will differentiate both sides with respect to λ, using the chain rule. We get that

$$\nabla f(\lambda x) \cdot x = k\lambda^{k-1} f(x)$$

Then, choosing $\lambda = 1$, we get that

$$\nabla f(x) x = k f(x)$$

$\square$