About TA sections:

TAs: Ekaterina Zubova (ez268@cornell.edu), Zheyang Zhu (zz792@cornell.edu)

Section time and location: 8:40am - 9:55am Uris Hall 262 (section 201), Goldwin Smith Hall 236 (section 202)

Office hours: Tuesdays 5-7 pm in Uris Hall 451 (Ekaterina), Thursdays 5-7 pm in Uris Hall 429 (Zheyang). Other times available by appointment (just send us an email!)

Our plan for today:[1]

[1] Materials adapted from notes provided by a previous Teaching Assistant, Zhuoheng Xu.

1 State Variables vs. Control Variables

1.1 Definitions

Recall the simple neoclassical model:

$$\max_{\{c_t,k_{t+1},n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

subject to:

$$F(k_t,n_t) = c_t + k_{t+1} - (1-\delta)k_t$$

$$c_t \geq 0, \quad k_t \geq 0, \quad 0 \leq n_t \leq 1$$

$$k_0 \text{ given}$$

State variables are the variables that the decision-maker faces in period t, i.e., they are predetermined and represent the current state of the system or economy, which the decision-maker cannot change in the current period. In order to make decisions in period t, the decision-maker has to have these state variables in hand.

Control variables (also called "choice" variables) are the variables that determine the (expected) payoff in the current period and the (expected) state next period. In other words, control variables are the decision variables that the agent optimizes over in the current period to influence future states and payoffs.

How to tell the difference?

- State variables determine what I am "endowed with" at the beginning of a period, before taking any actions.
- There are two key criteria for determining whether a variable is a state variable: 1) it must be *predetermined*, and 2) it must be *relevant to the decision to be made.*
- Usually, **random/exogenous variables** (e.g., idiosyncratic/aggregate productivity level in business cycles), **"dynamic" endogenous variables** (i.e., capital, human capital, knowledge...), are all state variables.
- Control variables are the endogenous decisions I have to make that determine the

"flow value" of my utility, considering what I am already endowed with.

Intuition: "Each day, I begin with the state variables at my disposal. I make decisions regarding the control variables, and then I end the day, ready to start again the next morning."

1.2 Examples

Example 1: The Simple Neoclassical Model

In the simple neoclassical model, find

(a) all state variables

$$k_t$$

(b) all control variables

$$c_t, k_{t+1}, \text{ and } n_t$$

Example 2: The Neoclassical Model With Productivity Uncertainty

Consider the following model:

$$\max_{\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

subject to:

$$z_t k_t^{\alpha} n_t^{1-\alpha} = c_t + k_{t+1} - (1-\delta)k_t$$

$$\ln(z_t) = \rho \cdot \ln(z_{t-1}) + \epsilon_t \quad \text{with} \quad E(\epsilon_t) = 0$$

$$c_t \geq 0, \quad k_t \geq 0, \quad 0 \leq n_t \leq 1, \quad 0 < \rho < 1$$

k_0 given, where z_t is the productivity level, and ϵ_t is the productivity shock.

(a) Is c_{t-1} a state variable?

(b) Is z_{t-1} a state variable? (Note: When I wake up in the morning, I observe z_t.)

(c) Find all state variables in this model.

2 Bellman Equation

2.1 Idea

The Bellman Equation is a recursive formula that serves as the foundation for dynamic programming. It calculates the expected total reward of taking an action from a given state by decomposing the problem into two components: the immediate reward (also known as the flow value) and the total future expected reward (referred to as the expected future stock value). This decomposition allows the decision-maker to evaluate the consequences of current actions not only based on immediate outcomes but also by considering their impact on future opportunities.

1. **Figure out state variables and control variables**: State variables represent the current situation that the decision-maker cannot change in the current period but will influence future decisions. Control variables are the choices the decision-maker can make in the current period to influence both immediate outcomes and future states.

2. **Rearrange the objective function and constraints and write down the Bellman equation**: The objective function, often representing utility or profit maximization, is rewritten to express the problem recursively. This involves linking the current period's decision to future periods, typically by expressing the value function, which summarizes the maximum achievable utility starting from a given state.

Think about why the Bellman Equation is particularly powerful:

1. **Computationally easy to handle**: The recursive structure of the Bellman Equation simplifies the problem by breaking it down into smaller, manageable subproblems that can be solved period by period.

2. **Easy to interpret**: Value = Welfare = NPV of total utility under the optimum = immediate payoff + discounted future payoffs. In words, the value of being in a certain state (welfare) is equivalent to the sum of the immediate payoff (usually utility in the current period) and the net present value (NPV) of future utility, assuming optimal decisions are made in all future periods.

2.2 Writing the Basic Neoclassical Growth Model into Bellman Form

The Social Planner solves the following problem:

$$V(k_0) = \max_{\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

subject to:

$$F(k_t, n_t) = c_t + k_{t+1} - (1-\delta)k_t$$

$$c_t \geq 0, \quad k_t \geq 0, \quad 0 \leq n_t \leq 1$$

k_0 given.

The reason why we can write the "value" or welfare as $V(k_0)$ is that: under optimal decisions, we know the initial capital stock k_0 essentially pins down all decisions in the economy, and therefore pins down the equilibrium welfare.

We can rewrite the problem as:

$$V(k_0) = \max_{\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

$$= \max_{\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty}} \left(U(c_0) + \sum_{t=1}^{\infty} \beta^t U(c_t) \right)$$

$$= \max_{\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty}} \left(U(c_0) + \beta \sum_{t=1}^{\infty} \beta^{t-1} U(c_t) \right)$$

$$= \max_{\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty}} \left(U(c_0) + \beta \sum_{t=1}^{\infty} \beta^{s} U(c_{s+1}) \right)$$

Let $s = t - 1$:

$$= \max_{\{c_0, k_1, n_0\}} \left(U(c_0) + \beta \max_{\{c_s, k_{s+1}, n_s\}_{s=0}^{\infty}} \sum_{s=0}^{\infty} \beta^s U(c_{s+1}) \right)$$

$$= \max_{\{c_0, k_1, n_0\}} \left(U(c_0) + \beta V(k_1) \right)$$

$$= \max_{\{c_0,k_1,n_0\}} (U(F(k_0,n_0) - k_1 + (1-\delta)k_0) + \beta V(k_1))$$

with inelastic labor supply we conclude:

$$= \max_{\{c_0,k_1\}} (U(F(k_0,1) - k_1 + (1-\delta)k_0) + \beta V(k_1))$$

Define $f(k) = F(k,1) + (1-\delta)k$:

$$= \max_{\{0<k_1\leq f(k_0)\}} (U(f(k_0) - k_1) + \beta V(k_1))$$

which is exactly what we want.

Note that this formula holds for arbitrary t, and denoting time $t+1$ variables using primed notations, we get:

$$v(k) = \max_{0\leq k'\leq f(k)} \{U(f(k) - k') + \beta v(k')\}$$

which is the form in the lecture slides.

What's next? Finding the policy function (optimal decision rule for endogenous state variables, k'), namely:

$$k' = g(k)$$

Two ways to do it:

1. Numerical Methods (e.g., Value Function Iteration, to be covered in the next section): generally applicable.

2. Guess and Verify: straightforward but not always possible.

3 Practice Question

3.1 Question 1 (Midterm 2021)

Problem 1. How to eat a pie optimally.

An agent must decide on the optimal strategy to eat a delicious pie. The pie is initially of size $W_0 > 0$ and a fraction $1 - R$, $0 < R < 1$, of what hasn't been eaten in a given day goes bad. The full pie eating problem is

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \delta^t u(c_t)$$

subject to the constraints

$$W_{t+1} = R(W_t - c_t)$$

$$0 \leq c_t \leq W_t$$

$$W_0 > 0, \text{given.}$$

1. (5 points) Write the Bellman equation associated with this problem. Your value function v should only be a function of the state variable W and your control variable should be W'.

4 Guess and Verify

4.1 Algorithm

Recall the Simple Neoclassical Model

$$\max_{\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log(c_t)$$

subject to:

$$k_t^{\alpha} n_t^{1-\alpha} = c_t + k_{t+1}$$

$$c_t \geq 0, \quad k_t \geq 0, \quad 0 \leq n_t \leq 1$$

k_0 given

The Bellman Equation

$$v(k) = \max_{0 \leq k' \leq k^{\alpha}} \{\log(k^{\alpha} - k') + \beta v(k')\}$$

Step 1: Guess a functional form of the value function $v(\cdot)$

$$v(k) = A + B \log(k)$$

Remark: Usually, on the exam, you don't have to come up with a guess yourself. You are given a guess and only need to check whether it is correct.

Step 2: Substitute the conjectured functional form $v'(\cdot)$ into the Bellman Equation

$$v(k) = \max_{0 \leq k' \leq k^{\alpha}} \{\log(k^{\alpha} - k') + \beta A + \beta B \log(k')\}$$

Step 3: Find the FOCs to obtain the policy function $g(\cdot)$

$$k' = g(k) = \frac{\beta B k^{\alpha}}{1 + \beta B}$$

Step 4: Substitute the policy function $g(\cdot)$ into the Bellman Equation to verify whether the conjectured functional form is correct

$$v(k) = -\log(1+\beta B) + \beta A + \beta B \log\left(\frac{\beta B}{1+\beta B}\right) + \alpha \log(k) + \alpha\beta B \log(k)$$

Step 5a: If the guess is wrong, go back to Step 1 and come up with a new functional form

Step 5b: If the guess is right, apply the method of undetermined coefficients to determine coefficients in the policy function $g(\cdot)$

$$B = \alpha(1+\beta B)$$

$$g(k) = \frac{\beta B k^\alpha}{1+\beta B} = \alpha\beta k^\alpha$$

4.2 Example

Consider the representative agent that has persistence of habit.

$$\max_{\{c_t, k_{t+1}\}_{t=0}^\infty} \sum_{t=0}^{\infty} \beta^t \left(\ln c_t + \gamma \ln c_{t-1}\right)$$

subject to:

$$c_t + k_{t+1} \leq A k_t^\alpha$$

(a) Write down the Bellman equation

(b) Prove the solution of Bellman's equation is of the form

$$v(k, c_{t-1}) = E + F \ln k + G \ln c_{t-1}$$

and verify that the optimal policy is of the form

$$\ln k_{t+1} = I + H \ln k_t$$

where E, F, G, H, and I are constants. Give explicit formulas for the constants in terms of the parameters, A, β, α, and γ.