ECON 6130: Neoclassical growth model

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Moving from endowment economies

So far, we have only studied economies with no production. These were useful to familiarize ourselves with

- Equilibrium concept
- Consumption smoothing across time and across histories
- Asset pricing
- Efficiency of allocations

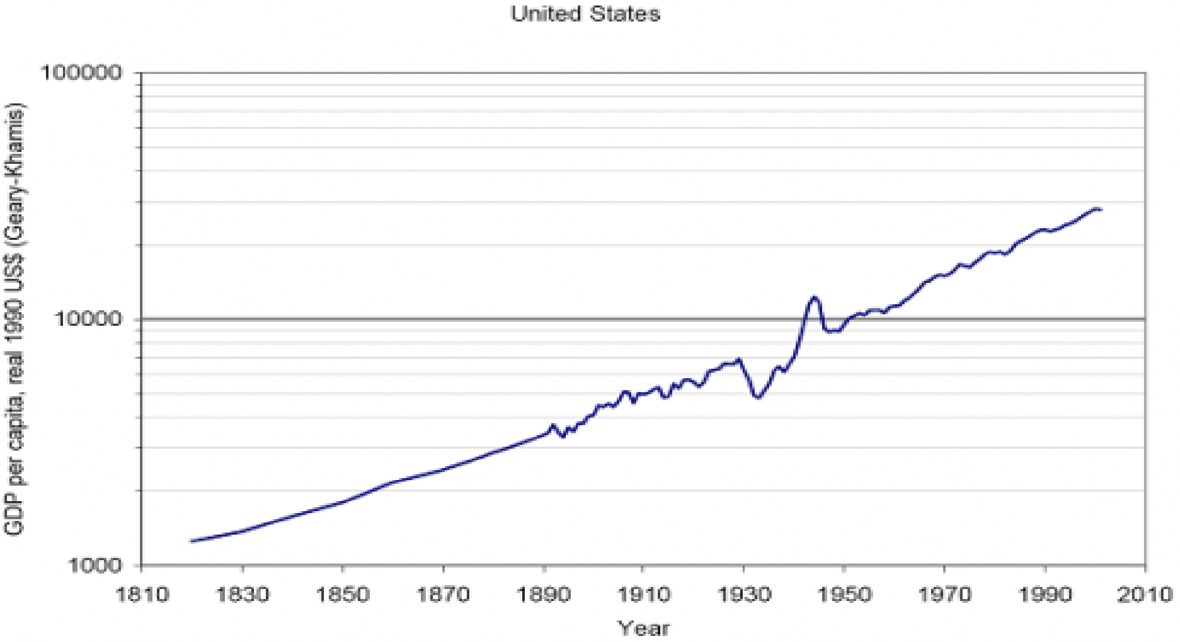
These models are obviously not very good at explaining growth. We now introduce production.

We will use the growth model as a gateway to study dynamic programming.

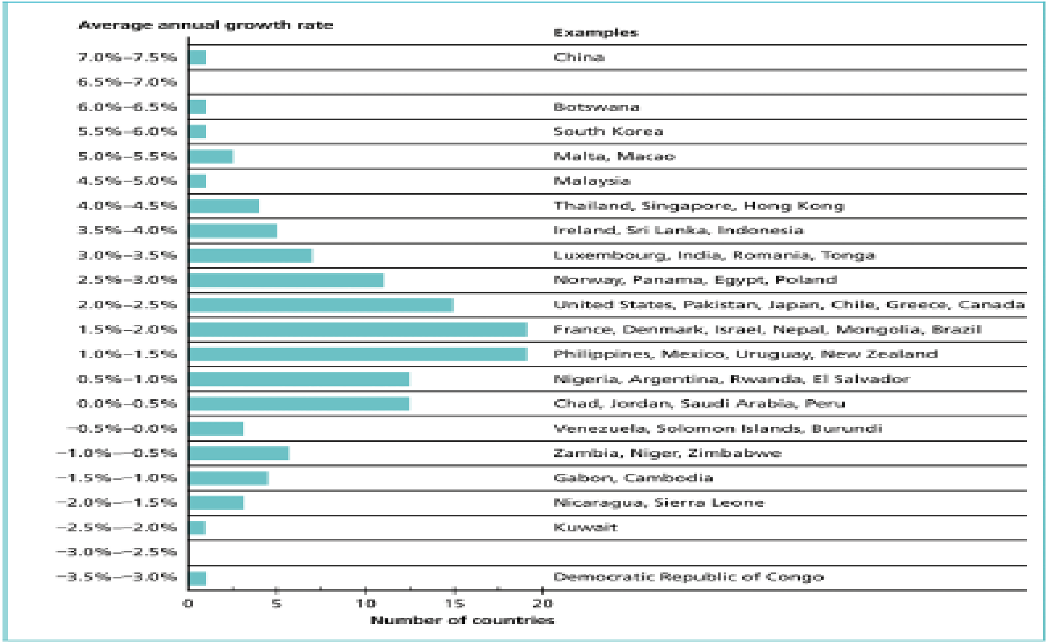
Main facts about long-run growth

Kaldor (1959) popularized the following six stylized facts concerning long run economic growth

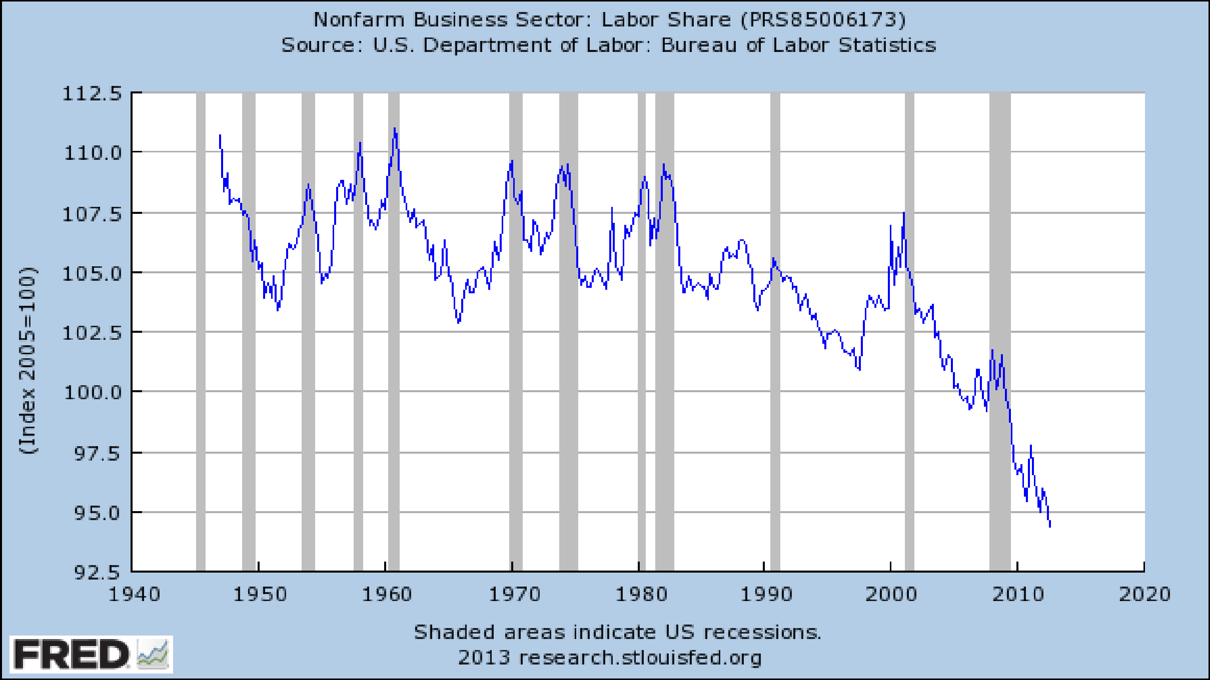
1. Output per capita, Y/N, grows at a constant rate
2. The capital to labor ratio, K/N, grows at constant rate
3. The interest rate, R, is fairly constant
4. The output to capital ratio, Y/K, is fairly constant
5. The share of value added going to labor and capital are fairly constant
6. There are wide dispersion in Y_i/N_i across countries



United States
100000
10000
1000
GDP per capita, real 1990 US$ (Geary-Khamis)
1810
1830
1850
1870
1890
1910
1930
1950
1970
1990
2010
Year



Sources: Heston, Summers, and Aten (2006), World Bank (2007a).


Nonfarm Business Sector: Labor Share (PRS85006173)
Source: U.S. Department of Labor: Bureau of Labor Statistics
112.5
110.0
107.5
105.0
102.5
100.0
97.5
95.0
92.5
(Index 2005=100)
1940
1950
1960
1970
1980
1990
2000
2010
2020
Shaded areas indicate US recessions.
2013 research.stlouisfed.org
FRED

Neoclassical growth model in discrete time

- Time is discrete, $t = 0, 1, 2, \ldots$
- In each period, three goods are traded:
  - labor services n_t
  - capital services k_t
  - final good output y_t that can be consumed (c_t) or invested (i_t)
- Aggregate Production function F
  - output $y_t = F(k_t, n_t)$ is consumed or invested $y_t = c_t + i_t$
  - investment increases capital stock which depreciate at rate $\delta > 0$

$$k_{t+1} = (1 - \delta)k_t + i_t$$

- Preferences: large number of identical, infinitely lived households:

$$u(\{c_t\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta^t U(c_t)$$

- Endowments: initial capital k_0 given and one unit of time each period.

Optimal growth

For now, we will concerned ourselves with *optimal growth*. We will study the problem of a social planner who maximizes total welfare.

Definition 1 (Feasible allocation)

An allocation $\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty}$ *is feasible if, for all* $t \geq 0$

$$\begin{aligned} F(k_t, n_t) &= c_t + k_{t+1} - (1-\delta)k_t \\ & c_t \geq 0,\ k_t \geq 0,\ 0 \leq n_t \leq 1 \\ & k_0 \textit{ given} \end{aligned}$$

Definition 2 (Pareto efficient allocation)

An allocation $\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty}$ *is Pareto efficient if it is feasible and there is no other feasible allocation* $\{\hat{c}_t, \hat{k}_{t+1}, \hat{n}_t\}_{t=0}^{\infty}$ *such that*

$$\sum_{t=0}^{\infty} \beta^t U(\hat{c}_t) > \sum_{t=0}^{\infty} \beta^t U(c_t)$$

Social planner problem

The SP solves:

$$w(k_0) = \max_{\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

subject to:

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How can we interpret $w(\bar{k}_0)$?

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How can we interpret $w(\bar{k}_0)$?

- Function that gives the *total lifetime* utility of the *representative household* with initial capital stock $\bar{k}_0$ when the social planner is behaving *optimally*.

Assumptions

We make the following assumptions:

- Utility function
  1. U is continuously differentiable, strictly increasing, strictly concave and bounded
  2. Inada conditions: $\lim_{c \to 0} U'(c) = \infty$ and $\lim_{c \to \infty} U'(c) = 0$
  3. $\beta \in (0, 1)$
- Production function
  1. F is continuously differentiable and homogenous of degree 1, strictly increasing and strictly concave
  2. $F(0, n) = F(k, 0) = 0$ for all $k, n > 0$
  3. Inada condition: $\lim_{k \to 0} F_k(k, 1) = \infty$ and $\lim_{k \to \infty} F_k(k, 1) = 0$

These assumptions imply:

- From the structure of U, $n_t = 1$ for all t
- We can write

$$f(k) = F(k, 1) + (1 - \delta)k$$

  What properties does f have?

Simplified social planner problem

Since $c_t = f(k_t) - k_{t+1}$, we can write the SP problem as

$$w(k_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(f(k_t) - k_{t+1})$$

subject to:

$$0 \leq k_{t+1} \leq f(k_t)$$
$$k_0 \text{ given}$$

- Why do we care about this problem?

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- Why do we care about this problem? It turns out that the welfare theorems apply here. By solving the SP problem we solve for the competitive equilibrium.
- How do we solve it? This is an infinite dimensional optimization problem. We will use *dynamic programming* to rewrite the problem in a much simpler form.

Dynamic programming

Main idea: Use the stationary nature of the economic environment to rewrite the problem in a *recursive* way.

$$
\begin{aligned}
w(k_0) &= \max_{\substack{\{k_{t+1}\}_{t=0}^{\infty} \text{ s.t.} \\ 0 \le k_{t+1} \le f(k_t),\, k_0 \text{ given}}} \sum_{t=0}^{\infty} \beta^t U(f(k_t) - k_{t+1}) \\
&= \max_{\substack{k_1 \text{ s.t.} \\ 0 \le k_1 \le f(k_0),\, k_0 \text{ given}}} \Bigg(U(f(k_0) - k_1) \\
&\quad + \beta \left(\max_{\substack{\{k_{t+1}\}_{t=1}^{\infty} \text{ s.t.} \\ 0 \le k_{t+1} \le f(k_t),\, k_1 \text{ given}}} \sum_{t=1}^{\infty} \beta^{t-1} U(f(k_t) - k_{t+1}) \right) \Bigg)
\end{aligned}
$$

Dynamic programming

Intuitively, it looks like:

$$w(k_0) = \max_{\substack{0 \le k_1 \le f(k_0) \\ k_0 \text{ given}}} U(f(k_0) - k_1) + \beta w(k_1)$$

- When is the intuitive suggestion correct?
- Why is this new problem better than the old one?

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- When is the intuitive suggestion correct?
- Why is this new problem better than the old one? much simpler than finding $\{k_{t+1}\}_{t=0}^{\infty}$

Recursive formulation

Denote by $v(\cdot)$ the value function for this new formulation of the problem:

$$v(k) = \max_{0 \leq k' \leq f(k)} \left\{ U(f(k) - k') + \beta v(k') \right\} \tag{*}$$

Interpretation: $v(k)$ is the discounted lifetime utility of the representative agent, from the current period onward, if the social planner has initial capital stock k and allocates consumption optimally.

- (*) is the recursive formulation of the planner's problem.
- (*) is a functional equation called the Bellman equation.
- k is called the state variable. It completely describes the economy today.
- k' is called the control variable. It is decided today by the planner.
- To solve (*) we need a value function and a policy function $k' = g(k)$.

Recursive formulation

This new approach raises questions:

- Under what conditions does a solution to (*) exist and, if so, is it unique?

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- Under what conditions can we solve (*) and be sure that we have solved the sequential problem? (i.e. $v = w$ and $g(k)$ generates the optimal $\{k_{t+1}\}_{t=0}^{\infty}$).

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Proving these results requires heavier mathematics that we will introduce later. For now, we will look at a few examples and show the link between the social planner problem and competitive equilibrium.

An example of a recursive problem

Let $U(c) = \log(c)$, $F(k, n) = k^\alpha n^{1-\alpha}$ and $\delta = 1$. Then $f(k) = k^\alpha$ and

$$v(k) = \max_{0 \leq k' \leq k^\alpha} \left\{ \log(k^\alpha - k') + \beta v(k') \right\}$$

Guess and verify

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Guess that $v(k)$ has the form

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for some constant A and B that we have to determine.

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Now the maximization problem (taking k as given is)

$$\max_{0 \le k' \le k^\alpha} \left\{ \log(k^\alpha - k') + \beta \left(A + B\log(k') \right) \right\}$$

and the FOC is

$$k' = \frac{\beta B k^\alpha}{1 + \beta B}$$

Guess and verify

The second step is to plug back the optimal k' into the Bellman equation:

$$\begin{aligned} v(k) &= \max_{0 \le k' \le k^\alpha} \{\log(k^\alpha - k') + \beta v(k')\} \\ &= \log(k^\alpha - k') + \beta(A + B\log(k')) \\ &= \log\left(\frac{k^\alpha}{1+\beta B}\right) + \beta A + \beta B \log\left(\frac{\beta B k^\alpha}{1+\beta B}\right) \\ &= -\log(1+\beta B) + \beta A + \beta B \log\left(\frac{\beta B}{1+\beta B}\right) + \alpha\log(k) + \alpha\beta B\log(k) \end{aligned}$$

Was our guess correct?

Guess and verify

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&= -\log(1+\beta B) + \beta A + \beta B \log\left(\frac{\beta B}{1+\beta B}\right) + \alpha\log(k) + \alpha\beta B\log(k)
\end{aligned}
$$

Was our guess correct? Yes!

$$
\begin{aligned}
B &= \alpha(1+\beta B) \\
A &= \frac{1}{1-\beta}\left(\frac{\alpha\beta}{1-\alpha\beta}\log(\alpha\beta) + \log(1-\alpha\beta)\right)
\end{aligned}
$$

Is the solution unique?

Guess and verify

We now need to find the allocation. Remember that $g(k) = k'$:

$$
\begin{aligned}
g(k) &= \frac{\beta B k^{\alpha}}{1 + \beta B} \\
&= \alpha \beta k^{\alpha}
\end{aligned}
$$

How can we interpret this policy rule?

Guess and verify

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How can we interpret this policy rule? Save a constant fraction $\alpha\beta$ of output k^{α} and consume what's left.
We can construct the whole sequence $\{k_{t+1}\}_{t=0}^{\infty}$

$$\begin{aligned} k_1 &= g(k_0) = \alpha\beta k_0^{\alpha} \\ k_2 &= g(k_1) = \alpha\beta k_1^{\alpha} = (\alpha\beta)^{1+\alpha} k_0^{\alpha^2} \\ k_3 &= g(k_2) = \ldots \end{aligned}$$

How useful is the guess and verify approach? Unfortunately, it works in very few cases.

Value function iteration

1. Guess an arbitrary function $v_0(k)$, say $v_0(k) = 0$
2. Solve
$$v_1(k) = \max_{0 \le k' \le k^\alpha} \left\{ \log(k^\alpha - k') + \beta v_0(k') \right\}$$
   The solution is $k' = g_1(k) = 0$ for all k. Therefore
$$v_1(k) = \log(k^\alpha - 0) = \alpha \log(k)$$
3. Since we know v_1, now we can solve
$$v_2(k) = \max_{0 \le k' \le k^\alpha} \left\{ \log(k^\alpha - k') + \beta v_1(k') \right\}$$
4. Repeat for
$$v_{n+1}(k) = \max_{0 \le k' \le k^\alpha} \left\{ \log(k^\alpha - k') + \beta v_n(k') \right\}$$
   to get $\{v_n\}_{n=0}^{\infty}$ and $\{g_n\}_{n=0}^{\infty}$.
5. Will these sequences converge to the optimum solution g^* and v^*?

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5. Will these sequences converge to the optimum solution g^* and v^*? Yes by the CMT.

Value function iteration a numerical example

A computer can only deal with finite-dimensional objects. We can only *approximate* the value function.

Here is an example from Dirk Krueger's notes.

- *Discretize* the space: $k, k' \in K = \{0.04, 0.08, 0.12, 0.16, 0.2\}$
- Value functions v_n:

$$(v_n(0.04), v_n(0.08), v_n(0.12), v_n(0.16), v_n(0.2))$$

- Pick values for the parameters. Say, $\alpha = 0.3$ and $\beta = 0.6$.

Value function iteration a numerical example

Numerical algorithm

1. Initial guess $v_0(k) = 0$ for all $k \in K$.
2. Solve
$$v_1(k) = \max_{0 \leq k' \leq k^{0.3}} \left\{ \log(k^{0.3} - k') + 0.6 \times 0 \right\}$$
Optimal policy $k'(k) = g_1(k) = 0.04$ for all $k \in K$. Plugging back in:
$$v_1(0.04) = \log(0.04^{0.3} - 0.04) = -1.077$$
$$v_1(0.08) = \log(0.08^{0.3} - 0.04) = -0.847$$
$$v_1(0.12) = \log(0.12^{0.3} - 0.04) = -0.715$$
$$v_1(0.16) = \log(0.16^{0.3} - 0.04) = -0.622$$
$$v_1(0.20) = \log(0.20^{0.3} - 0.04) = -0.55$$

Value function iteration a numerical example

Next iteration

$$v_2(k) = \max_{0 \leq k' \leq k^{0.3}} \left\{ \log(k^{0.3} - k') + 0.6 v_1(k') \right\}$$

Start with $k = 0.04$:

$$v_2(0.04) = \max_{0 \leq k' \leq 0.04^{0.3}} \left\{ \log(0.04^{0.3} - k') + 0.6 v_1(k') \right\}$$

Value function iteration a numerical example

Let's try different values for k'.

If $k' = 0.04$, then

$$v_2(0.04) = \log(0.04^{0.3} - 0.04) + 0.6(-1.08) = -1.72$$

If $k' = 0.08$, then

$$v_2(0.04) = \log(0.04^{0.3} - 0.08) + 0.6(-0.85) = -1.71$$

If $k' = 0.12$, then

$$v_2(0.04) = \log(0.04^{0.3} - 0.12) + 0.6(-0.72) = -1.77$$

If $k' = 0.16$, then

$$v_2(0.04) = \log(0.04^{0.3} - 0.16) + 0.6(-0.62) = -1.88$$

If $k' = 0.20$, then

$$v_2(0.04) = \log(0.04^{0.3} - 0.20) + 0.6(-0.55) = -2.04$$

Therefore, for $k = 0.04$ the optimal choice is $k'(0.04) = g_2(0.04) = 0.08$ and $v_2(0.04) = -1.71$

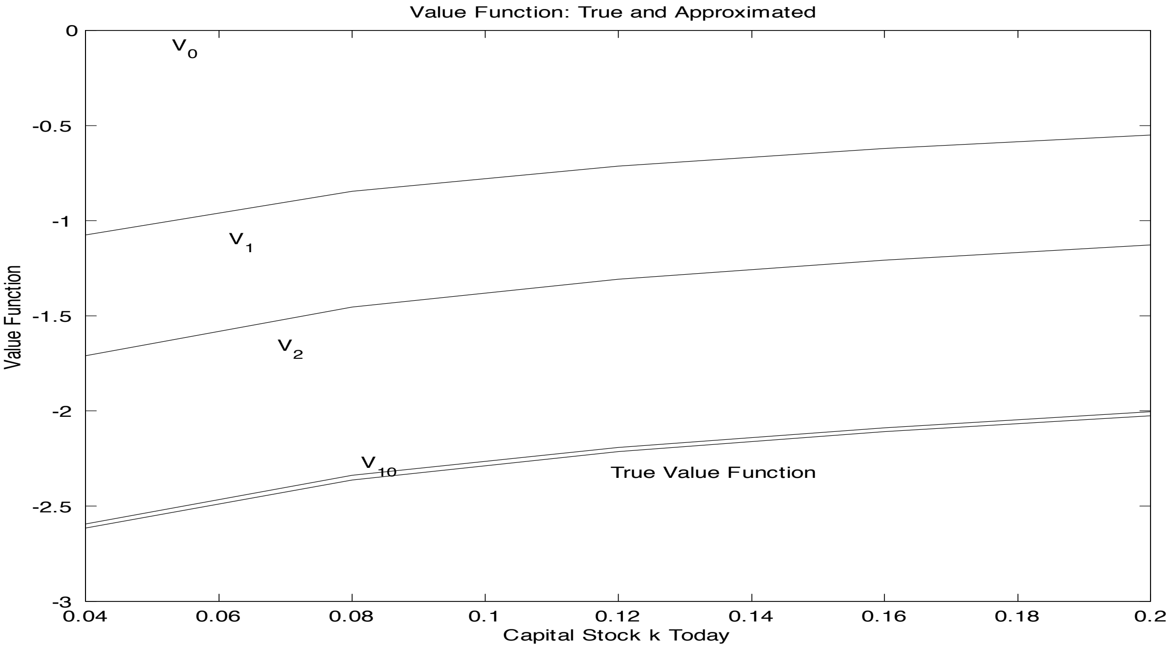
Value function iteration a numerical example

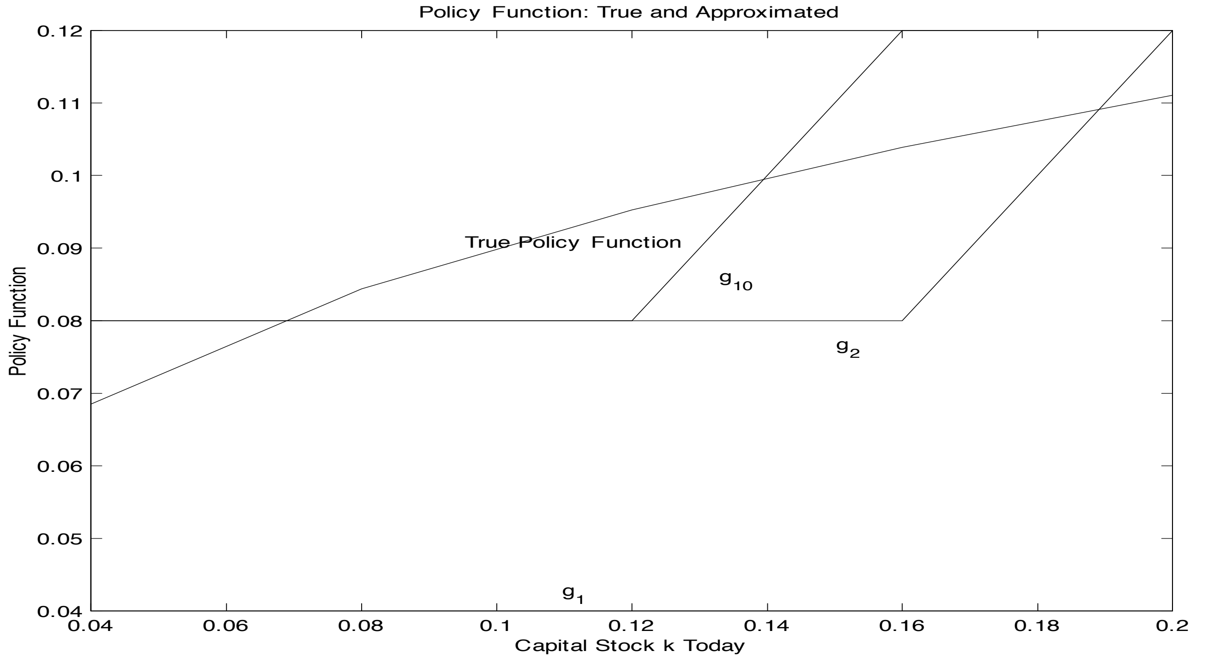
Summary of the second iteration: Table below shows the value of

$$(k^{0.3} - k') + 0.6v_1(k')$$

for different values of k and k'.

(k, k')	0.04	0.08	0.12	0.16	0.2
0.04	-1.72	-1.71*	-1.77	-1.88	-2.04
0.08	-1.49	-1.45*	-1.48	-1.55	-1.64
0.12	-1.36	-1.31*	-1.32	-1.37	-1.44
0.16	-1.27	-1.21*	-1.21	-1.25	-1.31
0.20	-1.20	-1.13	-1.13*	-1.16	-1.20


Value Function: True and Approximated
0
-0.5
-1
-1.5
-2
-2.5
-3
Value Function
V0
V1
V2
V10
True Value Function
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
0.2
Capital Stock k Today



Policy Function: True and Approximated
0.12
0.11
0.1
0.09
0.08
0.07
0.06
0.05
0.04
Policy Function
True Policy Function
g_{10}
g_2
g_1
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
0.2
Capital Stock k Today

Euler equation approach

Back to original problem:

$$w(k_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(f(k_t) - k_{t+1})$$

subject to:

$$0 \leq k_{t+1} \leq f(k_t)$$
$$k_0 \text{ given}$$

We cannot use the standard Kuhn-Tucker theorem to solve the optimization problem. Why?

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We cannot use the standard Kuhn-Tucker theorem to solve the optimization problem.
Why? Infinite-dimensional object
But we can solve the SP problem if there is a final period, T.

Euler equation approach

In which case:

$$w^T(k_0) = \max_{\{k_{t+1}\}_{t=0}^T} \sum_{t=0}^{T} \beta^t U(f(k_t) - k_{t+1})$$

subject to:

$$0 \leq k_{t+1} \leq f(k_t) \text{ and } k_0 \text{ given}$$

- We obviously have $k_{T+1} = 0$ (right?).
- The problem is now optimization of a continuous function in a finite-dimensional space on a compact set: a solution exists (Extreme Value Theorem).
- Since the constraint set is convex (right?) and U is strictly concave (by assumption) there is a unique optimum and the FOCs are necessary and sufficient.

Lagrangian

We can use the usual tools:

$$\begin{aligned} L = {} & U(f(k_0) - k_1) + \dots \\ & + \beta^t U(f(k_t) - k_{t+1}) + \beta^{t+1} U(f(k_{t+1}) - k_{t+2}) + \dots \\ & + \beta^T U(f(k_T) - k_{T+1}) \end{aligned}$$

FOCs

$$\frac{\partial L}{\partial k_{t+1}} = -\beta^t U'(f(k_t) - k_{t+1}) + \beta^{t+1} U'(f(k_{t+1}) - k_{t+2}) f'(k_{t+1}) = 0$$

Lagrangian

$$\underbrace{U'(f(k_t) - k_{t+1})}_{\substack{\text{Cost in utility of saving} \\ \text{1 unit more capital for } t+1}} = \underbrace{\beta U'(f(k_{t+1}) - k_{t+2})}_{\substack{\text{Discounted add. utility} \\ \text{from 1 more unit of cons.}}} \underbrace{f'(k_{t+1})}_{\substack{\text{Add. prod. with one} \\ \text{more unit of cap. in } t+1}}$$

- This equation is called the Euler equation
- System of T second order difference equations with $T+1$ unknowns $\{k_{T+1}\}_{t=0}^{T}$
- With $k_{T+1} = 0$ we can solve for $\{k_{t+1}\}_{t=0}^{T}$ (can be a huge pain)

Lagrangian

Going back to our example with log utility: $U(c) = \log(c)$ and $f(k) = k^\alpha$.
The Euler equation is

$$\frac{1}{k_t^\alpha - k_{t+1}} = \frac{\beta\alpha k_{t+1}^{\alpha-1}}{k_{t+1}^\alpha - k_{t+2}}$$

$$k_{t+1}^\alpha - k_{t+2} = \alpha\beta k_{t+1}^{\alpha-1}(k_t^\alpha - k_{t+1})$$

Lagrangian

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Trick: Define $z_t \equiv \frac{k_{t+1}}{k_t^\alpha}$. Interpretation?

Lagrangian

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$$k_{t+1}^\alpha - k_{t+2} = \alpha\beta k_{t+1}^{\alpha-1}(k_t^\alpha - k_{t+1})$$

Trick: Define $z_t \equiv \frac{k_{t+1}}{k_t^\alpha}$. Interpretation?

$$1 - z_{t+1} = \alpha\beta\left(\frac{1}{z_t} - 1\right)$$
$$z_{t+1} = 1 + \alpha\beta - \frac{\alpha\beta}{z_t}$$

Why is that a nicer equation?

Lagrangian

We know that $z_T = 0$. Solve backwards from T. Since

$$z_t = \frac{\alpha\beta}{1 + \alpha\beta - z_{t+1}}$$

we get

$$z_t = \alpha\beta \frac{1 - (\alpha\beta)^{T-t}}{1 - (\alpha\beta)^{T-t+1}}$$

and therefore

$$k_{t+1} = \alpha\beta \frac{1 - (\alpha\beta)^{T-t}}{1 - (\alpha\beta)^{T-t+1}} k_t^{\alpha}$$

$$c_t = \frac{1 - \alpha\beta}{1 - (\alpha\beta)^{T-t+1}} k_t^{\alpha}$$

Notice that

$$\lim_{T \to \infty} k_{t+1} = \alpha\beta k_t^{\alpha}$$

Looking familiar?

Graphical analysis

Drawing a little graph can bring big insights into the behavior of the economy.

$$z_{t+1} = 1 + \alpha\beta - \frac{\alpha\beta}{z_t}$$

Plotting z_{t+1} against z_t informs us about the dynamics of the system.

- Since $k_{t+1} \geq 0$ we have $z_t \geq 0$
- $\lim_{z_t \to 0} 1 + \alpha\beta - \frac{\alpha\beta}{z_t} = -\infty$
- $\lim_{z_t \to \infty} 1 + \alpha\beta - \frac{\alpha\beta}{z_t} = 1 + \alpha\beta > 1$
- $z_{t+1} = 0$ for $z_t = \frac{\alpha\beta}{1+\alpha\beta} < 1$

Steady state

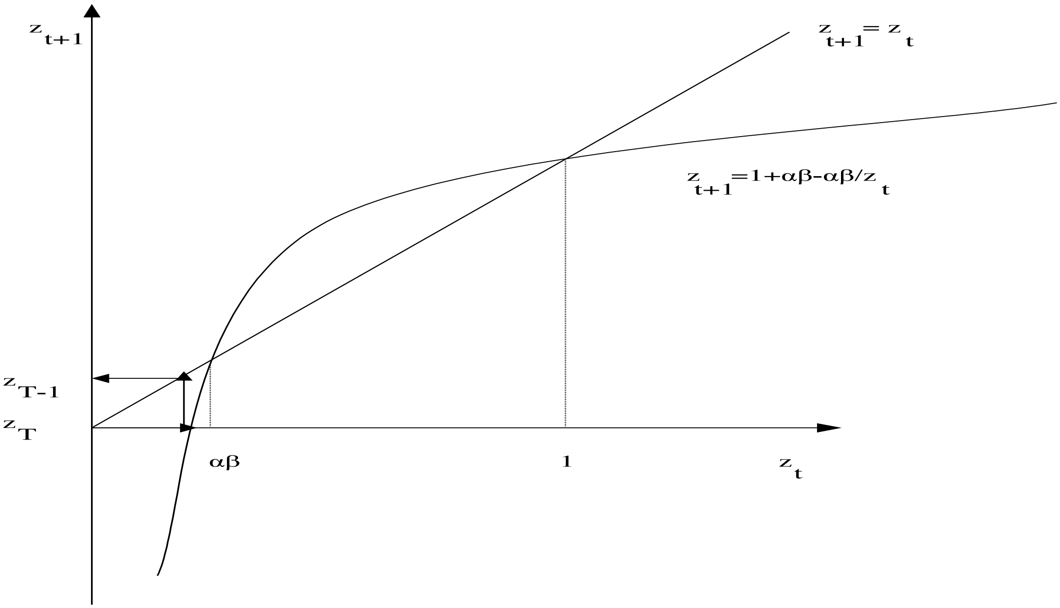
Define a steady state as:

$$z_{t+1} = z_t = z$$

There are two steady states in this economy:

$$z = 1 + \alpha\beta - \frac{\alpha\beta}{z}$$
$$(z-1)(z-\alpha\beta) = 0$$

Therefore $z = 1$ and $z = \alpha\beta$ are steady states.


z_{t+1}
$z_{t+1} = z_t$
$z_{t+1} = 1+\alpha\beta-\alpha\beta/z_t$
z_{T-1}
z_T
$\alpha\beta$
1
z_t

Going back to the infinite horizon case

$$w(k_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(f(k_t) - k_{t+1})$$

subject to:

$$0 \leq k_{t+1} \leq f(k_t)$$
$$k_0 \text{ given}$$

The Euler equation

$$U'(f(k_t) - k_{t+1}) = \beta U'(f(k_{t+1}) - k_{t+2}) f'(k_{t+1})$$

As before we have a second order difference equation but we are missing a terminal condition.

Transversality condition

We impose on additional condition on the problem:

$$\lim_{t\to\infty} \underbrace{\beta^t U'(f(k_t) - k_{t+1}) f'(k_t)}_{\substack{\text{value in discounted utility terms}\\ \text{of one more unit of capital}}} \underbrace{k_t}_{\substack{\text{capital}\\ \text{stock}}} = 0$$

- Transversality plays the role of the missing terminal condition. It is an optimality condition.
- Meaning: shadow value of capital has to converge to zero.
- Mathematically it is a condition coming from the use of the Separating Hyperplane Theorem to find optimality conditions in an infinite-dimensional context. See note on Chris Sims website.
- Can also be: $\lim_{t\to\infty} \lambda_t k_{t+1} = 0$. Where λ is LM on $c_t + k_{t+1} = f(k_t)$.

Transversality condition

Theorem 1

Let U, β and F satisfy our earlier assumptions. Then an allocation $\{k_{t+1}\}_{t=0}^{\infty}$ that satisfies the Euler equations and the transversality condition solves the sequential social planners problem, for a given k_0.

See SLP Theorem 4.15 for a proof.

- Does not work for log utility (since not bounded) but a similar theorem exist for this case.
- The theorem gives *sufficient* conditions for optimality.
- The conditions of the theorem are *necessary* for the log-case (Ekelund and Scheinkman, 1985)

Transversality condition

Going back to our log example: $U(c) = \log(c)$ and $f(k) = k^\alpha$. The TVC becomes

$$\lim_{t\to\infty} \beta^t U'(f(k_t) - k_{t+1}) f'(k_t) k_t = \lim_{t\to\infty} \frac{\alpha\beta^t k_t^\alpha}{k_t^\alpha - k_{t+1}} = \lim_{t\to\infty} \frac{\alpha\beta^t}{1 - z_t}$$

The Euler equation is still

$$z_{t+1} = 1 + \alpha\beta - \frac{\alpha\beta}{z_t}$$

How do we solve this?

Transversality condition

Going back to our log example: $U(c) = \log(c)$ and $f(k) = k^{\alpha}$. The TVC becomes

$$\lim_{t\to\infty} \beta^t U'(f(k_t) - k_{t+1}) f'(k_t) k_t = \lim_{t\to\infty} \frac{\alpha \beta^t k_t^{\alpha}}{k_t^{\alpha} - k_{t+1}} = \lim_{t\to\infty} \frac{\alpha\beta^t}{1 - z_t}$$

The Euler equation is still

$$z_{t+1} = 1 + \alpha\beta - \frac{\alpha\beta}{z_t}$$

How do we solve this?
Guess z_0, iterate and check if TVC holds.

Transversality condition

We have already done part of the work for the previous graph:

1. if $z_0 < \alpha\beta$: in finite time $z_t < 0$ which violates $k_{t+1} \geq 0$
2. if $z_0 > \alpha\beta$: we go to $\lim_{t\to\infty} z_t = 1$ which violates TVC (take a few steps to show)
3. if $z_0 = \alpha\beta$: then $z_t = \alpha\beta$ for all $t > 0$. This satisfies Euler equation and the TVC

$$\lim_{t\to\infty} \frac{\alpha\beta^t}{1 - z_t} = \frac{\alpha\beta^t}{1 - \alpha\beta} = 0$$

The theorem tells us that $z_t = \alpha\beta$ is an optimal solution.
The log-case is basically the only example that can be done by hand. In general, we need to use computation methods.

Modified golden rule

Steady-state (SS): social optimum or CE with $c_t = c^*$ and $k_{t+1} = k^*$.
The Euler equation:

$$U'(f(k_t) - k_{t+1}) = \beta U'(f(k_{t+1}) - k_{t+2})f'(k_{t+1})$$
$$U'(c_t) = \beta U'(c_{t+1})f'(k_{t+1})$$

At a SS:

$$f'(k) = \frac{1}{\beta} \equiv 1 + \rho$$

where ρ is called the time discount rate. Since $f'(k) = F_k(k,1) + 1 - \delta$, we obtain the *modified golden rule*

$$F_k(k^*, 1) - \delta = \rho$$

In our example

$$\alpha(k^*)^{\alpha-1} = \rho + 1 = \frac{1}{\beta} \text{ and } k^* = (\alpha\beta)^{\frac{1}{1-\alpha}}$$

The planner's optimal sequence will converge to k^* regardless of k_0.

Modified golden rule

Why is it called the *modified* golden rule?
The resource constraint is

$$
\begin{aligned}
c_t &= f(k_t) - k_{t+1} \\
c &= f(k) - k
\end{aligned}
$$

Therefore to maximize consumption per capita we need

$$
\begin{aligned}
f'(k^g) &= 1 \\
F_k(k^g, 1) - \delta &= 0
\end{aligned}
$$

where k^g is called the golden rule capital stock.
Why does the SP find it optimal to pick $k^* < k^g$ in the long run?

Modified golden rule

Why is it called the *modified* golden rule?
The resource constraint is

$$\begin{aligned} c_t &= f(k_t) - k_{t+1} \\ c &= f(k) - k \end{aligned}$$

Therefore to maximize consumption per capita we need

$$\begin{aligned} f'(k^g) &= 1 \\ F_k(k^g, 1) - \delta &= 0 \end{aligned}$$

where k^g is called the golden rule capital stock.
Why does the SP find it optimal to pick $k^* < k^g$ in the long run? Because the agent is impatient.

Balanced growth path

What do you think of our *growth* model so far?

Balanced growth path

What do you think of our *growth* model so far?

Let's add population growth ($N_t = (1+n)^t$) and labor-augmenting technological progress:

$$F(K_t, N_t(1+g)^t)$$

What's the utility function now? Either (c_t is per capita):

- per capita lifetime utility

$$\sum_{t=0}^{\infty} \beta^t U(c_t)$$

- or lifetime utility of the entire dynasty

$$\sum_{t=0}^{\infty} (1+n)^t \beta^t U(c_t)$$

Resource constraint

$$(1+n)^t c_t + K_{t+1} = F(K_t, (1+n)^t (1+g)^t) + (1-\delta) K_t$$

Balanced growth path

Define

$$\tilde{c}_t = \frac{c_t}{(1+g)^t}$$
$$\tilde{k}_t = \frac{k_t}{(1+g)^t} = \frac{K_t}{(1+n)^t(1+g)^t}$$

We can rewrite the resource constraint as:

$$\tilde{c}_t + (1+n)(1+g)\tilde{k}_{t+1} = F(\tilde{k}_t, 1) + (1-\delta)\tilde{k}_t$$

In order to obtain a balanced growth path, we assume CRRA utility $U(c) = \frac{c^{1-\sigma}}{1-\sigma}$.

$$\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} = \sum_{t=0}^{\infty} \tilde{\beta}^t \frac{\tilde{c}_t^{1-\sigma}}{1-\sigma}$$

where $\tilde{\beta} = \beta(1+g)^{1-\sigma}$

Balanced growth path

The social planner solves

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \tilde{\beta}^t \frac{(f(\tilde{k}_t) - (1+g)(1+n)\tilde{k}_{t+1})^{1-\sigma}}{1-\sigma}$$

subject to

$$0 \leq (1+g)(1+n)\tilde{k}_{t+1} \leq f(\tilde{k}_t)$$
$$k_0 \text{ given}$$

A *balanced growth path* is a socially optimal allocation where all variables grow at a constant rate. Here it corresponds to a steady state for $\{\tilde{c}_t, \tilde{k}_{t+1}\}$.

Balanced growth path

Euler equations

$$(1+n)(1+g)(\tilde{c}_t)^{-\sigma} = \tilde{\beta}(\tilde{c}_{t+1})^{-\sigma}\left(F_k(\tilde{k}_{t+1},1)+(1-\delta)\right)$$

Steady state on $\{\tilde{c},\tilde{k}\}$

$$(1+n)(1+g) = \tilde{\beta}\left(F_k(\tilde{k}^*,1)+(1-\delta)\right)$$

Defining $\tilde{\beta} \equiv \frac{1}{1+\tilde{\rho}}$ we find

$$(1+n)(1+g)(1+\tilde{\rho}) = \left(F_k(\tilde{k}^*,1)+(1-\delta)\right)$$

which is (approximately)

$$F_k(\tilde{k}^*,1)-\delta \approx n+g+\tilde{\rho}$$

Competitive equilibrium

So far we have been interested in the social planner's problem. Now we decentralize the Pareto allocation to a competitive equilibrium.

- Arrow-Debreu market structure
- Perfect competition
- Ownership
  - Households own firms (receive their profits)
  - Households own capital (they rent it to firms)
- Goods:
  - Final output y_t: Used for consumption and investment. Its price is p_t (quoted in period 0).
  - Labor n_t: Let w_t be the price of one unit of labor delivered in period t (quoted in period 0) in terms of the period t consumption good. w_t is called the real wage. The nominal wage is $w_t p_t$.
  - Capital services k_t: Let r_t be the rental price of one unit of capital services delivered in period t, quoted in period 0, in terms of the period t consumption good. r_t is the real rental rate, the nominal rate is $p_t r_t$

Firms

Firms behave competitively in output and factor markets.
The representative firm's problem is, given a sequence of price $\{p_t, w_t, r_t\}_{t=0}^{\infty}$:

$$\pi = \max_{\{y_t, n_t, k_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} p_t(y_t - r_t k_t - w_t n_t)$$

subject to

$$y_t = F(k_t, n_t) \text{ for all } t \geq 0$$
$$y_t, n_t, k_t \geq 0$$

Households

Households own capital stock and supply labor and capital services. They decide how much to consume and how much to save (through capital accumulation). Taking prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$ as given the representative household solves

$$\max_{\{c_t, i_t, x_{t+1}, k_t, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

subject to

$$\begin{aligned} \sum_{t=0}^{\infty} p_t(c_t + i_t) &\leq \sum_{t=0}^{\infty} p_t(r_t k_t + w_t n_t) + \pi \\ x_{t+1} &= (1-\delta)x_t + i_t \\ 0 \leq n_t &\leq 1,\ 0 \leq k_t \leq x_t \\ c_t, x_{t+1} &\geq 0,\ x_0 \text{ given} \end{aligned}$$

Here we are being very careful. We will have $k_t = x_t$.

Definition on an equilibrium

Definition 3

A Competitive Equilibrium (Arrow-Debreu) is a set of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$ *and allocations for the firm* $\{y_t^d, n_t^d, k_t\}_{t=0}^{\infty}$ *and the household* $\{c_t, i_t, x_{t+1}, k_t^s, n_t^s\}_{t=0}^{\infty}$ *such that*

1. *Given prices, the allocation of the representative firm solves the firm's problem.*
2. *Given prices, the allocation of the representative household solves the household's problem.*
3. *Markets clear:*

$$
\begin{aligned}
y_t &= c_t + i_t \text{ (Goods market)} \\
n_t^d &= n_t^s \text{ (Labor market)} \\
k_t^d &= k_t^s \text{ (Capital services market)}
\end{aligned}
$$

Characterizing the equilibrium

In equilibrium:

$$k_t = k_t^d = k_t^s$$
$$n_t = n_t^d = n_t^s$$

All prices must be strictly positive: $p_t, r_t, w_t > 0$

Firm's problem

The firm problem is static

$$\max_{k_t, n_t \geq 0} p_t(F(k_t, n_t) - r_t k_t - w_t n_t)$$

Marginal product pricing

$$\begin{aligned} r_t &= F_k(k_t, n_t) \\ w_t &= F_n(k_t, n_t) \end{aligned}$$

Using constant return to scale and Euler's theorem

$$\pi_t = p_t(F(k_t, n_t) - F_k(k_t, n_t)k_t - F_n(k_t, n_t)n_t) = 0$$

Indeterminacy of Number of Firms

Constant return to scale imply marginal products are homogeneous of degree 0. Differentiate

$$F(\lambda k, \lambda n) = \lambda F(k, n)$$

with respect to, say, k

$$\lambda F_k(\lambda k, \lambda n) = \lambda F_k(k, n)$$
$$F_k(\lambda k, \lambda n) = F_k(k, n)$$

Now take $\lambda = 1/n$:

$$F_k(k/n, 1) = F_k(k, n)$$

All firms operate with same capital-labor ratio

$$r_t = F_k(k, n) = F_k(k/n, 1)$$

Indeterminacy of Number of Firms

As a consequence, total output could be produced by one representative firm or n_t firms with one worker:

$$F(k_t, n_t) = n_t F(k_t/n_t, 1)$$

Both number of firms as well as output per firm are indeterminate and irrelevant in equilibrium. Only determinate things are k/n and total output y.

Household's problem

We have $n_t = 1$, $k_t = x_t$ and $i_t = k_{t+1} - (1-\delta)k_t$
The budget constraint holds with equality, we can write:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

subject to

$$\sum_{t=0}^{\infty} p_t(c_t + k_{t+1} - (1-\delta)k_t) = \sum_{t=0}^{\infty} p_t(r_t k_t + w_t)$$
$$c_t, k_{t+1} \geq 0, \; k_0 \text{ given}$$

Household's problem

Using μ as the LM of the budget constraint, the FOC wrt to c_t, c_{t+1} and k_{t+1} are

$$\begin{aligned} \beta^t U'(c_t) &= \mu p_t \\ \beta^{t+1} U'(c_{t+1}) &= \mu p_{t+1} \\ \mu p_t &= \mu(1-\delta+r_{t+1})p_{t+1} \end{aligned}$$

Which yield

$$\begin{aligned} \frac{\beta U'(c_{t+1})}{U'(c_t)} &= \frac{p_{t+1}}{p_t} = \frac{1}{1-\delta+r_{t+1}} \\ \frac{(1-\delta+r_{t+1})\beta U'(c_{t+1})}{U'(c_t)} &= 1 \end{aligned}$$

Household's problem

Using our previous notation ($f(k) = F(k, 1) + (1 - \delta)k$) and the marginal pricing equation:

$$r_t = F_k(k_t, 1) = f'(k_t) - (1 - \delta)$$

and goods market clearing

$$c_t = f(k_t) - k_{t+1}$$

we obtain

$$\frac{f'(k_{t+1})\beta U'(f(k_{t+1}) - k_{t+2})}{U'(f(k_t) - k_{t+1})} = 1$$

Which is exactly the same Euler equation as in the SP problem.

Household's problem

TVC for the household:

$$\lim_{t\to\infty} p_t k_{t+1} = 0$$

Using the FOC:

$$\begin{aligned}
\lim_{t\to\infty} p_t k_{t+1} &= \frac{1}{\mu} \lim_{t\to\infty} \beta^t U'(c_t) k_{t+1} \\
&= \frac{1}{\mu} \lim_{t\to\infty} \beta^{t-1} U'(c_{t-1}) k_t \\
&= \frac{1}{\mu} \lim_{t\to\infty} \beta^{t-1} \beta U'(c_t)(1-\delta+r_t) k_t \\
&= \frac{1}{\mu} \lim_{t\to\infty} \beta^t U'(f(k_t)-k_{t+1}) f'(k_t) k_t
\end{aligned}$$

Which is the same as the Planner's TVC.
We have loosely shown that the welfare theorems hold.

Rest of the economy

Notice that once we have determined the equilibrium capital stock we are done

$$
\begin{aligned}
c_t &= f(k_t) - k_{t+1} \\
y_t &= F(k_t, 1) \\
i_t &= y_t - c_t \\
n_t &= 1 \\
r_t &= F_k(k_t, 1) \\
w_t &= F_n(k_t, 1)
\end{aligned}
$$

Sequential markets equilibrium

Household problem

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

subject to

$$c_t + k_{t+1} - (1-\delta)k_t = w_t + r_t k_t$$

$$c_t,\ k_{t+1} \geq 0$$

$$k_0 \text{ given}$$

Firm's problem

$$\max_{k_t, n_t \geq 0} F(k_t, n_t) - w_t n_t - r_t k_t$$

Sequential markets equilibrium

Definition 4

A sequential markets equilibrium is prices $\{w_t, r_t\}_{t=0}^{\infty}$*, allocations for representative household* $\{c_t, k_{t+1}^s\}_{t=0}^{\infty}$ *and for representative firm* $\{n_t^d, k_t^d\}_{t=0}^{\infty}$ *such that*

1. *Given* k_0 *and* $\{w_t, r_t\}_{t=0}^{\infty}$*, household allocations solves household maximization problem.*
2. *For each* $t \geq 0$*, given* (w_t, r_t) *firm allocation* (n_t^d, k_t^d) *solves firms' maximization problem.*
3. *Markets clear: for all* $t \geq 0$

$$
\begin{aligned}
n_t^d &= 1 \\
k_t^d &= k_t^s \\
F(k_t^d, n_t^d) &= c_t + k_{t+1}^s - (1-\delta)k_t^s
\end{aligned}
$$

Recursive competitive equilibrium

- State variables (k, K). Control variables (c, k')
- Bellman equation

$$
\begin{aligned}
v(k, K) &= \max_{c, k' \geq 0} U(c) + \beta v(k', K') \\
c + k' &= w(K) + (1 + r(K) - \delta)k \\
K' &= H(K)
\end{aligned}
$$

- $K' = H(K)$ is the aggregate law of motion.
- Solution is a value function v and policy functions $c = C(k, K)$ and $k' = G(k, K)$.
- Firms

$$
\begin{aligned}
w(K) &= F_l(K, 1) \\
r(K) &= F_k(K, 1)
\end{aligned}
$$

Recursive competitive equilibrium

Definition 5

A RCE is a value function $v : \mathbb{R}^2_+ \to \mathbb{R}$ and policy functions $C, G : \mathbb{R}^2_+ \to \mathbb{R}_+$ for the representative household, pricing functions $w, r : \mathbb{R}_+ \to \mathbb{R}_+$ and an aggregate law of motion $H : \mathbb{R}_+ \to \mathbb{R}_+$ such that

1. *Given w, r and H, v solves the Bellman equation and C, G are the associated policy function.*
2. *The pricing functions satisfy the firms FOC*
3. *Consistency*
$$H(K) = G(K, K)$$
4. *For all $K \in \mathbb{R}_+$*
$$c(K, K) + G(K, K) = F(K, 1) + (1 - \delta)K$$