Econ 6170: Final

9 December 2023

You have the full allocated time (2.5 hours) to complete the following problems. You are to work alone. This test is not open book. Please write out your answer neatly below each question, and use a new sheet of paper if you need more space than provided. When using extra sheets, make sure to write out your name and the relevant question number. In your answers, you are free to cite results that you can recall from class or previous problem sets unless explicitly stated otherwise. There are seven questions and each question is worth five points, and so the exam is out of 35 points. There are no bonus points.

Question 1 Suppose $X \subseteq \mathbb{R}^d$ is nonempty and let $f : X \to \mathbb{R}$.

(i) Using either the sequential characterisation or the ϵ-δ criterion of continuity of f, prove that: if f is continuous at $x \in X$, then, for all open sets $O \subseteq \mathbb{R}$ such that $f(x) \in O$, there is an open ball with radius $\epsilon > 0$ centred at x, $B_\epsilon(x)$, such that $f(z) \in O$ for all $z \in B_\epsilon(x) \cap X$.

(ii) Explicitly prove that: If X is compact and f is continuous, then $f(X) := \{f(x) : x \in X\}$ is bounded.

(iii) Give an example in which $f(X)$ is not bounded but X is bounded and f is continuous.

(iv) Give an example in which $f(X)$ is not bounded but X is closed and f is continuous.

Hint: You will be given partial credit for writing down definitions of what you are being asked to prove (this applies to all other questions too!).

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Question 1 continued

Question 2

(i) Define quasiconvexity and quasiconcavity.

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be a monotone function. Prove or disprove that: f is quasiconcave and/or quasiconvex.

(iii) Prove or disprove: If $f : \mathbb{R} \to \mathbb{R}$ is quasiconcave and quasiconvex, then f is monotone.

(iv) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is both strictly quasiconcave and strictly quasiconvex.

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Question 2 continued

Question 3 Let $f : \mathbb{R}^d_+ \to \mathbb{R}_+$ be a continuous function and define a correspondence $\Gamma : \mathbb{R}^d_+ \rightrightarrows \mathbb{R}_+$ by $\Gamma(\theta) := [0, f(\theta)]$. Show that Γ is continuous.

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Question 3 continued

Question 4 Let $X \subseteq \mathbb{R}^d$ and $Y \subseteq \mathbb{R}^m$ be two open sets. Let $f : X \to Y$.

(i) Give a definition of the directional derivative of f in the direction $\mathbf{v} \in \mathbb{R}^d \backslash \{\mathbf{0}\}$ at $\mathbf{x}_0 \in X$.

(ii) How is the directional derivative of f related to the partial derivative of f? Be precise.

Suppose $d = m = 1$ and $X := (a, b)$ and that f is convex. We say that $s \in \mathbb{R}$ is a *subgradient* of f at $x_0 \in X$ if

$$f(y) \geq f(x_0) + s(y - x_0) \ \ \forall y \in X.$$

(iii) Suppose f is differentiable. What is the subgradient of f at any $x \in X$?

(iv) Let $y_0, z_0 \in (a, b)$ with $z_0 > y_0$. Show that

$$f'\left(y_0^-\right) \leq f'\left(y_0^+\right) \leq \frac{f(z_0) - f(y_0)}{z_0 - y_0} \leq f'\left(z_0^-\right) \leq f'\left(z_0^+\right),$$

where $f'(x^-) := \lim_{h \nearrow 0} \frac{f(x+h) - f(x)}{h}$ and $f'(x^+) := \lim_{h \searrow 0} \frac{f(x+h) - f(x)}{h}$ are the left- and right-derivatives of f at $x \in X$, respectively. **Hint:** Recall the Chordal Slope lemma: For any convex $f : (a, b) \to \mathbb{R}$ and any $y < x < z$,

$$\frac{f(x) - f(y)}{x - y} \leq \frac{f(z) - f(y)}{z - y} \leq \frac{f(z) - f(x)}{z - x}.$$

Use this to first prove that $f'(y_0^-) \leq f'(y_0^+)$ and $f'(z_0^-) \leq f'(z_0^+)$. Then use the lemma again to prove the rest of the inequalities.

(v) Suppose f has a kink at $x_0 \in X$. Use the result above to show that any $s \in [f'(x_0^-), f'(x_0^+)]$ is a subgradient. **Hint:** Drawing helps!

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Question 4 continued

Question 5 Let $f : \mathbb{R}^d \to \mathbb{R}$ and $h : \mathbb{R}^d \to \mathbb{R}^K$, where h_k is $\mathbf{C}^1$ for each $k \in \{1, \ldots, K\}$. Suppose $\mathbf{x}^*$ is a local maximum or minimum of f on the constraint set

$$\Gamma := \left\{ \mathbf{x} \in \mathbb{R}^d : h(\mathbf{x}) = \mathbf{0} \right\}.$$

(i) State the (rest of the) Theorem of Lagrange.

(ii) Suppose $K = 1 < d = 2$. Prove the Theorem of Lagrange. **Hint:** Write out the two first-order conditions and apply the implicit function theorem to the appropriate one to obtain the Lagrange multiplier. Then, use the fact that $\mathbf{x}^*$ is a local maximum to show that the constructed Lagrange multiplier also satisfies the other first-order condition.

(iii) What does the Theorem of Lagrange say when (a) $K = d$ and (b) $K > d$?

(iv) Prove or disprove that the converse of the Theorem of Lagrange is true.

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Question 5 continued

Question 6 Fix some $Y \subseteq \mathbb{R}^d$ that is nonempty and has a nonempty interior. We say that a (production) vector $\mathbf{y} \in Y$ is *efficient* if there is no $\mathbf{y}' \in Y$ such that $\mathbf{y}' \geq \mathbf{y}$ and $\mathbf{y}' \neq \mathbf{y}$. A production vector $\mathbf{y} \in Y$ is *profit maximising for some* $\mathbf{p} \in \mathbb{R}^d_{++}$ *if*

$$\mathbf{p} \cdot \mathbf{y} \geq \mathbf{p} \cdot \mathbf{y}' \; \forall \mathbf{y}' \in Y.$$

(i) Prove or disprove: (a) If $\mathbf{y} \in Y$ is efficient, then $\mathbf{y}$ is a boundary point of Y; (b) If $\mathbf{y} \in Y$ is a boundary point of Y, then $\mathbf{y}$ is efficient.

(ii) Prove that: If $\mathbf{y} \in Y$ is profit maximising for some $\mathbf{p} \in \mathbb{R}^d_{++}$, then $\mathbf{y}$ is efficient.

(iii) State a separating hyperplane theorem.

(iv) Suppose that Y is convex. Prove that every efficient production vector $\mathbf{y} \in Y$ is a profit-maximising production vector for some $\mathbf{p} \in \mathbb{R}^d_{+}$ (i.e., $\mathbf{p} \neq \mathbf{0}$ and $\mathbf{p} \geq \mathbf{0}$). **Hint:** Apply the separating hyperplane theorem to the set Y and $P_{\mathbf{y}} := \{\mathbf{y}' \in Y : \mathbf{y}' \gg \mathbf{y}\}$, where $(y'_i)^d_{i=1} = \mathbf{y}' \gg \mathbf{y} = (y_i)^d_{i=1}$ means that $y'_i > y_i$ for all $i = 1, \ldots, d$. Try drawing the case of $d = 2$.

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Question 6 continued

Question 7 Suppose $(X, \geq)$ is a nonempty convex sublattice of $(\mathbb{R}^d, \geq)$. We say that $f : X \to \mathbb{R}$ is *C-supermodular* if

$$f\left(x \vee y - tv\right) - f\left(y\right) \geq f\left(x\right) - f\left(x \wedge y + tv\right) \ \forall t \in \left[0, 1\right],$$

where $v = y - x \wedge y = x \vee y - x$ for all $x, y \in X$.

(i) Suppose $d = 2$ and draw the points x, y, $x \wedge y$, and $x \vee y$ for the case in which x and y are not ordered. Draw in the points $b = x \vee y - tv$ and $a = x \wedge y + tv$ for some $t \in (0, 1)$. Interpret the condition for C-supermodularity in this diagram.

(ii) Argue that C-supermodularity is a stronger property than supermodularity.

(iii) Prove that: If f is supermodular and $f(x_i, x_{-i})$ is a concave function of x_{-i} for all $i \in \{1, \ldots d\}$, then f is C-supermodular.

(iv) Given an example of f that is not concave but $f(x_i, x_{-i})$ is a concave function of x_{-i} for all $i \in \{1, \ldots d\}$. **Hint:** It suffices to consider $d = 2$.

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Question 7 continued