ECON 6130 — Prelim exam — Fall 2024

October 15, 2024

Name: ____________________________

This is a closed-book exam. Some questions are harder. Don't worry if you cannot answer them all, and plan your time accordingly. Good luck!

Optimal cake-eating strategy

You have unfortunately been shipwrecked on a deserted island. The only food that you have is a cake of size $W_0 > 0$. You need to optimally decide how much of the cake to eat every day. If at the beginning of period t there is W_t units of cake left and that you eat C_t of those, the size of the cake in $t+1$ is given by

$$W_{t+1} = R\left(W_t - C_t\right), \tag{1}$$

where $R > 0$ is given. If $R < 1$, you can think of R as the fraction of the cake that does not spoil between periods. If $R > 1$, you have a magical cake that grows on its own over time. The problem you are facing is therefore

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u\left(C_t\right),$$

subject to (1) and $0 \leq C_t \leq W_t$, and where u is strictly increasing, strictly concave and satisfies the Inada condition.

1. (5 points) The Bellman equation for this problem is

$$V\left(W\right) = \max_{0 \leq C \leq W} u\left(C\right) + \beta V\left(R\left(W - C\right)\right).$$

Explain why the Bellman equation takes this form. In the language of dynamic programming, what roles do W and C play here? Define an operator T that characterizes that Bellman equation. A solution V to the Bellman equation should be a fixed point of T.

2. (10 points) Suppose that u is bounded. Use Blackwell's theorem to show T is a contraction mapping. How many fixed points does T have?

Solution: We need to show that B has the monotonicity and discounting properties. For monotonicity, suppose that there are two functions f and g such that $f \leq g$. Then

$$\begin{aligned} Tf(W) &= \max_{0 \leq C \leq W} u(C) + \beta f(R(W-C)) = u(c_f(W)) + \beta f(R(W - c_f(W))) \\ &\leq u(c_f(W)) + \beta g(R(W - c_f(W))) \leq \max_{0 \leq C \leq W} u(C) + \beta g(R(W-C)) = Tg(W). \end{aligned}$$

For discounting,

$$T(f+\alpha)(W) = \max_{0 \leq C \leq W} u(C) + \beta f(R(W-C)) + \beta\alpha = Tf(W) + \beta\alpha.$$

So Blackwell's conditions hold, T is a contraction mapping, and the contraction mapping theorem implies that it has a unique fixed point.

3. (5 points) Use the envelope theorem and the first-order condition of the recursive problem to derive two equations that link $V'(W)$, $u'(C)$ and $V'(R(W-C))$.

   Solution: The envelope theorem implies

   $$V'(W) = \beta R V'(R(W-C)).$$

   The first-order condition implies

   $$u'(C) = R\beta V'(R(W-C)).$$

   Combining these two equations yields

   $$u'(C) = R\beta V'(R(W-C)) = V'(W).$$

4. (20 points) From now on, assume that the preferences are CRRA such that $u(C) = C^{1-\gamma}/(1-\gamma)$ for $\gamma > 0$ and $\gamma \neq 1$. Show that the value function takes the form

   $$V(W) = A\frac{W^{1-\gamma}}{1-\gamma},$$

   for some $A > 0$. What is the value of A?

   Solution: The equation that we just found implies that

   $$C^{-\gamma} = AW^{-\gamma} \Rightarrow C = A^{-\frac{1}{\gamma}}W.$$

Plugging into the Bellman equation we find,

$$\begin{aligned}
V(W) &= \max_{0\leq C\leq W} \frac{C^{1-\gamma}}{1-\gamma} + \beta\frac{A}{1-\gamma}\left(R(W-C)\right)^{1-\gamma} \\
&= \frac{1}{1-\gamma}A^{-\frac{1-\gamma}{\gamma}}W^{1-\gamma} + \beta\frac{A}{1-\gamma}\left(R\left(W-A^{-\frac{1}{\gamma}}W\right)\right)^{1-\gamma} \\
&= \frac{1}{1-\gamma}A^{-\frac{1-\gamma}{\gamma}}W^{1-\gamma} + \beta\frac{A}{1-\gamma}R^{1-\gamma}\left(1-A^{-\frac{1}{\gamma}}\right)^{1-\gamma}W^{1-\gamma}
\end{aligned}$$

Notice that our guess is correct. We can find A from

$$\begin{aligned}
A\frac{W^{1-\gamma}}{1-\gamma} &= \frac{1}{1-\gamma}A^{-\frac{1-\gamma}{\gamma}}W^{1-\gamma} + \beta\frac{A}{1-\gamma}R^{1-\gamma}\left(1-A^{-\frac{1}{\gamma}}\right)^{1-\gamma}W^{1-\gamma} \\
1-A^{-\frac{1}{\gamma}} &= \beta R^{1-\gamma}\left(1-A^{-\frac{1}{\gamma}}\right)^{1-\gamma} \\
1 &= \beta R^{1-\gamma}\left(1-A^{-\frac{1}{\gamma}}\right)^{-\gamma} \\
1-\left(\beta R^{1-\gamma}\right)^{\frac{1}{\gamma}} &= A^{-\frac{1}{\gamma}}
\end{aligned}$$

5. (5 points) Provide an expression for the optimal policy function $C(W)$ in terms of the primitives of the problem. When β increases, do you eat a bigger share of the remaining cake in each period? What about when R increases? When $\gamma = 1$, does your expression depends on R? Why? In this case, does the time series of cake size $\{W_t\}_{t=0}^{\infty}$ depend on R? Why?

   Solution: From this last problem we see that

$$C = A^{-\frac{1}{\gamma}}W = \left(1-\left(\beta R^{1-\gamma}\right)^{\frac{1}{\gamma}}\right)W.$$

   When $\gamma = 1$ this expression does not depend on R. The substitution and income effect of R cancel each other. The time series does however depend on R since it affects the link between W_t and W_{t+1} given above.

6. (15 points) Now suppose that R is stochastic, and that the agent does not know R when choosing how much to consume. With probability p, R is equal to R_h, and with probability $1-p$ it is equal to R_l, with $R_h > R_l$. The draws are independent over time time. Write the Bellman equation associated with this problem. Solve for the value function. (Hint: You might want to make an educated guess...) Describe how the solution to the stochastic problem differs from the solution to the deterministic problem.

   Solution: The Bellman equation is

$$V(W) = \max_{0\leq C\leq W} u(C) + \beta\left[pV\left(R_h(W-C)\right) + (1-p)V\left(R_l(W-C)\right)\right].$$

The envelope theorem is

$$V'(W) = \beta\left[pR_h V'(R_h(W-C)) + (1-p)R_l V'(R_l(W-C))\right]$$

The first-order condition is

$$u'(C) = \beta\left[pR_h V'(R_h(W-C)) + (1-p)R_l V l(R_l(W-C))\right] = V'(W).$$

As in the deterministic problem, guess that

$$V(W) = A\frac{W^{1-\gamma}}{1-\gamma},$$

such that

$$V'(W) = AW^{-\gamma}.$$

The first-order condition implies that

$$C = A^{-\frac{1}{\gamma}}W.$$

Plugging back in the value function

$$\begin{aligned} V(W) &= \max_{0\le C\le W}\frac{C^{1-\gamma}}{1-\gamma} + \beta\left[p\frac{A}{1-\gamma}(R_h(W-C))^{1-\gamma} + (1-p)\frac{A}{1-\gamma}(R_l(W-C))^{1-\gamma}\right] \\ &= \frac{1}{1-\gamma}A^{-\frac{1-\gamma}{\gamma}}W^{1-\gamma} + \beta\left[p\frac{A}{1-\gamma}\left(R_h\left(W-A^{-\frac{1}{\gamma}}W\right)\right)^{1-\gamma} + (1-p)\frac{A}{1-\gamma}\left(R_l\left(W-A^{-\frac{1}{\gamma}}W\right)\right)^{1-\gamma}\right] \end{aligned}$$

So we see that our value function guess works, and that

$$A^{-\frac{1}{\gamma}} = 1-\beta^{\frac{1}{\gamma}}\left[p(R_h)^{1-\gamma} + (1-p)(R_l)^{1-\gamma}\right]^{\frac{1}{\gamma}}.$$