Econ 6190 Final Exam

2:00 pm - 4:30 pm, Wednesday, Dec 13, 2023

Instructions

This exam consists of two questions, not of equal length or difficulty. Answer all questions. Remember to always explain your answer. Good luck!

1. **[50 pts]** Suppose X is a random variable with the following pdf

$$f(x \mid \theta) = C(\theta) \exp^{Q(x)\theta}, \ x \in \mathbb{R},$$

where the functional forms of $C(\cdot)$ and $Q(\cdot)$ are both known, and $\theta \in \mathbb{R}$ is the unknown parameter of interest. As $f(x \mid \theta)$ is a pdf, note $C(\theta) \geq 0$ for all possible values of θ. We observe a random sample $\{X_1, X_2, \ldots, X_n\}$ from X. The goal here is to learn about θ. You may assume that all regularity conditions hold for this question, and in particular, that both $C(\cdot)$ and $Q(\cdot)$ are differentiable.

   (a) **[5 pts]** By using the properties of a pdf, express $C(\theta)$ in terms of an integral.

   (b) **[10 pts]** By using the Factorization Theorem, find a sufficient statistic for θ based on the random sample $\{X_1, X_2, \ldots, X_n\}$.

   (c) **[5 pts]** Find the log-likelihood function and derive the F.O.C that the Maximum Likelihood Estimator (MLE), say $\hat{\theta}_{MLE}$, should satisfy. (You do not need to solve for the estimator).

   (d) **[5 pts]** Explain how you would construct a method of moment estimator for θ, say, $\hat{\theta}_{MM}$.

   (e) **[10 pts]** Show that Cramer-Rao Lower Bound for estimating θ, say V_{CRLB}, equals $\frac{1}{n \cdot Var(Q(X))}$.

   (f) Suppose one finds the asymptotic distribution of the method of moment estimator $\hat{\theta}_{MM}$ as

$$\sqrt{n}\left(\hat{\theta}_{MM} - \theta\right) \xrightarrow{d} N(0, V),$$

   where V is the asymptotic variance of $\hat{\theta}_{MM}$. In addition, they finds that $\hat{V}$ is a consistent estimator for V.

i. **[5 pts]** Which of the two, V_{CRLB} and V, do you think is larger?

ii. **[10 pts]** Construct an asymptotically valid confidence interval with coverage probability 98% for $\beta = \exp(\theta)$. Explain your reasoning carefully.

2. **[50 pts]** Consider the normal sampling model, where $X \sim N(\mu, \sigma^2)$ with σ^2 **known** and with pdf $f(x \mid \mu) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$. A random sample of n observations, $\{X_1, X_2, \ldots, X_n\}$, is drawn from the distribution of X.

(a) Consider testing $\mathbb{H}_0 : \mu = \mu_0$ v.s. $\mathbb{H}_1 : \mu > \mu_0$ for some $\mu_0 \in \mathbb{R}$. Let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, and

$$T_1 = \frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}},$$

$$T_2 = \frac{\bar{X} - \mu_0}{\frac{s}{\sqrt{n}}}, \text{ where } s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

i. [**5 pts**] Construct a finite-sample valid one-sided test that controls size at 5% based on T_1. Call this Test A.

ii. [**5 pts**] Construct a finite-sample valid one-sided test that controls size at 5% based on T_2. Call this Test B.

iii. [**5 pts**] Derive the finite-sample power of Tests A and B when the alternative distribution has mean $\mu_1 = \mu_0 + \sigma$. What do you think would happen to the power of Tests A and B when you let $n \to \infty$?

iv. [**10 pts**] Derive the asymptotic distribution of T_2 when $n \to \infty$ under $\mathbb{H}_0$. Carefully prove any of your asymptotic statement.

(b) Suppose now one wishes to test $\mathbb{H}_0 : \mu = \mu_0$ v.s. $\mathbb{H}_1 : \mu \neq \mu_0$ for the same μ_0 used in (a).

i. [**5 pts**] Construct a finite-sample valid two-sided test that controls size at 5% based on T_1. Call this Test C.

ii. [**10 pts**] Derive the finite-sample power of Tests A and C when the alternative distribution has mean $\mu_1 = \mu_0 - \sigma$. Which test has a larger power as $n \to \infty$?

(c) [**10 pts**] Derive the most powerful test for testing $\mathbb{H}_0 : \mu = \mu_0$ v.s. $\mathbb{H}_1 : \mu = \mu_0 - \sigma$ among the set of all tests that control size at 5%. Is this test the same as any of the Tests A, B and C above?