Initial Assessment Solutions

Exercise 1. A definition of the exponential function is

$$e^x := \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

(i) Recall that a real-valued sequence is *convergent* if there exists $z \in \mathbb{R}$ such that, for all $\epsilon > 0$, for some $N \in \mathbb{N}$, it is the case that $|z_n - z| < \epsilon$ for all $n \in \mathbb{N} \setminus \{1, 2, \dots, N\}$. A sequence $(z_n)_n$ is *divergent* if it is not convergent. Write out a definition of a divergent sequence $(z_n)_n$ analogous to the definition of a convergent sequence using ϵ and N etc. Hint: Take the negation of the definition of convergence.

A real-valued sequence is divergent if for all $z \in \mathbb{R}$, there exists $\epsilon > 0$ such that for all $N \in \mathbb{N}$ there exists $n > N$, $n \in \mathbb{N}$ such that $|z_n - z| \geq \epsilon$.

(ii) Suppose a bank pays interest annually at rate $r > 0$. Consider investing \$1 with the bank. How much money would you have after $t \in \{0\} \cup \mathbb{N}$ years?

$(1+r)^t$

(iii) Suppose that the bank now pays interest every month and the monthly compound interest is $r/12$. How much money would you have after $t \in \{0\} \cup \mathbb{N}$ years?

$\left(1+\frac{r}{12}\right)^{12t}$

(iv) Suppose that the bank now pays interest continuously (compounded), show that you would have \$$e^{rt}$ after $t \in \{0\} \cup \mathbb{N}$.

Paying interest continuously means taking the payment frequency, n, to infinity. The bank therefore pays

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt} = \left(\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n\right)^t = (e^r)^t = e^{rt}$$

over t years.

(v) What does this tell you about the discount factor we often use in discrete- and continuous-time models?

Discounting is the reciprocal of interest calculation and so this gives us a relationship between the discount rate in discrete time, $\frac{1}{1+r}$, and in continuous time, e^{-r}.

Exercise 2. Recall that a binary relation $\succsim \subseteq X \times X$ on a nonempty set X is:

(a) *complete* if, for any $x, y \in X$, either $x \succsim y$ or $y \succsim x$;

(b) *transitive* if, for any $x, y, z \in X$, $x \succsim y$ and $y \succsim z$ implies $x \succsim z$.

(i) Define $\sim \subseteq X \times X$ such that, for any $x, y \in X$, $x \sim y$ if and only if $x \succsim y$ and $y \succsim x$. Prove that $\sim$ is transitive. Is $\sim$ always complete? If yes, prove it; if not, give an example.

Suppose $x \sim y$ and $y \sim z$. We have, then, that $x \succsim y$ and $y \succsim z$, so by transitivity of $\succsim$, $x \succsim z$. We also have $z \succsim y$ and $y \succsim x$, similarly implying $z \succsim x$. Therefore, $x \sim z$, and $\sim$ is transitive.

$\sim$ is not necessarily complete. For example, if $X = \mathbb{R}$ and $\succsim$ is the natural order on $\mathbb{R}$, $\geq$, then $\sim$ is the equality relation. But it is not the case that either $x = y$ or $y = x$: take, e.g., 1 and 2.

(ii) Suppose $X = \mathbb{R}^n$ and that $\succsim$ additionally satisfies the following condition: for any $x, y \in X$,

$$x \succsim y \iff \alpha x + (1-\alpha)z \succsim \alpha y + (1-\alpha)z \tag{1}$$

for all $z \in X$ and all $\alpha \in (0,1)$. Show that, for any $x, y \in X$ and any $\alpha > 0$, $x \succsim y \iff \alpha x \succsim \alpha y$. Hint: Consider first the case in which $\alpha \in (0,1)$, then $\alpha = 1$. Finally, consider the case in which $\alpha > 1 \iff \alpha^{-1} < 1$.

Suppose $x \succsim y$. If $\alpha \in (0,1)$ then we can apply (1) with $z = 0$, to obtain $\alpha x \succsim \alpha y$. If $\alpha = 1$, then we want $x \succsim y$, which is true by assumption.

Now suppose $\alpha > 1$. As noted in the hint, this is equivalent to $\alpha^{-1} < 1$. By way of contradiction, suppose $\alpha y \succ \alpha x$. Then, by the first part of this proof, $\alpha^{-1}\alpha y \succ \alpha^{-1}\alpha x$, or $x \succ y$. This contradicts our assumption that $x \succsim y$, so $\alpha x \succsim \alpha y$ by completeness.

The reverse direction ($\Longleftarrow$) follows trivially from the first direction.

(iii) Show that the equivalence class $[0]_\sim := \{x \in X \mid x \sim 0\}$ is a linear subspace of X; i.e., for any $x, y \in X$, $\alpha x + \beta y \in X$ for all $\alpha, \beta \in \mathbb{R}_+$. Bonus: why don't we need to consider $\alpha, \beta \in \mathbb{R}$?

If $x \sim 0$ then $x \succsim 0$ and $0 \succsim x$, so by (ii), $\alpha x \succsim 0$ and $0 \succsim \alpha x$, i.e., $\alpha x \sim 0$. Similarly, $\beta y \sim 0$. If $\alpha = 0$ or $\beta = 0$, we are done. Otherwise, by (1),

$$\frac{1}{2}\alpha x + \frac{1}{2}\beta y \succsim 0 + \frac{1}{2}\beta y \succsim 0$$

and

$$0 \succsim 0 + \frac{1}{2}\beta y \succsim \frac{1}{2}\alpha x + \frac{1}{2}\beta y$$

so

$$\frac{1}{2}\alpha x + \frac{1}{2}\beta y \sim 0$$

and thus

$$\alpha x + \beta y \sim 0$$

We can ignore $\alpha, \beta < 0$ because $x \in X$ implies $-x \in X$, so $x, y \in X$ implies, for example,

$$\alpha x - \beta y = \alpha x + \beta(-y) \in X$$

To see that $x \in X$ implies $-x \in X$, note that

$$x \succsim 0 \implies \frac{1}{2}x + \frac{1}{2}(-x) \succsim \frac{1}{2}(-x) \implies 0 \succsim -x$$

Similarly, $0 \succsim x$ implies that $-x \succsim 0$. It follows that $x \sim 0$ implies $-x \sim 0$.

Exercise 3. Consider again the linear system of equations

$$Xb = y$$

where $X \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^{n+1}$, and $y \in \mathbb{R}^{m+1}$. We showed in class that the normal system associated with the above is given by

$$X^\top y = X^\top X\hat{b}$$

We now wish to show that the normal system has a solution. To that end, recall that the *image* of a matrix A, denoted $\operatorname{im} A$, is its column space (i.e., the set of all linear combinations of columns of A). The *null space* of $A \in \mathbb{R}^{c\times d}$, denoted $\operatorname{null} A$, consists of $b \in \mathbb{R}^{d\times 1}$ such that $Ab = 0$.

(i) Show that

$$\operatorname{null}\left(X^\top X\right) = \operatorname{null} X$$

Hint: Recall that two sets X and Y are equal if and only if $X \subseteq Y$ and $Y \subseteq X$. Recall also that $z^\top A^\top Az = \|Az\|^2$.

If $b \in \operatorname{null} X$, then $Xb = 0$, so $X^\top Xb = X^\top 0 = 0$. Thus, $b \in \operatorname{null}(X^\top X)$. So $\operatorname{null} X \subseteq \operatorname{null}(X^\top X)$.

If $b \in \operatorname{null}(X^\top X)$, then $X^\top Xb = 0$ and so $b^\top X^\top Xb = 0$. By the hint, this means $\|Xb\|^2 = 0$ and so $\|Xb\| = 0$, which can only be true if $Xb = 0$. Therefore, $b \in \operatorname{null} X$ so that $\operatorname{null}(X^\top X) \subseteq \operatorname{null} X$.

(ii) Show that

$$\operatorname{im}\left(X^\top X\right) = \operatorname{im}\left(X^\top\right)$$

Hint: Use part (i) and the fact that $\operatorname{null}\left(A^\top\right) = \left(\operatorname{im}(A)\right)^\perp$ and $\left(M^\perp\right)^\perp = M$.

If $\operatorname{null}\left(X^\top X\right) = \operatorname{null} X$ then $\left(\operatorname{im}\left(X^\top X\right)\right)^\perp = \left(\operatorname{im}\left(X^\top\right)\right)^\perp$. Taking orthogonal complements on each side,

$$\operatorname{im}\left(X^\top X\right) = \operatorname{im}\left(X^\top\right)$$

(iii) Use parts (i) and (ii) to conclude that $X^\top y \in \operatorname{im}\left(X^\top X\right)$; i.e., a solution to the normal system exists.

$X^\top y \in \operatorname{im}\left(X^\top\right) = \operatorname{im}\left(X^\top X\right)$

Exercise 4. Let $g, f : \mathbb{R}^2 \to \mathbb{R}$ be defined via

$$\begin{aligned} g(x,y) &:= -1 + x^2 + y^2 \\ f(x,y) &:= x^2 - y \end{aligned}$$

Consider the following problem:

$$\max_{(x,y)\in\mathbb{R}^2} f(x,y) \text{ s.t. } g(x,y) \le 0$$

(i) Write the Lagrangian, $\mathcal{L}$, for this problem. Denote the Lagrange multiplier using λ.

$$\mathcal{L}(x, y, \lambda) = (x^2 - y) + \lambda\left(1 - x^2 - y^2\right)$$

(ii) Write down the KKT conditions (the derivatives of $\mathcal{L}$ with respect to x and y; nonnegativity of the Lagrange multiplier, complementary slackness, and the constraint itself).

$$\begin{aligned} 2x - 2\lambda x &= 0 \\ -1 - 2\lambda y &= 0 \\ \lambda &\geq 0 \\ \lambda \cdot (1 - x^2 - y^2) &= 0 \\ 1 - x^2 - y^2 &\geq 0 \end{aligned}$$

(iii) Assuming that the constraint qualification holds (which it does), use the KKT conditions to solve the problem. Hint: You may find that there are multiple values of (x, y, λ) that satisfy the KKT conditions—in that case, recall that you can choose the one(s) that maximise the objective.

The first condition tells us that either $\lambda = 1$ or $x = 0$. The second conditions tells us that $\lambda y = -1/2$. This implies, in particular, that neither λ nor y can equal 0. It follows, by the fourth condition, that $x^2 + y^2 = 1$. If $x = 0$ then $y = \pm 1$ and $\lambda = \pm 1/2$. The case $\lambda = -1/2$ and $y = 1$ is precluded by the nonnegativity constraint on λ. So $(0, -1, 1/2)$ is a candidate solution. It gives $f(0, -1) = 1$. If $\lambda = 1$ then $y = -1/2$ and $x = \pm\sqrt{1 - y^2} = \pm\sqrt{3/4}$. At these two solutions $f(\pm\sqrt{3/4}, -1/2) = 5/4 > 1 = f(0, -1)$. So our solutions are $(\sqrt{3/4}, -1/2, 1)$ and $(\sqrt{3/4}, -1/2, 1)$.