ECON 6170 Section 5

TA: Patrick Ferguson

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Midterm 1 Practice Questions

Exercise 1 (2023 Midterm 1 Q1)**.** Prove either that the following statements are true or false.

(i) The set $S = \{(x, y) \in \mathbb{R}^2 \mid x > 0,\ y \geq 1/x^2\}$ is open.

(ii) The set $S = \{(x, y) \in \mathbb{R}^2 \mid x > 0,\ y \geq 1/x^2\}$ is closed.

(iii) A closed subset of a compact set $S \subseteq \mathbb{R}^d$ is compact.

(i) False.

Take $(x, y) \in S$ such that $y = 1/x^2$. Then $(x, y) \in S$. Consider the sequence in $S^\complement$, $(x, y - \frac{1}{n})$. Clearly this sequence converges to (x, y), so $S^\complement$ is not closed. It follows that S is not open.

(ii) True.

Let (x_n, y_n) be a sequence in S converging to some (x, y). Note that it cannot be that $x = 0$, for then $y_n \to \infty$ and (x_n, y_n) doesn't converge, a contradiction. Therefore, $x > 0$. Given $y_n \geq 1/x_n^2 \to 1/x^2$, we must have $y \geq 1/x^2$. Thus, $(x, y) \in S$.

Note that proving S is closed also suffices to prove that it is not open, given that it is neither $\emptyset$ nor $\mathbb{R}^d$.

(iii) True.

S is compact $\implies$ S is bounded $\implies$ every subset of S is bounded $\implies$ every closed subset of S is compact.

Exercise 2 (2023 Midterm 1 Q2)**.** Let $(x_n)_n$ be a sequence in $\mathbb{R}$. A point $s \in \mathbb{R}$ is a *limit point* of $(x_n)_n$ if there exists a subsequence of $(x_n)_n$ that converges to s. Let S be the set of limit points of $(x_n)_n$.

(i) Prove that there is a subsequence $(x_{n_k})_k$ that converges to $\limsup_{n\to\infty} x_n$.

(ii) Prove that $\limsup_{n\to\infty} x_n = \sup S$.

To save on time, you may assume the sequence $(x_n)_n$ is bounded. **Hint:** If you can't prove (i), assume it and use it to prove (ii).

(i) We know that $(s_m)_{m=1}^{\infty} := (\sup\{x_n \mid n \geq m\})_{m=1}^{\infty}$ is a sequence converging to $s := \limsup x_n$. By definition of a supremum, because the sequence in the question is bounded, there exists $x_{n_1} \in [s_1 - 1, s_1]$. Similarly, there exists $x_{n_2} \in [s_{n_1+1} - \frac{1}{2}, s_{n_1+1}]$ such that $n_2 > n_1$. Proceeding similarly, we obtain a subsequence of (x_n), (x_{n_k}) such that $s_{n_{k-1}+1} - \frac{1}{k} \leq x_{n_k} \leq s_{n_{k-1}+1}$ and both bounding sequences converge to s, so $x_{n_k} \to s$ also.

(ii) By (i), we only need to prove that no subsequence (x_{n_k}) converges to a point greater than $s := \limsup x_n$. Suppose such a subsequence did exist. Suppose $x_{n_k} \to s + \varepsilon$. Then infinitely many terms of (x_n) lie above $s + \varepsilon/2$. It follows that infinitely many $\sup\{x_n \mid n \geq m\}$ lie above $s + \varepsilon/2$, so $(\sup\{x_n \mid n \geq m\})_{m=1}^{\infty}$ doesn't converge to s, a contradiction.

Exercise 3 (2023 Midterm 1 Q3)**.** A *boundary point* of a set $S \subseteq \mathbb{R}^d$ is a point $x \in \mathbb{R}^d$ such that every open ball centred at x intersects both S and S^C. Define

$$\text{bd}(S) := \{x \in \mathbb{R}^d \mid x \text{ is a boundary point of } S\}.$$

(i) Show that $\text{bd}(S) = \text{bd}(S^\mathsf{C})$.

(ii) Prove or disprove: If $x \in S$ is an isolated point, then x is a boundary point of S.

(iii) Show that the set $S \subseteq \mathbb{R}^d$ is closed if and only if it contains all its boundary points.

Hint: Recall that a point $x \in S$ is *isolated* if there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \cap S = \{x\}$.

(i) $x \in \text{bd}(S) \iff$ every open ball centered at x intersects both S and $S^\mathsf{C} \iff x \in \text{bd}(S^\mathsf{C})$.

(ii) True. Clearly every open ball centred at x contains an element of S, x itself. There is some ε such that $B_\varepsilon(x)$ contains no other elements of S. But $B_\varepsilon(x)$ is not a singleton, so it must contain elements of S^C.

(iii) Suppose $\text{bd}\, S \subseteq S$. Suppose S is not closed. Then there exists a sequence of elements of S that converges to $x \in S^\mathsf{C}$. Then every open ball centred at x contains elements of S, so x is a boundary point of S that lies in S^C, a contradiction.

Suppose S is closed. Then S^C is open, so $x \in S^\mathsf{C}$ implies that some $B_\varepsilon(x) \subseteq S^\mathsf{C}$, meaning that x is not a boundary point of S^C. But this means that x is not a boundary point of S either. Therefore, $\text{bd}(S) \subseteq S$.

Exercise 4 (2023 Midterm 2 Q2)**.**

(i) Suppose $f : \mathbb{R} \to \mathbb{R}$ is concave and $g : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function. Prove that $g \circ f$ is quasiconcave.

(ii) Define $h : \mathbb{R}_+ \to \mathbb{R}$ by

$$h(x) := \begin{cases} 0 & \text{if } x \in [0,1] \\ (x-1)^2 & \text{if } x > 1 \end{cases}$$

Show that h is quasiconcave.

(iii) Show that h, defined above, is not a strictly increasing function of a concave function.

Hint: Prove by contradiction and use the fact that every local maximum of a concave function is a global maximum.

(i) Because f is concave and thus quasiconcave,

$$f(\alpha x + (1-\alpha)y) \geq \min\{f(x), f(y)\}$$

Because g is increasing,

$$(g \circ f)(\alpha x + (1-\alpha)y) = g(f(\alpha x + (1-\alpha)y))$$
$$\geq g(\min\{f(x), f(y)\}) = \min\{(g \circ f)(x), (g \circ f)(y)\}$$

(ii) h is nondecreasing and thus quasiconcave.

(iii) BWOC, suppose h is a strictly increasing function of a concave function. Write $h = g \circ f$. Then the local maxima and global maxima of h are the same as those of f. h has a local maximum at $x = 1/2$, so f must have a local maximum at $x = 1/2$. But f is concave, so f has a *global* maximum at $x = 1/2$. It follows that h also has a global maximum at $x = 1/2$. But h has no global maximum ($\lim_{x\to\infty} h = \infty$), so this is a contradiction.

Exercise 5 (2023 Final Q6)**.** Fix some $Y \subseteq \mathbb{R}^d$ that is nonempty and has a nonempty interior. We say that a (production) vector $y \in Y$ is *efficient* if there is no $y' \in Y$ such that $y' \geq y$ and $y' \neq y$. A production vector $y \in Y$ is *profit-maximising for some* $p \in \mathbb{R}^d_{++}$ if

$$p \cdot y \geq p \cdot y'$$

for all $y' \in Y$.

(i) Prove or disprove: (a) If $y \in Y$ is efficient, then y is a boundary point of Y; (b) if $y \in Y$ is a boundary point of Y, then y is efficient.

(ii) Prove that: if $y \in Y$ is profit-maximising for some $p \in \mathbb{R}^d_{++}$, then y is efficient.

(iii) State a separating hyperplane theorem.

(iv) Suppose that Y is convex. Prove that every efficient production vector $y \in Y$ is a profit-maximising production vector for some $p \in \mathbb{R}^d_+$ (i.e., $p \neq 0$ and $p \geq 0$). **Hint:** Apply the separating hyperplane theorem to the set Y and $P_y := \{y' \in Y \mid y' \gg y\}$, where $(y'_i)_{i=1}^d = y' \gg y = (y_i)_{i=1}^d$ means that $y'_i > y_i$ for all $i = 1, \ldots, d$. Try drawing the case of $d = 2$.

(i) (a) True. If $y \in Y$ is efficient but not a boundary point of Y, then $y + \varepsilon \mathbf{1} \in Y$ for some sufficiently small positive ε. This contradicts efficiency of y.

(b) False. Let $Y = [0,1]^2$. Then $(0,0)$ is a boundary point of Y, but it is not efficient.

(ii) Suppose y is profit-maximising for p but is not efficient. Then there exists $y' \in Y$ such that $y' \geq y$ and $y' \neq y$. Because p is strictly positive, this means $p \cdot y' > p \cdot y$, a contradiction.

(iii) Suppose X and Y are two nonempty, disjoint and convex subsets of $\mathbb{R}^d$. Then, X and Y are separated by a hyperplane.

(iv) We're given that Y is nonempty and convex. P_y contains $y + \mathbf{1}$ so it is nonempty. If $y' \gg y$ and $y'' \gg y$, then $\alpha y' + (1-\alpha)y'' \gg y$, so P_y is convex. Because y is efficient, Y and P_y must be disjoint. It follows by the Separating Hyperplane Theorem that Y and P_y are separated by a hyperplane. That is, there exists $p \neq 0$ such that

$$p \cdot y' \geq p \cdot y'' \tag{1}$$

for every $y' \in P_y$ and every $y'' \in Y$. In particular,

$$p \cdot \left(y + \frac{1}{n} \mathbf{1} \right) \geq p \cdot y''$$

for every $y'' \in Y$ and every $n \in \mathbb{N}$. Taking $n \to \infty$,

$$p \cdot y \geq p \cdot y''$$

for every $y'' \in Y$. All that remains is to show that $p \geq 0$. Suppose $p_j < 0$ for some j. Let $y'_j > y_j$ and $y'_i = y_i$ for all $i \neq j$. Then $y' \in P_y$ and $y \in Y$, but $p \cdot y' < p \cdot y$, contradicting (1).