ECON 6190 Section 1

Aug. 30, 2024

Yiwei Sun

Logistics

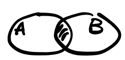
- TA: – Section: PS problems
  - OH: M 12-1pm ; W 9-10AM @ uris 457
  - Email: ys556@cornell.edu
- Problem set: 10% – 10 in total
  - due date check canvas (Friday before canvas)

    1 point (full mark): submitted in time and no substantial methodological errors or major conceptual misunderstandings of the materials
    0.6 point: submitted in time but there are substantial methodological errors or major conceptual misunderstandings of the materials
    0 point: no submission or submitted but no attempt
  - Submit on Canvas
- Supplemental Material for Probability Theory
  - Tak's notes
  - Previous class slides on canvas (Topic 0-2)
  - Hansen chapter 1-5.

1. Basic Probability Theory

1.1 Properties of probability function:

- $\underbrace{P(A \cup B)}_{\le 1} = P(A) + P(B) - \underbrace{P(A \cap B)}_{\ge 0}$



- Boole's inequality: $P(A \cup B) \leq P(A) + P(B)$
- Bonferroni's inequality: $P(A \cap B) \geq P(A) + P(B) - 1$

  $P(A \cap B) = P(A) + P(B) - \underbrace{P(A \cup B)}_{\le 1}$

1.2 Conditional Probability

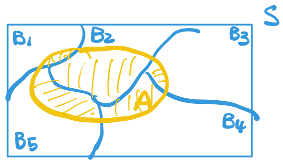
$\boxed{\text{DEF}}$ If $P(B) > 0$, the conditional probability of A given B is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

1.3 Law of Total Probability

Theorem If $\{B_1, B_2, \dots\}$ is partition of sample space S, and $P(B_i) > 0, \forall i$,

then $$P(A) = \sum_{i=1}^{n} P(B_i)\, P(A \mid B_i)$$



$$P(A) = P\left(\bigcup_{i=1}^{n} (A \cap B_i)\right)$$

$$\overset{B_i \text{ disjoint}}{=} \sum_{i=1}^{n} P(A \cap B_i)$$

$$\overset{\text{cond. Prob}}{=} \sum_{i=1}^{n} P(B_i) P(A \mid B_i)$$

1.4. Baye's Rule:

Theorem If $P(A) > 0$, and $P(B) > 0$, then

$$P(A \mid B) = \frac{P(B \mid A) P(A)}{P(B \mid A) P(A) + P(B \mid A^c) P(A^c)}$$

Proof. By def of cond. Prob. & Law of total probability,

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

cond. prob.

$$= \frac{P(B \mid A) P(A)}{P(B \mid A) P(A) + P(B \mid A^c) P(A^c)}$$

$\{A, A^c\}$ is a partition of S & Law of total prob.

2. Random Variables

DEF A RV is a function from the sample space S to the real line $\mathbb{R}$.

DEF A Cumulative distribution function (CDF) of a RV X is

$$F_X(x) = P_X(X \le x), \ \forall x \in \mathbb{R}.$$

Properties of CDF:

- $F_X(x)$ is nondecreasing: $x_1 < x_2 \Rightarrow F_X(x_1) \le F_X(x_2)$.
- $\lim_{x \to -\infty} F_X(x) = 0$, $\lim_{x \to \infty} F_X(x) = 1$.
- $F_X(\cdot)$ is right continuous.

- <u>Continuous</u> RV if $F_X(x)$ is continuous
  - → probability density function (pdf) $f_X(x) = \frac{d}{dx} F_X(x)$.

    By fundamental theorem of Calculus
    $$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt \ , \ \forall x.$$
- <u>Discrete</u> RV if $F_X(x)$ is a step function
  - → probability mass function (pmf) $f_X(x) = P(X=x)$, $\forall x$.

Transformation of RV, see Topic 1 Slides.

$\boxed{\text{DEF}}$ The expectation or mean of a RV $g(X)$ is

$$E[g(x)] = \int_{-\infty}^{\infty} g(x)\,dF(x)$$

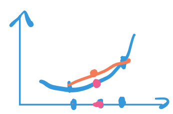
$$= \begin{cases} \int_{-\infty}^{\infty} g(x) f_X(x)\,dx & \text{if cont.} \\ \sum_{x \in \mathcal{X}} g(x) P(X=x) & \text{if discrete} \end{cases}$$

$\mathcal{X}$: the support of a random variable X.

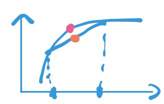
<u>Properties</u>:

- linear operator: $E[aX+b] = aE[X]+b$
- ✱ Jensen's inequality:

  if $g(x)$ is convex, $g(E[X]) \leq E[g(x)]$

  if $g(x)$ is concave, $E[g(x)] \leq g(E[X])$

$\boxed{\text{DEF}}$ The <u>variance</u> of X is $\sigma_X^2 = E[(X-E[X])^2]$

The <u>standard deviation</u> of X is $\sigma_X = \sqrt{\sigma_X^2}$.

<u>Theorem</u> $\quad var(X) = E[X^2] - (E[X])^2$

$\quad var(aX+b) = a^2 var(X)$

3. Random Vectors (Bivariate)

Joint distribution function: $F_{X,Y}(x,y) = P(X \le x, Y \le y)$
$= P(\{X \le x\} \cap \{Y \le y\})$

Joint probability mass function: $f(x,y) = P(X=x, Y=y)$

Joint probability density function: $f(x,y) = \frac{\partial}{\partial x \partial y} F(x,y)$, if $F(x,y)$ is cont. & diff.

Marginal distribution X: $F_X(x) = P(X \le x) = P(X \le x, Y \le \infty)$

Marginal pmf is $f_X(x) = P(X=x) = \sum_{y \in \mathbb{R}} f(x,y)$, $\forall x \in \mathbb{R}$

Marginal pdf is $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$.

4. Conditional distribution

Conditional distribution function of $Y|X$ is:

X discrete $F_{Y|X}(y|x) = P(Y \le y | X = x) = \frac{P(Y \le y, X = x)}{P(X = x)}$

X continuous $F_{Y|X}(y|x) = \lim_{\varepsilon \to 0} P(Y \le y \mid x - \varepsilon \le X \le x + \varepsilon)$

Conditional pdf/pmf is $f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$, $f_X(x) > 0$.

$\boxed{\text{DEF}}$ Two random variable X, Y are statistically independent if

$F(x,y) = F_X(x) F_Y(y)$. written $X \perp Y$.

$\Rightarrow f(x,y) = f_X(x) f_Y(y)$ $\Rightarrow f_{Y|X}(y|x) = f_Y(y)$

Law of Iterated Expectation

Theorem If $E[|Y|] < \infty$, $E[Y] = E[E[Y|X]]$.

marginal X Y|X

$$E[Y] = \iint y\, f(x,y)\, dy\, dx$$
$$= \iint y\, f_X(x) f_{Y|X}(y|x)\, dy\, dx$$
$$= \int f_X(x) \underbrace{\int y\, f_{Y|X}(y|x)\, dy}_{E[Y|X=x] \text{ by def.}}\, dx$$
$$= \int f_X(x) \underbrace{E[Y|X=x]}_{\text{only a function } X}\, dx = E[E[Y|X]]$$

2. [Hansen 1.16] Suppose that the unconditional probability of a disease is 0.0025. A screening test for this disease has a detection rate of 0.9, and has a false positive rate of 0.01. Given that the screening test returns positive, what is the conditional probability of having the disease?

event D : have disease

event P : positive screening test

$P(D) = 0.0025$

$P(P \mid D) = 0.9$

$P(P \mid D^c) = 0.01$

$$P(D \mid P) = \frac{P(D \cap P)}{P(P)} = \frac{P(P \mid D)P(D)}{P(P \mid D)P(D) + P(P \mid D^c)P(D^c)}$$

$$= \frac{0.9 \times 0.0025}{0.9 \times 0.0025 + 0.01 \times 0.9975}$$

$$= 0.184.$$

5. [Hong 5.54] Suppose (X, Y) have a joint pdf

$$f_{XY}(x, y) = \begin{cases} xe^{-y}, & \text{if } 0 < x < y < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the conditional pdf $f_{Y|X}(y|x)$ of Y given $X = x$.

(b) Find the conditional mean $\mathbb{E}(Y|X = x)$.

(c) Find the conditional variance $\text{var}(Y|X = x)$.

(d) Are X and Y independent? Give your reasoning.

(a) $f_X(x) = \int_x^\infty xe^{-y}\,dy = -xe^{-y}\Big|_x^\infty = 0 - (-xe^{-x}) = xe^{-x}$, for $x>0$

then $f_{Y|X}(y|x) = \dfrac{f_{XY}(x,y)}{f_X(x)}$

$$= \begin{cases} e^{x-y}, & \text{if } 0 < x < y < \infty. \\ 0, & \text{otherwise.} \end{cases}$$

(b) $E[Y \mid X = x] = \int_x^\infty y\, f_{Y|X}(y|x)\, dy$

$$= \int_x^\infty y\, e^{x-y}\, dy$$

$$= e^x \underbrace{\int_x^\infty y\, e^{-y}\, dy}$$

$\int u\,dv = uv - \int v\,du$

Integration by parts: $\frac{d(-e^{-y})}{dy} = e^{-y} \Rightarrow e^{-y}dy = d(-e^{-y})$

$\Rightarrow \int_x^\infty y\, d(-e^{-y})$

$$= e^x \left\{ y(-e^{-y})\Big|_x^\infty - \left(-\int_x^\infty e^{-y}\,dy\right) \right\}$$

$$= e^x \left\{ 0 - (-xe^{-x}) + \underbrace{\left(-e^{-y}\Big|_x^\infty\right)}_{0-(-e^{-x})} \right\}$$

$$= e^x \left\{ xe^{-x} + e^{-x} \right\} = x+1$$

(c) $\mathrm{var}(Y \mid X=x) = E\left[(Y - E[Y \mid X=x])^2 \mid X=x\right]$

$$= \underbrace{E[Y^2 \mid X=x]}_{?} - \left(\underbrace{E[Y \mid X=x]}_{\checkmark}\right)^2$$

$$E[Y^2 \mid X=x] = \int_x^\infty y^2 f_{Y|X}(y|x)\,dy$$

$$= x^2 + 2x + 2$$

$$\mathrm{var}(Y \mid X=x) = E[Y^2 \mid X=x] - (E[Y \mid X=x])^2$$

$$= x^2 + 2x + 2 - (x+1)^2 = 1.$$

(d) $E[Y \mid X=x] = x+1$.

NOT Statistically independent.

If so, $E[Y \mid X=x]$ shouldn't be a function of x.