ECON 6170 Section 4

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Convexity

Definition. A subset X of $\mathbb{R}^d$ is *convex* if for any $x, y \in X$ and $\alpha \in [0,1]$, $\alpha x + (1-\alpha)y \in X$.

Remark 1. Visually, this means that the line segment between any two points in X is also in X.

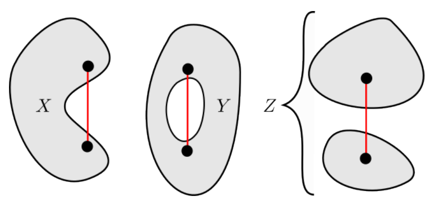


Figure 1: X, Y and Z are all non-convex, as the red line segments lie outside the sets. Note that Z is the finite union of convex sets.

Remark 2. To show a set, X, is convex we can

(i) Take two arbitrary points, $x, y \in X$, and an arbitrary $\alpha \in [0,1]$, and show that $\alpha x + (1-\alpha)y \in X$.

(ii) Show that X is the intersection of sets we know to be convex, e.g., intervals.

Section Exercise 1. Are the following sets convex? Prove your answer.

(i) $\mathbb{R}$

(ii) A line, $\ell := \{(x,y) \mid ax + by = c\}$, in $\mathbb{R}^2$

(iii) The unit circle centered at the origin, $S := \{(x,y) \mid x^2 + y^2 = 1\}$

(iv) The open unit disc[1] centred at the origin, $B := \{(x,y) \mid x^2 + y^2 < 1\}$

(v) The complement of a convex set

(vi) A singleton (a set with exactly one element)

(vii) A finite set with more than one element

(i) Yes. If $x, y \in \mathbb{R}$ and $\alpha \in [0,1]$, then $\alpha x + (1-\alpha)y \in \mathbb{R}$.

[1] "Disc" is a term for a ball in $\mathbb{R}^2$; compare circle versus sphere.

(ii) Yes. If (x, y) satisfies $ax + by = c$, and (z, w) satisfies $az + bw = c$, and $\lambda \in [0, 1]$, then $(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w)$ satisfies

$$\begin{aligned} a(\lambda x + (1-\lambda)z) + b(\lambda y + (1-\lambda)w) &= \lambda(ax+by) + (1-\lambda)(az+bw) \\ &= \lambda c + (1-\lambda)c \\ &= c \end{aligned}$$

(iii) No. Both $(1, 0)$ and $(-1, 0)$ are on the unit circle, but $\frac{1}{2}(1, 0) + \frac{1}{2}(-1, 0) = (0, 0)$ is not.

(iv) Yes. If $x^2 + y^2 < 1$ and $z^2 + w^2 < 1$, then

$$\begin{aligned} (\alpha x + (1-\alpha)z)^2 + (\alpha y + (1-\alpha)w)^2 &= \alpha^2 x^2 + 2\alpha(1-\alpha)xz + (1-\alpha)^2 z^2 + \alpha^2 y^2 \\ &\quad + 2\alpha(1-\alpha)yw + (1-\alpha)^2 w^2 \\ &= \alpha^2(x^2+y^2) + (1-\alpha)^2(z^2+w^2) + 2\alpha(1-\alpha)(xz+yw) \\ \\ &< \alpha^2 + (1-\alpha)^2 + 2\alpha(1-\alpha)(xz+yw) \end{aligned}$$

It suffices to show that $xz + yw < 1$, for then $\alpha^2 + 2\alpha(1-\alpha) + (1-\alpha)^2 = (\alpha + 1 - \alpha)^2 = 1$. To show $xz + yw < 1$, it suffices to show that $xz + yw > x^2 + y^2$ and $xz + yw > z^2 + w^2$ together imply a contradiction. Note that we can sum these inequalities to get $x^2 - 2xz + z^2 + yw - 2yw + w^2 < 0$, which we can rewrite as $(x-z)^2 + (y-w)^2 < 0$, which is impossible.

(v) Possibly, but not in general. The complement of $(-\infty, 0)$ is $[0, \infty)$, which is convex. However, the complement of $(-1, 1)$ is $(-\infty, 1] \cup [1, \infty)$, which is non-convex.

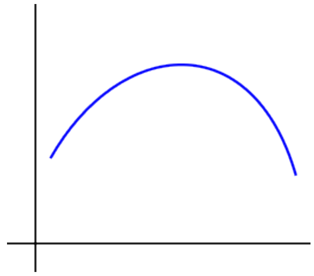
(vi) Yes, as $\alpha x + (1-\alpha)x = x \in \{x\}$ for all $\alpha \in [0, 1]$.

(vii) No. Let x and y be distinct elements of the set. Then, $\{\alpha_n x + (1-\alpha_n)y \mid n \in \mathbb{N}\}$ is an infinite set, so at least one of its elements cannot be in the finite set.

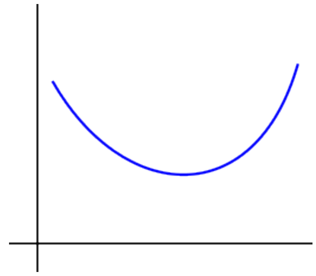
Convex and Quasiconvex Functions

Remark 3. Visually, a function $f : X \subseteq \mathbb{R}^d \to \mathbb{R}$ is...

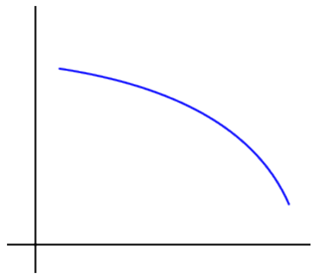
- ➢ Concave iff its subgraph is convex.
- ➢ Convex iff its epigraph is convex.
- ➢ Quasiconcave iff the preimage of every interval $[r, \infty)$ under f is convex.
- ➢ Quasiconvex iff the preimage of every interval $(-\infty, r]$ under f is convex.



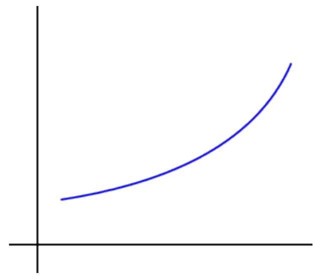
concave



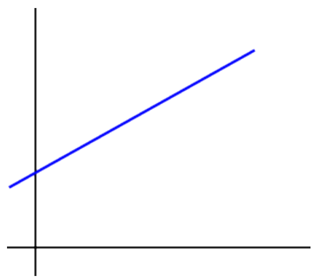
convex



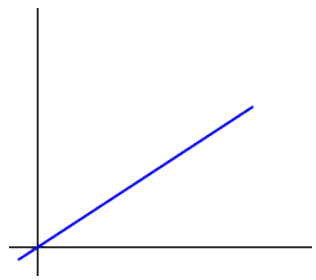
concave and quasiconvex



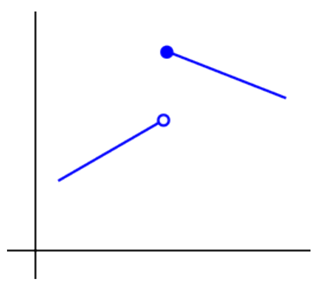
convex and quasiconcave



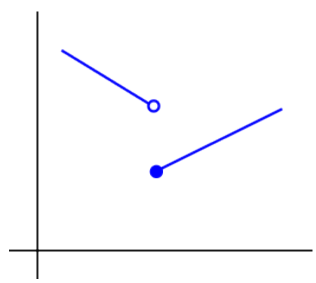
affine



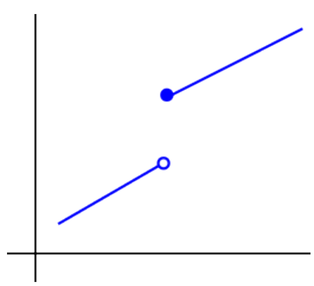
linear ($\Longrightarrow$ affine)



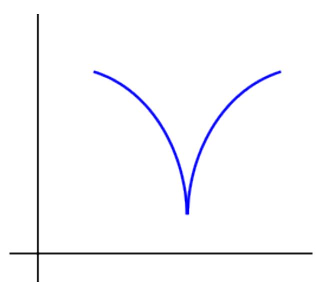
quasiconcave



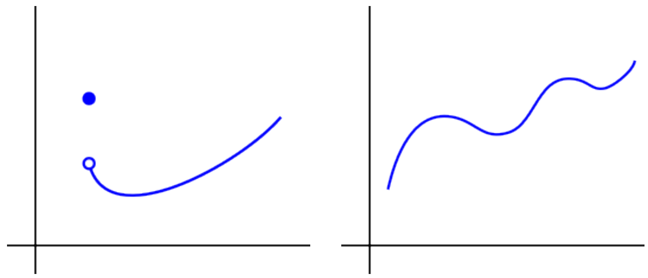
quasiconvex



quasiconcave and quasiconvex



quasiconvex



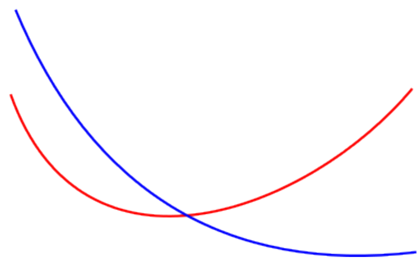
More Convexity Exercises

Section Exercise 2.

(i) Show that if $f, g : \mathbb{R}^k \to \mathbb{R}$ are convex, then so too is $\max\{f, g\}$.

(ii) Provide a counterexample to show that the previous result doesn't hold if we replace max with min.

(iii) Using the claim in part (i), show that if f and g are concave, then so too is $\min\{f, g\}$.



(i) Let $h := \max\{f, g\}$. Fix $x, y \in \mathbb{R}^k$ and $\alpha \in [0, 1]$. We're given that $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$ and $g(\alpha x + (1-\alpha)y) \leq \alpha g(x) + (1-\alpha) g(y)$.

$$\begin{aligned} h(\alpha x + (1-\alpha)y) &= \max\{f(\alpha x + (1-\alpha)y), g(\alpha x + (1-\alpha)y)\} \\ &\leq \max\{\alpha f(x) + (1-\alpha) f(y), \alpha g(x) + (1-\alpha) g(y)\} \\ &\leq \max\{\alpha f(x), \alpha g(x)\} + \max\{(1-\alpha) f(y), (1-\alpha) g(y)\} \qquad (**) \\ &= \alpha \max\{f(x), g(x)\} + (1-\alpha) \max\{f(y), g(y)\} \\ &= \alpha h(x) + (1-\alpha) h(y) \end{aligned}$$

so $\max\{f, g\}$ is convex. One step we might be unsure of is $(**)$. This step uses the claim that $\max\{x+y, z+w\} \leq \max\{x, z\} + \max\{y, w\}$. We can confirm this by supposing, without loss of generality, that $x + y \geq z + w$. Then clearly $\max\{x+y, z+w\} = x + y \leq \max\{x, z\} + \max\{y, w\}$.[2]

(ii) Take $f, g : \mathbb{R} \to \mathbb{R}$ given by $f(x) := x$ and $g(x) := -x$, respectively. Then $\min\{f, g\}(x) := -|x|$.

[2]Solution suggested by Spencer Dean: $\{(x, y) \mid y \geq f(x)\} \cap \{(x, y) \mid y \geq g(x)\} = \{(x, y) \mid y \geq \max\{f, g\}(x)\}$. Convexity of f and g implies that the first two sets are convex. So their intersection, the epigraph of $\max\{f, g\}$ must also be convex. It follows that $\max\{f, g\}$ is a convex function.

(iii) If f and g are concave, then $-f$ and $-g$ are convex, and thus so too is $\max\{-f,-g\}$. But $\max\{-f,-g\} = -\min\{f,g\}$, so $\min\{f,g\}$ is concave.

Exercise 4. Show that if S is open then so too is $\operatorname{co}(S)$.

This holds trivially for the empty set. Suppose, then, that S is nonempty and open. Let $z \in \operatorname{co}(S)$. Then we can write

$$z = \sum_{i=1}^{n} \alpha_i x_i$$

for some $x_i \in S$ and $\alpha_i \in [0,1]$ that sum to 1. Openness of S implies that for each x_i, there exists ε_i such that $B_{\varepsilon_i}(x_i) \subseteq S$. Let $\varepsilon = \min\{\varepsilon_i \mid i = 1, \ldots, n\}$. Then we can write

$$B_\varepsilon(x_i) \in S$$

for all i. Take $w \in B_\varepsilon(z)$. We want to show that $w \in \operatorname{co}(S)$, which would imply $B_\varepsilon(z) \subseteq \operatorname{co}(S)$. This, in turn, would be sufficient to prove openness of $\operatorname{co}(S)$. Write

$$w = z + w - z = \sum_{i=1}^{n} \alpha_i x_i + w - z = \sum_{i=1}^{n} \alpha_i (x_i + w - z) =: \sum_{i=1}^{n} \alpha_i y_i$$

where $y_i := x_i + w - z$ for all i. Thus w is a convex combination of $y_1, \ldots, y_n$, so if $y_1, \ldots, y_n \in S$, we would have $w \in \operatorname{co}(S)$. But for all i,

$$\|y_i - x_i\| = \|x_i + w - z - x_i\| = \|w - z\| < \varepsilon$$

so $y_i \in B_\varepsilon(x_i) \subseteq S$.