ECON 6170 Section 3

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Summary of $\mathbb{R}^d$

Definition 1.

$$\mathbb{R}^d := \{(x_1, \ldots, x_d) \mid x_i \in \mathbb{R} \text{ for all } i\}$$

We call elements of $\mathbb{R}^d$ vectors. When $d \geq 2$, we call elements of $\mathbb{R}$ scalars.

Definition 2. We define two operations on $\mathbb{R}^d$:

(i) Vector addition:

$$x + y = (x_1 + y_1, \ldots, x_d + y_d)$$

(ii) Scalar multiplication:

$$\alpha x = (\alpha x_1, \ldots, \alpha x_d)$$

where $\alpha \in \mathbb{R}$.

These reduce to ordinary addition and multiplication when $d = 1$.

Definition 3. The (Euclidean) **norm** of a vector $x \in \mathbb{R}^d$ is defined by

$$\|x\| := \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}$$

Remark 1. The norm reduces to absolute value when $d = 1$.

Proposition 1. *The norm satisfies the following properties*

(i) $\|x\| \geq 0$, *and* $\|x\| = 0$ *iff* $x = 0$.

(ii) *For* $\alpha \in \mathbb{R}$, $\|\alpha x\| = |\alpha| \cdot \|x\|$.

(iii) $\|x + y\| \leq \|x\| + \|y\|$.

Definition 4. The (Euclidean) **distance** between two vectors, x and y is defined by

$$d(x, y) := \|x - y\| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_d - y_d)^2}$$

Remark 2. We can generalise boundedness, convergence and continuity to $\mathbb{R}^d$ by replacing absolute values $|\cdot|$ with Euclidean norms $\|\cdot\|$.

Definition 5. A set $X \in \mathbb{R}^d$ is bounded if there exists $M \in \mathbb{R}$ such that such that $\|x\| \leq M$ for all $x \in X$.

Remark 3. Note that for $d \geq 2$, the natural order on $\mathbb{R}$ can be extended in several ways, none of which have all the desirable properties of their restriction to $\mathbb{R}$. Hence, we won't often see terms like maximum, minimum, supremum, infimum, monotone, *et cetera*, for sets and sequences in $\mathbb{R}^d$.[1]

Definition 6. A sequence (x_n) converges in $\mathbb{R}^d$ if

For all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $\|x_n - x\| < \epsilon$.

Proposition 2. *Suppose that (x_n) and (y_n) are sequences in $\mathbb{R}^d$ that converge to x and y, respectively. Suppose also that (α_n) is a sequence in $\mathbb{R}$ that converges to α. Then*

(i) $x_{n,j} \to x_j$ *where* $x_{n,j}$ *is the j-th entry of* x_n *and* x_j *is the j-th entry of* x.[2]

(ii) $\|x_n\| \to \|x\|$

(iii) $x_n + y_n \to x + y$

(iv) $\alpha_n x_n \to \alpha x$

(v) $\frac{1}{\alpha_n} \to \frac{1}{\alpha}$ *if* $\alpha_n, \alpha \neq 0$ *for all* n.

Open, Closed and Compact Sets

Definition 7. A set $U \subseteq \mathbb{R}^k$ is open if for every $x \in U$ there exists some $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U$.

Definition 8. A set $C \subseteq \mathbb{R}^d$ is closed if its complement $C^{\mathsf{C}} := \mathbb{R}^d \setminus C$ is an open set.

	closed	*not closed*
open	clopen: $\mathbb{R}^d, \emptyset$	nontrivial open sets
not open	nontrivial closed sets	"most" sets

Note: Nontrivial here simply means neither the empty set nor $\mathbb{R}^n$.

Remark 4. To show a set U is open, do one of:

1. Take an arbitrary $x \in U$ and find some $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U$.
2. Show U^{C} is closed.
3. Show U is the union of sets we know to be open (e.g., open intervals).
4. Show U is the *finite* intersection of sets we know to be open.

To show C is closed, do one of:

[1] Keep in mind that norms take values in $\mathbb{R}$, so we *can* order norms in a natural way.

[2] The use of x_j and x_n here is an abuse of notation. Hopefully the distinction is clear from the context.

1. Take an arbitrary convergent sequence of points in C, and show that the sequence converges to a point in C.
2. Show C^{C} is open.
3. Show C is the intersection of sets we know to be closed (e.g., closed intervals).
4. Show C is the *finite* union of sets we know to be closed.

Section Exercise 1. Are the following sets open or closed or neither (as subsets of $\mathbb{R}$ unless otherwise specified)?

(i) The set of irrational numbers, $\mathbb{Q}^{\mathsf{C}}$

(ii) $\mathbb{Q}$

(iii) A cofinite set; that is, a set having a finite complement.

(iv) $\bigcap_{n\in\mathbb{N}}[-1/n, 1/n]$

(v) $\bigcap_{n\in\mathbb{N}}(-1/n, 1/n)$

(vi) $\mathbb{N}$

(vii) $\{1/n \mid n \in \mathbb{N}\}$

(viii) A line in $\mathbb{R}^2$

(ix) The finite Cartesian product of open subsets of $\mathbb{R}$, $\bigtimes_{n=1}^{N} U_n$

(x) The finite Cartesian product of closed subsets of $\mathbb{R}$, $\bigtimes_{n=1}^{N} C_n$

(i) Neither. Every nonempty open interval contains some $q \in \mathbb{Q}$, so $\mathbb{Q}^{\mathsf{C}}$ is not open. Consider the sequence $(\frac{\sqrt{2}}{n})_{n=1}^{\infty}$. This is a sequence of irrational numbers[3] that converges to the rational number 0, so $\mathbb{Q}^{\mathsf{C}}$ is not closed.

(ii) Neither. It follows that $\mathbb{Q} = \mathbb{Q}^{\mathsf{CC}}$ is neither closed nor open.

(iii) Open. We saw in the lectures that a finite set is closed, so its complement must be open.

(iv) Closed. The intersection of closed intervals is closed. In this case, the set of all points satisfying $-1/n \leq x \leq 1/n$ for all $n \in \mathbb{N}$ is just $\{0\}$. For if $|x| > 0$ then $|x| > |1/n|$ for sufficiently large n.

(v) Closed. Even though the constituent intervals are open, this is an *infinite* intersection, so we cannot infer openness of the intersection. In fact, this intersection is again $\{0\}$, as $-1/n < 0 < 1/n$ for all $n \in \mathbb{N}$ and $|x| > 0$ implies $|x| \geq |1/n|$ for large n.

(vi) Closed. If (x_n) is a convergent sequence of natural numbers, then $|x_n - x| < 1$ for some x and sufficiently large n. It follows that (x_n) is eventually constant with tail $(x, x, \ldots)$. Since the sequence is in $\mathbb{N}$, it must be that $x \in \mathbb{N}$. Alternatively, observe that $\mathbb{N}^{\mathsf{C}} = (-\infty, 1) \cup (1, 2) \cup (2, 3) \cup \ldots$, a union of open intervals.

[3] If i is an irrational number, such as $\sqrt{2}$, and r is a rational number, such as $1/n$, $n \in \mathbb{N}$, then $i \cdot r$ is irrational. For otherwise we could write $ir = p/q$ and $r = s/t$ for integers p, q, s, t, and thus $i = pt/qs$ would be rational.

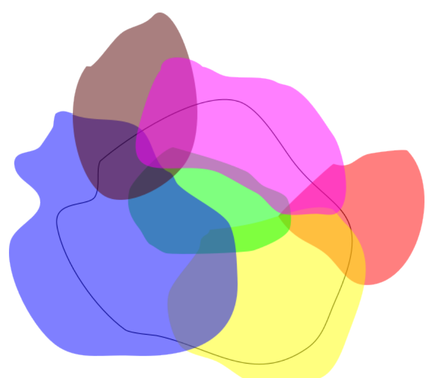
(vii) Neither. $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ contains a sequence $(\frac{1}{n})_n$ converging to 0, which is outside the set, so it is not closed. The sequence $(1 + 1/n)$ lies outside the set but converges to 1, which is inside the set, so the complement of the set is not closed, so the set itself isn't open.

(viii) Closed. We define a line in $\mathbb{R}^2$ as a set $\ell = \{(x, y) \in \mathbb{R}^2 \mid ax + by = c\}$, where a, b and c are specified real numbers with a and b not both zero. If $((x_n, y_n))_{n=1}^{\infty}$ is a sequence on the line such that $x_n \to x$, $y_n \to y$, and for all n, $ax_n + by_n = c$, then $\lim_n(ax_n + by_n) = a \lim_n x_n + b \lim_n y_n = \lim_n c$, so $ax + by = c$. In other words, $(x, y) \in \ell$.

(ix) Open. Suppose $x \in \bigtimes_n U_n$ then $x_n \in U_n$ for all n. It follows that there exists ϵ_n such that $(x_n - \epsilon_n, x_n + \epsilon_n) \subseteq U_n$. It follows that $\bigtimes_n(x_n - \epsilon_n, x_n + \epsilon_n) \subseteq \bigtimes_n U_n$. Let $\epsilon = \min_n \epsilon_n$. Suppose $\|y - x\| < \epsilon$. Then, in particular, $|y_n - x_n| \leq \|y - x\| < \epsilon \leq \epsilon_n$. Since y is just an arbitrary element of $B_\epsilon(x)$, it must be the case that $B_\epsilon(x) \subseteq \bigtimes_n(x_n - \epsilon_n, x_n + \epsilon_n) \subseteq \bigtimes_n U_n$.

(x) Closed. Let (x_k) be a sequence of terms in $\bigtimes_{n=1}^{N} C_n$ with limit x. Each term in (x_k) is a length-N vector, so (x_k) induces N scalar sequences, $(x_{k,n})$ consisting of, e.g., the sequence of n-th entries of (x_k). For each n, $x_{k,n} \to x_n$, the n-th entry of x. Because C_n is closed, $x_n \in C_n$. It follows that $x \in \bigtimes_n C_n$, as required.

Definition 9. A set A is compact if every open cover has a finite subcover. That is, if for *any* collection $\mathcal{U}$ of open sets such that $A \subseteq \bigcup_{U \in \mathcal{U}} U$, there is a *finite* subset $\{U_1, U_2, \ldots, U_m\}$ that still covers A: $A \subseteq \bigcup_{i=1}^{m} U_i$.



The collection of six coloured sets form a cover of the set inside the black border. If all the coloured sets are open, then this is an *open* cover. Note that the brown set is redundant, in that the parts of the underlying set it covers are also covered by the blue, green, or pink sets. Thus, the subcollection consisting of the remaining five coloured sets is a subcover of the original cover.

Theorem 17 (Heine-Borel)**.** *A set $A \subseteq \mathbb{R}^k$ is compact if and only if it is closed and bounded.*

Remark 5. If we want to identify if a subset of $\mathbb{R}^k$ is compact, identifying if it is closed and bounded is often the easiest way to do so. In the context of Euclidean space, therefore, Definition 16 is best thought of as a useful property of compact sets. Heine-Borel does not generalize to metric spaces, however.

Section Exercise 2. Which of the sets in Section Exercise 1 are compact?

The bounded, closed sets, which will be (iv) and (v). (x) will be compact if and only if the C_n are all bounded.