7. Comparative Statics

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1 Implicit Function Theorem (again)

Berge's theorem of the maximum gave us conditions under which the solution correspondence and value function were continuous. The implicit function theorem gave us (stronger) conditions under which the solution function are differentiable and told us how to compute those derivatives. We can then use the derivatives to conduct comparative statics.

Example 1. For concreteness, let $X \subseteq \mathbb{R}^d$ and $\Theta \subseteq \mathbb{R}^m$ be open and convex and that $f : X \times \Theta \to \mathbb{R}$ is $\mathbf{C}^2$ and that $f(\cdot, \theta)$ is strictly concave on X for each $\theta \in \Theta$. Suppose we want to maximise f on X for some given $\theta \in \Theta$; i.e.,

$$f^*(\boldsymbol{\theta}) \coloneqq \max_{\mathbf{x} \in X} f(\mathbf{x}, \boldsymbol{\theta}).$$

We know that the first-order condition characterises the unique global maximum; i.e., $\mathbf{x}_0 \in X$ is a global maximum of $f(\cdot, \boldsymbol{\theta}_0)$ if and only if

$$\nabla_{\mathbf{x}} f(\mathbf{x}_0, \boldsymbol{\theta}_0) = \mathbf{0}_{1\times d}.$$

Define $h : \Theta \times X \to \mathbb{R}^d$ via $h(\boldsymbol{\theta}, \mathbf{x}') \coloneqq \nabla_{\mathbf{x}} f(\mathbf{x}', \boldsymbol{\theta})$. Then, we have that $h(\boldsymbol{\theta}_0, \mathbf{x}_0) = \mathbf{0}$. Suppose that $D_{\mathbf{x}} h(\boldsymbol{\theta}_0, \mathbf{x}_0) = D^2_{\mathbf{x}} f(\mathbf{x}_0, \boldsymbol{\theta}_0)$ is invertible, then since h is $\mathbf{C}^1$ (because f is $\mathbf{C}^2$), the implicit function theorem gives us that there exist open balls centred at $\boldsymbol{\theta}_0$ and $\mathbf{x}_0$, denoted B_Θ and B_X respectively, and a differentiable solution function $x^* : B_\Theta \to B_X$ such that $x^*(\boldsymbol{\theta}_0) = \mathbf{x}_0$ and

$$h(\boldsymbol{\theta}, x^*(\boldsymbol{\theta})) = \nabla_{\mathbf{x}} f(\mathbf{x}^*(\boldsymbol{\theta}), \boldsymbol{\theta}) = \mathbf{0} \ \forall \boldsymbol{\theta} \in B_\Theta.$$

Moreover, the theorem also gives us how the solution changes with with parameter $\boldsymbol{\theta}$ (around $\boldsymbol{\theta}_0$):

$$\begin{aligned} Dx^*(\boldsymbol{\theta}) &= -\left(D_{\mathbf{x}} h(\boldsymbol{\theta}, x^*(\boldsymbol{\theta}))\right)^{-1} D_{\boldsymbol{\theta}} h(\boldsymbol{\theta}, x^*(\boldsymbol{\theta})) \\ &= -\left(D^2_{\mathbf{x}} f(x^*(\boldsymbol{\theta}), \boldsymbol{\theta})\right)^{-1} D_{\boldsymbol{\theta}} \nabla_{\mathbf{x}} f(x^*(\boldsymbol{\theta}), \boldsymbol{\theta}). \end{aligned}$$

Exercise 1 (PS11)**.** Consider the problem of maximising an objective function $f : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$

*Thanks to Giorgio Martini, Nadia Kotova and Suraj Malladi for sharing their lecture notes. The note is also based on a class I took from John Quah.

subject to K equality constraints; i.e.,

$$\max_{\mathbf{x}\in X} f(\mathbf{x},\boldsymbol{\theta}) \text{ s.t. } h_k(\mathbf{x},\boldsymbol{\theta}) = 0 \; k\in\{1,\ldots,K\},$$

where $h_k : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$ for each $k \in \{1,\ldots,K\}$. Suppose that f and h_k's are all $\mathbf{C}^2$ and concave, constraint qualification (for equality constraints) is satisfied, and that there is a unique solution for all $\boldsymbol{\theta} \in \Theta$. Apply the implicit function theorem on the first-order condition of the Lagrangian to give an expression for how the solution varies with $\boldsymbol{\theta}$. What must be true to apply the same argument when there are inequality constraints?

If we know that some (sub)set of inequality constraints are known to be bound at the global maximum of the problem, we can treat them as equality constraints and apply the argument above.

2 Envelope Theorem

Take the example above and let us ask how the maximised objective $f^*(\boldsymbol{\theta})$ varies with $\boldsymbol{\theta}$. Note that

$$f^*(\boldsymbol{\theta}) = f(x^*(\boldsymbol{\theta}),\boldsymbol{\theta}),$$

where we obtained the solution function $x^*(\cdot)$ via the implicit function theorem. Since both f and x^* are differentiable (in the neighbourhood of $\boldsymbol{\theta}_0$),

$$\nabla f^*(\theta) = \nabla_{\mathbf{x}} f(x^*(\boldsymbol{\theta}),\boldsymbol{\theta})\nabla x^*(\boldsymbol{\theta}) + \nabla_{\boldsymbol{\theta}} f(x^*(\boldsymbol{\theta}),\boldsymbol{\theta}).$$

But the first-order condition tells us that the first-term equals zero. Hence,

$$\nabla f^*(\theta) = \nabla_{\boldsymbol{\theta}} f(x^*(\boldsymbol{\theta}),\boldsymbol{\theta}).$$

In words, above tells us that if we want to know how the maximised objective changes with parameter θ, then it suffices to consider only the direct effect of θ on f and not the indirect effect of θ on f via $x^*(\cdot)$.

Remark 1. Where is the "envelope" in the envelope theorem? For each fixed $x \in X \subseteq \mathbb{R}$ (with $\Theta \subseteq \mathbb{R}$ too), consider the graph of $f(x,\cdot)$. Then, the graph of $f^*(\theta)$ is given by the upper envelope of graphs of $\{f(x,\cdot)\}_{x\in X}$—this upper envelope is the envelope!

Theorem 1 (Envelope Theorem)**.** *Let $X \subseteq \mathbb{R}^d$ and $\Theta \subseteq \mathbb{R}^m$ be convex. Suppose that $f : X\times\Theta \to \mathbb{R}$ is $\mathbf{C}^1$,*

$$f^*(\boldsymbol{\theta}) \coloneqq \max_{\mathbf{x}\in X} f(\mathbf{x},\boldsymbol{\theta})$$

is differentiable on $\operatorname{int}(\Theta)$*, and*

$$x^*(\boldsymbol{\theta}) \coloneqq \arg\max_{\mathbf{x}\in X} f(\mathbf{x},\boldsymbol{\theta})$$

is well-defined and differentiable. Then,

$$\nabla f^*(\boldsymbol{\theta}) = \nabla_{\boldsymbol{\theta}} f(x^*(\boldsymbol{\theta}),\boldsymbol{\theta}).$$

Proof. Define $\phi : X\times\Theta \to \mathbb{R}$ as $\phi_x(\boldsymbol{\theta}) \coloneqq f^*(\boldsymbol{\theta}) - f(\mathbf{x},\boldsymbol{\theta})$. By construction $\phi_x(\boldsymbol{\theta}) \geq 0$ for all $x \in X$

and $\phi_{x^*(\boldsymbol{\theta})}(\boldsymbol{\theta}) = 0$. Moreover, $\phi_{\mathbf{x}}$ is continuous (why?) and attains a minimum (of zero) at θ such that $\mathbf{x} = x^*(\boldsymbol{\theta})$. Since $\phi_{x(\boldsymbol{\theta})}$ is differentiable, its derivative at $\boldsymbol{\theta}$ must be zero (why?) which gives

$$\nabla \phi_{\mathbf{x}}(\boldsymbol{\theta}) = \nabla f^*(\boldsymbol{\theta}) - \nabla_{\boldsymbol{\theta}} f(\mathbf{x}, \boldsymbol{\theta}) = 0 \Rightarrow \nabla f^*(\boldsymbol{\theta}) = \nabla_{\boldsymbol{\theta}} f(x^*(\boldsymbol{\theta}), \boldsymbol{\theta}). \qquad \blacksquare$$

Remark 2. As stated above, the envelope theorem has an endogenous requirement for f^* to be differentiable. But since $f^*(\cdot) = f(x^*(\cdot), \cdot)$, for f^* to be differentiable, it suffices that f is differentiable and that $x^*(\cdot)$ is differentiable. While the latter requirement also appear endogenous, we can appeal to the implicit function theorem to obtain conditions that ensure that $x^*(\cdot)$ is differentiable as we did in the example above (wait, what were they all again?).

Exercise 2 (PS11)**.** State and prove the Envelope Theorem for the equality constrained optimisation problem from Exercise 1. **Hint:** You may make the same endogenous assumption as in the theorem above and an additional endogenous assumption regarding the Lagrange multipliers. What further assumptions on the Lagrangian can you make to replace the endogenous assumptions?

3 Monotone Comparative Statics

Consider the problem:

$$\max_{x \in X} \; u(x, \theta) - c(x),$$

where we think of $x \in \mathbb{R}$ is a choice variable, $\theta \in \mathbb{R}$ is a parameter, $u : \mathbb{R}^2 \to \mathbb{R}$ is the utility and $c : \mathbb{R} \to \mathbb{R}$ is the cost function. If u and c are differentiable, the first-order condition is

$$\frac{\partial u}{\partial x}(x^*(\theta), \theta) = \frac{\partial c}{\partial x}(x^*(\theta)).$$

If u and c are twice continuously differentiable and $\frac{\partial^2 u}{\partial x \partial x}(x^*(\theta), \theta) \neq c''(x^*(\theta))$, the implicit function theorem implies that $x^*(\cdot)$ is continuously differentiable and that

$$\frac{\partial x^*}{\partial \theta}(\theta) = \frac{\frac{\partial^2 u}{\partial x \partial \theta}(x^*(\theta), \theta)}{c''(x^*(\theta)) - \frac{\partial^2 u}{\partial x \partial x}(x^*(\theta), \theta)}.$$

If c is convex and u is concave in x, and $\frac{\partial^2 u}{\partial x \partial \theta} > 0$, we can conclude that $x^*(\theta)$ is (locally) increasing in θ.

From this example, you might be tempted to think that smoothness or concavity were important to identifying the effect of θ on x^*. But, recall that applying a strictly increasing transformation to an objective does not alter the maximisers. So continuity, differentiability and concavity of f must have little to do with whether x^* is increasing in θ: Because the transformation can have jumps, kinks and strictly increasing transformation of concave functions need not be concave, yet any comparative statics conclusions that apply to f also applies to the transformed f! We will see below that what's actually driving the result is related to the fact $\frac{\partial^2 u}{\partial x \partial \theta} > 0$; more generally, it is an ordinal condition called *single-crossing property* that drives this comparative static result.

3.1 Partial orders

Definition 1. Given sets X and Y, $R \subseteq X \times Y$ is a *binary relation from X to Y*. Write

$$xRy \Leftrightarrow (x, y) \in R,$$
$$\neg xRy \Leftrightarrow (x, y) \notin R.$$

The *inverse* of a binary relation R from X to Y is a relation from Y to X defined as

$$R^{-1} := \{(y, x) \in Y \times X : (x, y) \in R\}.$$

A *binary relation on X* is $R \subseteq X \times X$ and we say that it is

- ▷ *reflexive* if $\forall x \in X$, xRx;
- ▷ *symmetric* if $\forall x, y \in X$, $xRy \Leftrightarrow yRx$;
- ▷ *transitive* if $\forall x, y, z \in X$, $(xRy \wedge yRz) \Rightarrow xRz$;
- ▷ *antisymmetric if* $\forall x, y \in X$, $(xRy \wedge yRx) \Rightarrow x = y$ (i.e., rules out ties);
- ▷ *complete* if $\forall x, y \in X$, either xRy or yRx (i.e., every pair is ordered).[1]

A binary relation $\geq$ on X is:

- ▷ a *partial order* if it is reflexive, transitive and antisymmetric, and $(X, \geq)$ is a *partially ordered set* (*poset*);
- ▷ a *total order* if it is complete, transitive and antisymmetric and $(X, \geq)$ is a *totally ordered set;*[2]

Remark 3. Totally ordered set is a special case of partially ordered sets. Based on a partially ordered set $(X, \geq)$, we may define

$$\begin{aligned}
&\leq := \geq^{-1} \\
&= := \leq \cap \geq \\
&> := \left\{(x, y) \in X^2 : (x \geq y) \text{ and } \neg (y \geq x)\right\}, \\
&< := >^{-1}.
\end{aligned}$$

If $\geq$ is a binary relation on X, then $\geq$ is a binary relation on any $S \subseteq X$.

Example 2. $(\mathbb{R}, \geq)$ is a total order defined as:

$$\geq := \left\{(x, y) \in \mathbb{R}^2 : y - x \text{ is nonnegative}\right\}.$$

However, $(\mathbb{R}^d, \geq)$ (with $n \in \mathbb{N}\backslash\{1\}$) is a partial order defined as

$$\geq := \left\{\left((x_i)_{i=1}^d, (y_i)_{i=1}^d\right) \in \mathbb{R}^d \times \mathbb{R}^d : y_i - x_i \text{ is nonnegative } \forall i \in \{1, \ldots, d\}\right\}.$$

[1] Note that completeness implies reflexiveness.

[2] Some people refer to total order as linear order.

Observe that above is not complete (e.g., with $d = 2$, $(1, 2)$ and $(2, 1)$ are not ordered). Given a set X, $(2^X, \subseteq)$ is a partial order.

Remark 4. Given any poset $(X, \geq)$, we can define upper and lower bounds in the usual way; e.g., $u \in X$ is an upper bound of $S \subseteq X$ is $u \geq s$ for all $s \in S$. We can also define the supremum and the infimum as the least upper bound and the greatest upper bound respectively; e.g., $\sup S \in X$ is the least upper bound if (i) $\sup S$ is an upper bound of S and (ii) $\sup S \leq u$ for any upper bound u of S.

Exercise 3. Suppose $(X, \geq)$ is a poset. Show that if $S \subseteq X$ has a maximum or a minimum, then it is unique.

Example 3. Suppose $X \coloneqq \mathbb{R} \backslash \{0\}$ and $S \coloneqq \{x \in \mathbb{R} : x < 0\}$. Then, $S \subseteq X$, $(X, \leq)$ is a partially ordered set, and, for example, $1 \in X$ is an upper bound of S. However, there is no least upper bound.

Say that a partially ordered set $(X, \leq)$ has the *least upper bound (* resp. *greatest lower bound) property* if any nonempty subset of X bounded from above (resp. below) has a least upper (resp. greater lower) bound.

3.2 Lattices

Definition 2. Let $(X, \geq)$ be a partially ordered set (poset). Given any $x, y \in X$, the *join* of x and y is

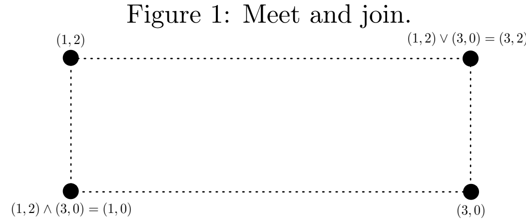
$$x \vee y \coloneqq \sup \{x, y\}$$

and the *meet* of x and y is

$$x \wedge y \coloneqq \inf \{x, y\}.$$

Example 4. Suppose $X = \mathbb{R}^2$ and let $x = (1, 2)$ and $y = (3, 0)$. Then, $x \vee y = (3, 2)$ and $x \wedge y = (1, 0)$. The four points x, y, $x \vee y$ and $x \wedge y$ forms a rectangle in $\mathbb{R}^2$.

Figure 1: Meet and join.



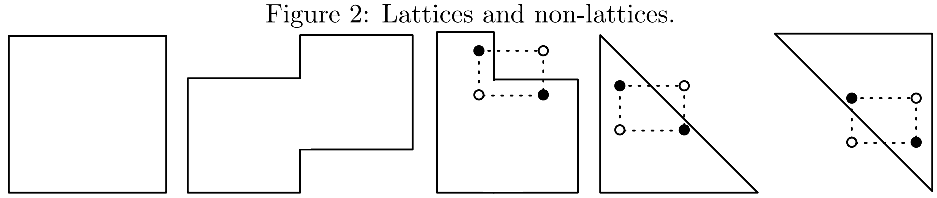
Definition 3. A poset set $(X, \geq)$ is...

- ▷ a *lattice* if join and meet of any two elements of X are contained in X; i.e., $x \vee y \in X$ and $x \wedge y \in X$ for all $x, y \in X$.
- ▷ a *complete lattice* if every nonempty subset $S \subseteq X$ has an infimum and a supremum in X; i.e., $\sup S \in X$ and $\inf S \in X$ for all nonempty $S \subseteq X$.

A subset $S \subseteq X$ is...

- ▷ a *sublattice of* $X \subseteq \mathbb{R}^d$ if the join and meet of any two element are contained in S (not just in X); i.e., $x \vee y \in S$ and $x \wedge y \in S$ for all $x, y \in S$.
- ▷ a *subcomplete sublattice* if it contains the supremum and the infimum of every subset; i.e., $\sup T \in S$ and $\inf T \in S$ for all $T \subseteq S$.

Figure 2: Lattices and non-lattices.



Remark 5. By the completeness axiom, $(\mathbb{R}, \geq)$ is a complete lattice. For any given set X, the poset $(2^X, \subseteq)$ is a lattice with

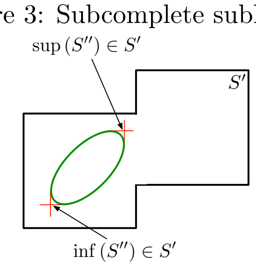
$$A \vee B = A \cup B,$$
$$A \wedge B = A \cap B$$

for any $A, B \in 2^X$.

Example 5. Define $S_1 \coloneqq \{(1,1),(1,2),(2,1),(2,2)\}$ and $S_2 \coloneqq \{(0,0),(2,1),(1,2),(3,3)\}$. Then, S_1 is the corner of a rectangle and so it is sublattice of $X = \mathbb{R}^2$. The set S_1 is also a lattice. In contrast, S_2 is not a sublattice of $\mathbb{R}^2$ since $(1,2) \vee (2,1) = (2,2) \notin S_2$ but it is a lattice (check!).

Example 6 (Subcomplete sublattice)**.** Consider $S_1 \coloneqq (0,1) \times (0,1)$ is a sublattice of poset $(\mathbb{R}^2, \geq)$. However, since $\sup S_1 = (1,1) \notin S_1$, it is not a subcomplete sublattice. On the other hand, $S_2 \coloneqq [0,1] \times [0,1]$ is a subcomplete lattice.

Figure 3: Subcomplete sublattice.



Exercise 4. Let $(X, \geq)$ be a lattice. Prove the following: (i) $x \vee y = x$ if and only if $x \geq y$; (ii) $x \wedge y = x$ if and only if $x \leq y$; (iii) $\neg(x \geq y) \Leftrightarrow x \not\geq y$ implies $x \vee y > x$; (iv) $\neg(x \leq y) \Leftrightarrow x \not\leq y$ implies $x \wedge y < x$.

Proposition 1. *Let $(X, \geq)$ be a sublattice of $(\mathbb{R}^d, \geq)$. If X is compact, then $(X, \geq)$ is a subcomplete sublattice.*

Proof. Let $\pi_i : \mathbb{R}^d \to \mathbb{R}$ be the projection onto the ith coordinate. Let X be a sublattice (of $\mathbb{R}^d$) and suppose X is compact. Fix $S \subseteq X$ and let

$$\mathbf{x}^i \in \arg\max_{\mathbf{x} \in \mathrm{cl}(S)} \pi_i(\mathbf{x}),$$

(how do we know $\mathbf{x}^i$ exists?). Let

$$\mathbf{x}^* = \mathbf{x}^1 \vee \mathbf{x}^2 \vee \cdots \vee \mathbf{x}^d = \left(x_i^i\right)_{i=1}^d.$$

Observe that, $\mathbf{x}^i \in X$ and so $\mathbf{x}^* \in X$ since X is a sublattice. We now show that $\mathbf{x}^* = \sup S$. First, $\mathbf{x}^*$ is an upper bound on S because if $\mathbf{z} \in S$, then $z_i \leq x_i^i = x_i^*$ for all $i \in \{1, \dots, d\}$. So $\mathbf{z} \leq \mathbf{x}^*$. Second, $\mathbf{x}^*$ is the least upper bound because if $\mathbf{z}$ is an upper bound of S, then $\mathbf{z}$ is also an upper bound of $\mathrm{cl}(S)$ (why?). So $\mathbf{x}^i \in \mathrm{cl}(S)$ means that $x_i^i \leq z_i$ for all $i \in \{1, \dots, d\}$; i.e., $\mathbf{x}^* \leq \mathbf{z}$. Hence, we conclude that X is a subcomplete sublattice. ■

Remark 6. The converse is also true so that, in fact, a poset $(X \subseteq \mathbb{R}^d, \geq)$ is a subcomplete sublattice if and only if X is compact. For this reason, we say that a sublattice $X \subseteq \mathbb{R}^d$ *is* a *compact sublattice* if X is also compact under the Euclidean metric. For example, $[0,1]^d$ is a compact sublattice of $(\mathbb{R}^d, \geq)$.

Corollary 1. *Suppose $(X, \geq)$ is a compact sublattice of $\mathbb{R}^d$. Then, X has a greatest and a least element; i.e.,*

$$\sup X \in X \text{ and } \inf X \in X.$$

Proof. That $(X, \geq)$ is a compact sublattice of $\mathbb{R}^d$ means that $(X, \geq)$ is a subcomplete sublattice and, as such, $\sup T, \inf T \in X$ for all $T \subseteq X$. In particular, $\sup X, \inf X \in X$. ■

Remark 7. Recall that if $X \subseteq \mathbb{R}$ and X is compact, then $\sup X \in X$ and $\inf X \in X$. However, this is not always true in $\mathbb{R}^d$. Hence, we need the subcomplete sublattice property on X to ensure the result above.

Exercise 5. Give an example of $X \subseteq \mathbb{R}^2$ such that X is compact but $\sup X$ (or $\inf X$) is not contained in X.

3.3 Supermodularity and increasing differences

Definition 4. Let Z be a sublattice of $\mathbb{R}^d$ (omitting the order $\geq$ for brevity). A function $f : Z \to \mathbb{R}$ is *supermodular* (on Z) if

$$f(\mathbf{z}) + f(\mathbf{z}') \leq f(\mathbf{z} \vee \mathbf{z}') + f(\mathbf{z} \wedge \mathbf{z}') \ \forall \mathbf{z}, \mathbf{z}' \in Z.$$

If the inequality holds strictly for any non-ordered $\mathbf{z}, \mathbf{z}' \in Z$, then f is *strictly supermodular*. A function f is *submodular* if $-f$ is supermodular.

Remark 8. If $\mathbf{z}$ and $\mathbf{z}'$ are ordered, then the inequality above holds with equality. A univariate function, $f : Z \subseteq \mathbb{R} \to \mathbb{R}$ is necessarily supermodular.

Definition 5. Suppose that X and Θ are sublattices of $\mathbb{R}^d$ and $\mathbb{R}^m$ respectively. A function $f : X \times \Theta \to \mathbb{R}$ has *increasing differences in* $(\mathbf{x}, \boldsymbol{\theta})$ if

$$f\left(\mathbf{x}', \boldsymbol{\theta}'\right) - f\left(\mathbf{x}, \boldsymbol{\theta}'\right) \geq f\left(\mathbf{x}', \boldsymbol{\theta}\right) - f\left(\mathbf{x}, \boldsymbol{\theta}\right) \ \forall \left(\mathbf{x}', \boldsymbol{\theta}'\right) \geq \left(\mathbf{x}, \boldsymbol{\theta}\right).$$

f has strictly increasing differences in $(\mathbf{x}, \boldsymbol{\theta})$ if we can replace the weak inequalities with strict inequalities in the expression above.

Remark 9. Equivalently, f has (resp. strictly) increasing differences in $(\mathbf{x}, \boldsymbol{\theta})$ if, for any $\mathbf{x}' \geq \mathbf{x}$, the function $g : \Theta \to \mathbb{R}$ defined as

$$g\left(\boldsymbol{\theta}\right) \coloneqq f\left(\mathbf{x}', \boldsymbol{\theta}\right) - f\left(\mathbf{x}, \boldsymbol{\theta}\right)$$

is (resp. strictly) increasing in $\boldsymbol{\theta}$.

Let us interpret these. Suppose that f is the utility function for a player when choosing an "action" $\mathbf{x}$ when the state is $\boldsymbol{\theta}$. Then, $g(\boldsymbol{\theta})$ is the additional benefit that the player gets from choosing the "higher" action $\mathbf{x}$ over the "lower" action $\mathbf{x}'$; i.e., it is the marginal benefit from choosing the higher action. Thus, increasing differences is the condition that the marginal benefit from choosing the higher action increases in the state $\boldsymbol{\theta}$. Notice that we can account for the case in which the marginal benefit is decreasing in $\boldsymbol{\theta}$ by ensuring that $g(-\boldsymbol{\theta})$ is increasing in $-\boldsymbol{\theta}$.

Proposition 2. *Suppose that* $(X, \geq)$ *and* $(\Theta, \geq)$ *are sublattices of* $\mathbb{R}^d$ *and* $\mathbb{R}^m$*, respectively, and that* $f : X \times \Theta \to \mathbb{R}$ *is supermodular. Then,*

(i) f *is supermodular in* $\mathbf{x}$ *for each* $\boldsymbol{\theta} \in \Theta$*; i.e., for all* $\boldsymbol{\theta} \in \Theta$,

$$f\left(\mathbf{x}, \boldsymbol{\theta}\right) + f\left(\mathbf{x}', \boldsymbol{\theta}\right) \leq f\left(\mathbf{x} \vee \mathbf{x}', \boldsymbol{\theta}\right) + f\left(\mathbf{x} \wedge \mathbf{x}', \boldsymbol{\theta}\right) \ \forall \mathbf{x}, \mathbf{x}' \in X.$$

(ii) f *satisfies increasing differences in* $(\mathbf{x}, \boldsymbol{\theta})$.

Proof. (i) follows from the definition of supermodularity and letting $\mathbf{z} \coloneqq (\mathbf{x}, \boldsymbol{\theta}) \in X \times \Theta$ and $\mathbf{z}' \coloneqq (\mathbf{x}', \boldsymbol{\theta}) \in X \times \Theta$. (ii) Fix any $\mathbf{z}' \coloneqq (\mathbf{x}', \boldsymbol{\theta}') \geq (\mathbf{x}, \boldsymbol{\theta}) \eqqcolon \mathbf{z}$, $\mathbf{z}, \mathbf{z}' \in X \times \Theta$. Let $\mathbf{w} \coloneqq (\mathbf{x}, \boldsymbol{\theta}')$ and $\mathbf{w}' \coloneqq (\mathbf{x}', \boldsymbol{\theta})$. Then, $\mathbf{w} \vee \mathbf{w}' = (\mathbf{x}', \boldsymbol{\theta}') = \mathbf{z}'$ and $\mathbf{w} \wedge \mathbf{w}' = (\mathbf{x}, \boldsymbol{\theta}) = \mathbf{z}$. By supermodularity of f,

$$f\left(\mathbf{w}\right) + f\left(\mathbf{w}'\right) \leq f\left(\mathbf{w} \vee \mathbf{w}'\right) + f\left(\mathbf{w} \wedge \mathbf{w}'\right) = f\left(\mathbf{z}'\right) + f\left(\mathbf{z}\right),$$

which implies

$$f\left(\mathbf{x}', \boldsymbol{\theta}'\right) - f\left(\mathbf{x}, \boldsymbol{\theta}'\right) \geq f\left(\mathbf{x}', \boldsymbol{\theta}\right) - f\left(\mathbf{x}, \boldsymbol{\theta}\right);$$

so that f satisfies increasing differences in $(\mathbf{x}, \boldsymbol{\theta})$. ■

The result below gives a way of checking if f is supermodular when f is $\mathbf{C}^2$.

Proposition 3. *Suppose* Z *is a sublattice of* $\mathbb{R}^d$ *and* $f : Z \to \mathbb{R}$ *is* $\mathbf{C}^2$*. Then,* f *is supermodular on* $\operatorname{int}(Z)$ *if and only if*

$$\frac{\partial^2 f}{\partial z_i \partial z_j}\left(\mathbf{z}\right) \geq 0 \ \forall i, j \in \{1, \ldots, d\} : i \neq j.$$

Proof. We prove an interim result first. Suppose $Z \subseteq \mathbb{R}^d$ and for any $\mathbf{z} \in Z$, let $(z_i', z_j'; \mathbf{z}_{-ij})$ denote the vector $\mathbf{z}$ with z_i and z_j replaced by z_i' and z_j' respectively. Say that a function $f : Z \to \mathbb{R}$

satisfies *increasing differences on* Z if, for all $\mathbf{z} \in Z$, for all distinct $i, j \in \{1, \ldots, d\}$ and for all z_i' and z_j' such that $z_i' \geq z_i$, $z_j' \geq z_j$ and $(z_i', z_j'; \mathbf{z}_{-ij}) \in Z$, we have

$$f\left(z_i', z_j'; \mathbf{z}_{-ij}\right) - f\left(z_i', z_j; \mathbf{z}_{-ij}\right) \geq f\left(z_i, z_j'; \mathbf{z}_{-ij}\right) - f\left(z_i, z_j; \mathbf{z}_{-ij}\right).$$

In words, f has increasing differences on Z if it has increasing differences in each pair (z_i, z_j) holding all other coordinates fixed.

Lemma 1. *A function $f : Z \subseteq \mathbb{R}^d \to \mathbb{R}$ is* supermodular *on Z if and only if f has increasing differences on Z.*

Proof. Suppose that f is supermodular and fix $\mathbf{z} \in Z$, distinct $i, j \in \{1, \ldots, d\}$ and z_i' and z_j' such that $z_i' \geq z_i$, $z_j' \geq z_j$ and $(z_i', z_j'; \mathbf{z}_{-ij}) \in Z$. Let $\mathbf{w} := (z_i', z_j; \mathbf{z}_{-ij})$ and $\mathbf{w}' := (z_i, z_j'; \mathbf{z}_{-ij})$ and observe that

$$\mathbf{w} \vee \mathbf{w}' = \left(z_i', z_j'; \mathbf{z}_{-ij}\right), \; \mathbf{w} \wedge \mathbf{w}' = \mathbf{z} = \left(z_i, z_j; \mathbf{z}_{-ij}\right)$$

and so by supermodularity,

$$\begin{aligned} & f(\mathbf{w}) + f(\mathbf{w}') \leq f(\mathbf{w} \vee \mathbf{w}') + f(\mathbf{w} \wedge \mathbf{w}') \\ \Leftrightarrow \; & f\left(z_i', z_j; \mathbf{z}_{-ij}\right) + f\left(z_i, z_j'; \mathbf{z}_{-ij}\right) \leq f\left(z_i', z_j'; \mathbf{z}_{-ij}\right) + f\left(z_i, z_j; \mathbf{z}_{-ij}\right). \end{aligned}$$

Hence, f has increasing differences on Z.

Conversely, suppose that f has increasing differences on Z. Fix any $\mathbf{z}, \mathbf{z}' \in Z$. We want to show that

$$f(\mathbf{z}) + f(\mathbf{z}') \leq f(\mathbf{z} \vee \mathbf{z}') + f(\mathbf{z} \wedge \mathbf{z}').$$

If $\mathbf{z} \geq \mathbf{z}'$ or $\mathbf{z} \leq \mathbf{z}'$, the inequality holds with equality. So suppose that $\mathbf{z}$ and $\mathbf{z}'$ are not ordered. Rearrange the coordinates such that

$$\begin{aligned} \mathbf{z} \vee \mathbf{z}' &= \left(z_1', \ldots, z_k', z_{k+1}, \ldots, z_d\right), \\ \mathbf{z} \wedge \mathbf{z}' &= \left(z_1, \ldots, z_k, z_{k+1}', \ldots, z_d'\right). \end{aligned}$$

That $\mathbf{z}$ and $\mathbf{z}'$ are not ordered means we must have $0 < k < m$. Now, for $0 \leq i \leq j \leq m$, define

$$\mathbf{z}^{i,j} := \left(z_1', \ldots, z_i', z_{i+1}, \ldots, z_j, z_{j+1}', \ldots, z_m'\right).$$

Then, $\mathbf{z}^{0,k} = \mathbf{z} \wedge \mathbf{z}'$, $\mathbf{z}^{k,m} = \mathbf{z} \vee \mathbf{z}'$, $\mathbf{z}^{0,m} = \mathbf{z}$ and $\mathbf{z}^{k,k} = \mathbf{z}'$. Since f has increasing differences on $\mathbf{Z}$, for all $0 \leq i < k \leq j < m$,

$$f\left(\mathbf{z}^{i+1,j+1}\right) - f\left(\mathbf{z}^{i,j+1}\right) \geq f\left(\mathbf{z}^{i+1,j}\right) - f\left(\mathbf{z}^{i,j}\right).$$

Therefore, for $k \leq j < m$,

$$\begin{aligned} f\left(\mathbf{z}^{k,j+1}\right) - f\left(\mathbf{z}^{0,j+1}\right) &= \sum_{i=0}^{k-1}\left[f\left(\mathbf{z}^{i+1,j+1}\right) - f\left(\mathbf{z}^{i,j+1}\right)\right] \\ &\geq \sum_{i=0}^{k-1}\left[f\left(\mathbf{z}^{i+1,j}\right) - f\left(\mathbf{z}^{i,j}\right)\right] \\ &= f\left(\mathbf{z}^{k,j}\right) - f\left(\mathbf{z}^{0,j}\right). \end{aligned}$$

Observe that the left-hand side is greatest when $j = m-1$, while the right-hand side is smallest when $j = k$. Therefore,

$$f\left(\mathbf{z}^{k,m}\right) - f\left(\mathbf{z}^{0,m}\right) \geq f\left(\mathbf{z}^{k,k}\right) - f\left(\mathbf{z}^{0,k}\right),$$

which is, in fact, what we wanted to show. ■

By the lemma above, f is supermodular if and only if, for all $\mathbf{z} \in Z$, for all distinct $i, j \in \{1, \dots, d\}$, and for all $\epsilon > 0$ and $\delta > 0$, we have

$$f\left(z_i + \epsilon, z_j + \delta; \mathbf{z}_{-ij}\right) - f\left(z_i + \epsilon, z_j; \mathbf{z}_{-ij}\right) \geq f\left(z_i, z_j + \delta; \mathbf{z}_{-ij}\right) - f\left(z_i, z_j; \mathbf{z}_{-ij}\right).$$

Dividing both sides by δ and letting $\delta \searrow 0$, we realise that f is supermodular on Z if and only if, for all $\mathbf{z} \in Z$, for all distinct $i, j \in \{1, \dots, d\}$, and for all $\epsilon > 0$,

$$\frac{\partial f}{\partial z_j}\left(z_i + \epsilon, z_j; \mathbf{z}_{-ij}\right) \geq \frac{\partial f}{\partial z_j}\left(z_i, z_j; \mathbf{z}_{-ij}\right).$$

Subtracting the right-hand side from the left-hand side, dividing both sides by ϵ, and letting $\epsilon \searrow 0$ gives that f is supermodular on Z if and only if, for all $\mathbf{z} \in Z$, for all distinct $i, j \in \{1, \dots, d\}$, $\frac{\partial^2 f}{\partial z_j \partial z_i}(\mathbf{z}) \geq 0$, as we wanted. ■

Letting $Z \coloneqq X \times \Theta = \mathbb{R} \times \mathbb{R}$, this tells us that $f : \mathbb{R}^2 \to \mathbb{R}$ is supermodular if

$$\frac{\partial^2 f}{\partial x \partial \theta}(x, \theta) \equiv \frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial x}(x, \theta)\right) \geq 0.$$

Thinking of f as the utility function, the condition tells us that the marginal utility from x is increasing in the parameter θ. Alternatively, if we think of f as production and x as labour and θ as capital input, the condition above is that the two factors of productions are complementary. Rewriting using "standard" notation and using Young's Theorem , the above condition becomes

$$\frac{\partial}{\partial K}\left(\frac{\partial F}{\partial L}(K, L)\right) = \frac{\partial}{\partial L}\left(\frac{\partial F}{\partial K}(K, L)\right) \geq 0.$$

Thus, supermodularity can be thought as expressing the idea of complementaries.

Exercise 6 (PS11)**.** Suppose that X and Θ are open sublattices of $\mathbb{R}^d$ and $\mathbb{R}^m$ respectively. Prove that $f : X \times \Theta \to \mathbb{R}$ that is $\mathbf{C}^2$ has increasing differences in $(\mathbf{x}, \boldsymbol{\theta}) \in X \times \Theta$ if

$$\frac{\partial^2 f}{\partial x_i \partial \theta_j}(\mathbf{x}, \boldsymbol{\theta}) \geq 0 \ \forall (i, j) \in \{1, \dots, d\} \times \{1, \dots, m\}.$$

$$f\left(x_i', \theta_j'; \mathbf{x}_{-i}, \boldsymbol{\theta}_{-j}\right) - f\left(x_i, \theta_j'; \mathbf{x}_{-i}, \boldsymbol{\theta}_{-j}\right) \geq f\left(x_i, \theta_j'; \mathbf{x}_{-i}, \boldsymbol{\theta}_{-j}\right) - f\left(x_i, \theta_j; \mathbf{x}_{-i}, \boldsymbol{\theta}_{-j}\right).$$

Then, we can mimic the proof in Proposition 3 to obtain the result.

Let us now state the reason why we are interested in supermodularity and increasing differences.

Theorem 2 (Supermodular Monotone Comparative Statics Theorem)**.** *Let X be a compact sublattice of $\mathbb{R}^d$, Θ be a sublattice of $\mathbb{R}^m$, and $f : X \times \Theta \to \mathbb{R}$ be a continuous function on X for each $\theta \in \Theta$. Suppose that f satisfies increasing differences in $(\mathbf{x}, \boldsymbol{\theta})$ and is supermodular in $\mathbf{x}$ for each $\boldsymbol{\theta} \in \Theta$. Define $X^* : \Theta \to \mathbb{R}$ by*

$$X^*(\boldsymbol{\theta}) \coloneqq \underset{\mathbf{x} \in X}{\arg\max}\ f(\mathbf{x}, \boldsymbol{\theta}).$$

Then, for each $\boldsymbol{\theta} \in \Theta$, $X^*(\theta)$ is a nonempty compact sublattice of $\mathbb{R}^d$ and contains a greatest element, denoted $x^*(\boldsymbol{\theta})$, that is increasing in $\boldsymbol{\theta}$; i.e. for any $\boldsymbol{\theta}, \boldsymbol{\theta}' \in \Theta$ such that $\boldsymbol{\theta}' \geq \boldsymbol{\theta}$,

$$x^*\left(\boldsymbol{\theta}'\right) \geq x^*(\boldsymbol{\theta}).$$

If f further satisfies strictly increasing differences in $(\mathbf{x}, \boldsymbol{\theta})$, then, for any $\boldsymbol{\theta}, \boldsymbol{\theta}' \in \Theta$ such that $\boldsymbol{\theta}' \geq \boldsymbol{\theta}$,

$$\mathbf{x}' \geq \mathbf{x}$$

for any $\mathbf{x} \in X^*(\boldsymbol{\theta})$ and any $\mathbf{x}' \in X^*(\boldsymbol{\theta}')$.

Proof. Fix $\boldsymbol{\theta} \in \Theta$. That $X^*(\boldsymbol{\theta})$ is nonempty and is compact follows from the theorem of the maximum (check that you understand this). Take any distinct $\mathbf{x}, \mathbf{x}' \in X^*(\boldsymbol{\theta})$. If $\mathbf{x} \wedge \mathbf{x}' \notin X^*(\boldsymbol{\theta})$, we must have

$$f(\mathbf{x} \wedge \mathbf{x}', \boldsymbol{\theta}) < f(\mathbf{x}, \boldsymbol{\theta}) = f(\mathbf{x}', \boldsymbol{\theta}).$$

Supermodularity in $\mathbf{x}$ then implies

$$f(\mathbf{x} \vee \mathbf{x}', \boldsymbol{\theta}) > f(\mathbf{x}, \boldsymbol{\theta}) = f(\mathbf{x}', \boldsymbol{\theta}),$$

which contradicts the optimality of $\mathbf{x}$ and $\mathbf{x}'$. Similar argument establishes that $\mathbf{x} \vee \mathbf{x}' \in X^*(\boldsymbol{\theta})$. Thus, $X^*(\boldsymbol{\theta})$ is a sublattice of $\mathbb{R}^d$ and, as a nonempty, compact sublattice of $\mathbb{R}^d$, admits a greatest element $x^*(\boldsymbol{\theta})$.

Now suppose $\boldsymbol{\theta}' > \boldsymbol{\theta}$ and take $\mathbf{x} \in X^*(\boldsymbol{\theta})$ and $\mathbf{x}' \in X^*(\boldsymbol{\theta}')$. Then,

$$\begin{aligned} 0 &\leq f(\mathbf{x}, \boldsymbol{\theta}) - f(\mathbf{x} \wedge \mathbf{x}', \boldsymbol{\theta}) \\ &\leq f(\mathbf{x} \vee \mathbf{x}', \boldsymbol{\theta}) - f(\mathbf{x}', \boldsymbol{\theta}) \\ &\leq f\left(\mathbf{x} \vee \mathbf{x}', \boldsymbol{\theta}'\right) - f\left(\mathbf{x}', \boldsymbol{\theta}'\right) \\ &\leq 0 \end{aligned}$$

(check that you know why each line is true). Hence, above expressions must hold with equality. Now suppose $\mathbf{x} = x^*(\boldsymbol{\theta})$ and $\mathbf{x}' = x^*(\boldsymbol{\theta}')$. Since above expression all hold with equality, $\mathbf{x} \vee \mathbf{x}'$ must also be optimal at $\boldsymbol{\theta}'$. If it were not true that $\mathbf{x}' \geq \mathbf{x}$, then we would have $\mathbf{x} \vee \mathbf{x}' > \mathbf{x}'$ and this contradicts $\mathbf{x}'$ as the greatest element of $X^*(\boldsymbol{\theta}')$. Thus, we must have $\mathbf{x}' \geq \mathbf{x}$.

Optimality of x' at θ'; supermodularity in x; increasing differences in (x, θ); optimality of x at θ.

Finally, take any $\mathbf{x} \in X^*(\boldsymbol{\theta})$ and $\mathbf{x}' \in X^*(\boldsymbol{\theta}')$. If we did not have $\mathbf{x}' \geq \mathbf{x}$, then we must have $\mathbf{x} \vee \mathbf{x}' > \mathbf{x}'$ and $\mathbf{x} \wedge \mathbf{x}' < \mathbf{x}$. If f satisfies strictly increasing differences, then, since $\boldsymbol{\theta}' > \boldsymbol{\theta}$,

$$f\left(\mathbf{x}, \boldsymbol{\theta}'\right) - f\left(\mathbf{x} \wedge \mathbf{x}', \boldsymbol{\theta}'\right) > f\left(\mathbf{x}, \boldsymbol{\theta}\right) - f\left(\mathbf{x} \wedge \mathbf{x}', \boldsymbol{\theta}\right)$$

so that third inequality in the string of inequality above becomes strict—a contradiction. ■

3.4 Quasi-supermodularity and single-crossing differences

Definition 6. Let $X \subseteq \mathbb{R}^d$ be a sublattice of $(\mathbb{R}^d, \geq)$. A function $f : Z \to \mathbb{R}$ is *quasi-supermodular* if

$$f\left(\mathbf{z} \vee \mathbf{z}'\right) \geq f\left(\mathbf{z}'\right) \ \forall \mathbf{z}, \mathbf{z}' \in Z : f\left(\mathbf{z}\right) \geq f\left(\mathbf{z} \wedge \mathbf{z}'\right)$$

and

$$f\left(\mathbf{z} \vee \mathbf{z}'\right) > f\left(\mathbf{z}'\right) \ \forall \mathbf{z}, \mathbf{z}' \in Z : f\left(\mathbf{z}\right) > f\left(\mathbf{z} \wedge \mathbf{z}'\right).$$

Remark 10. As the name suggests, quasi-supermodularity is a weaker property than supermodularity. The latter property is that

$$f\left(\mathbf{z} \vee \mathbf{z}'\right) - f\left(\mathbf{z}'\right) \geq f\left(\mathbf{z}\right) - f\left(\mathbf{z} \wedge \mathbf{z}'\right)$$

Thus, supermodularity says makes a comparison between these two differences. However, quasi-supermodularity is the requirement that

$$f\left(\mathbf{z}\right) - f\left(\mathbf{z} \wedge \mathbf{z}'\right) \geq 0 \Rightarrow f\left(\mathbf{z} \vee \mathbf{z}'\right) - f\left(\mathbf{z}'\right) \geq 0.$$

Thus, it has nothing to say about the relative size between the two except that if the right-hand side is positive, then so must the left-hand side. Observe also that the implication above is unchanged by applying a strictly positive transformation; i.e., quasi-supermodularity is an ordinal property.

Proposition 4. *Let $(X, \geq)$ be a lattice and $f : X \to \mathbb{R}$ be quasi-supermodular. Then,*

$$X^* := \underset{\mathbf{x} \in X}{\arg\max} \, f\left(\mathbf{x}\right) \tag{1}$$

is a sublattice of X.

Proof. Suppose that $\mathbf{x}, \mathbf{x}' \in X^*$. We wish to show that $\mathbf{x} \vee \mathbf{x}' \in X^*$ and $\mathbf{x} \wedge \mathbf{x}' \in X^*$. Since $\mathbf{x}$ is a maximiser, $f(\mathbf{x}) \geq f(\mathbf{x} \wedge \mathbf{x}')$ so that, by quasi-supermodularity, we must have $f(\mathbf{x} \vee \mathbf{x}') \geq f(\mathbf{x}')$. Since $\mathbf{x}'$ is also a maximiser, we must also have $f(\mathbf{x}) = f(\mathbf{x}') \geq f(\mathbf{x} \vee \mathbf{x}')$. That is,

$$f\left(\mathbf{x}'\right) \geq f\left(\mathbf{x} \vee \mathbf{x}'\right) \geq f\left(\mathbf{x}'\right) \Leftrightarrow f\left(\mathbf{x} \vee \mathbf{x}'\right) = f\left(\mathbf{x}'\right) \Rightarrow \mathbf{x} \vee \mathbf{x}' \in X^*.$$

Towards a contradiction, suppose that $\mathbf{x} \wedge \mathbf{x}' \notin X^*$; i.e., $f(\mathbf{x}) > f(\mathbf{x} \wedge \mathbf{x}')$, then by quasi-supermodularity, we have $f(\mathbf{x} \vee \mathbf{x}') > f(\mathbf{x}')$, which contradicts that $\mathbf{x}' \in X^*$. Thus, $\mathbf{x} \wedge \mathbf{x}' \in X^*$. ■

Combined with Weierstrass theorem, the result helps us to establish that with the lattice structure in place, we can think of such a thing as the *largest* and the *smallest* maximisers.

Corollary 2. *Suppose $X \subseteq \mathbb{R}^n$ is a compact sublattice of $(\mathbb{R}^n, \geq)$ and $f : X \to \mathbb{R}$ is continuous and quasi-supermodular. Then, X^* as defined in (1) is a nonempty compact sublattice and, in particular,*

$$\sup X^*, \inf X^* \in X.$$

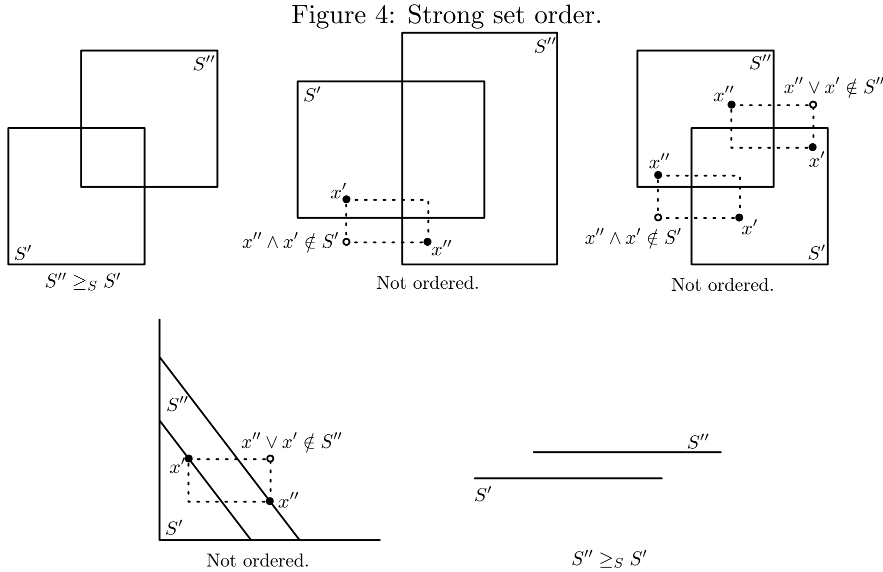
Proof. Since X is compact and f is continuous, by the Weierstrass Extreme Value Theorem, a solution exists so that $X^* \neq \varnothing$. By Proposition 4, X^* is a sublattice. Moreover, by Berge's theorem of maximum, X^* is compact. Thus, X^* is a compact sublattice. Then by Proposition 1, X^* contains its supremum and infimum. ■

Let us now introduce an order that allows us to compare subsets of a lattice.

Definition 7. Suppose $(X, \geq)$ is a lattice and let $S, S' \subseteq X$. The *strong set order*, $\geq_S$, is a binary relation $\geq_S$: $X \times X \to X$ defined by

$$S' \geq_S S \Leftrightarrow s \vee s' \in S',\ s \wedge s' \in S\ \forall s \in S\ \forall s' \in S'.$$

Figure 4: Strong set order.



Proposition 5. *Let $(X, \geq)$ be a lattice and suppose $S, S' \subseteq X$ such that $S \geq_S S'$.*

(i) for all $s \in S$, there exists $s' \in S'$ such that $s' \geq s$;

(ii) for all $s' \in S'$, there exists $s \in S$ such that $s' \geq s$.

Proof. (i) Let $z' \in S'$ and $s \in S$. Since $S' \geq_S S$, $z' \vee s \in S'$ so set $s' := z' \vee s$. (ii) Let $z \in S$ and $s' \in S'$. Since $S' \geq_S S$, $z \wedge s' \in S$ so set $s := z \wedge s'$. ■

That is, the strong set order implies ordering of the largest and the smallest elements in the sets.

Proposition 6. *Suppose S and S' are subcomplete sublattices in $\mathbb{R}^d$ such that $S' \geq_S S$ or $S' >_S S$. Then,*

$$\sup S' \geq \sup S \text{ and } \inf S' \geq \inf S.$$

Proof. By definition of a subcomplete sublattice, $\sup S, \inf S \in S$ and $\sup S', \inf S' \in S'$. By Proposition 5,

$$\sup S \in S \Rightarrow \exists \mathbf{s}' \in S',\ \mathbf{s}' \geq \sup S,$$
$$\inf S' \in S' \Rightarrow \exists \mathbf{s} \in S,\ \inf S \geq \mathbf{s}'.$$

Then, by definition of supremum and infimum $\sup S' \geq \mathbf{s}' \geq \sup S$ and $\inf S \geq \mathbf{s}' \geq \inf S'$. ■

Next, we show that if constraint sets are ordered with respect to strong set order, then so is the set of maximisers.

Theorem 3. *Let $(X, \geq)$ be a lattice and $f : X \to \mathbb{R}$ be quasi-supermodular. Suppose $\Gamma, \Gamma' \subseteq X$ such that $\Gamma' \geq_S \Gamma$, then*

$$X^*_{\Gamma'} \coloneqq \arg\max_{\mathbf{x} \in \Gamma'} f(\mathbf{x}) \geq_S \arg\max_{\mathbf{x} \in \Gamma} f(\mathbf{x}) =: X^*_{\Gamma} \tag{2}$$

Proof. Take any $\mathbf{x} \in X^*_{\Gamma}$ and $\mathbf{x}' \in X^*_{\Gamma'}$. We wish to show that $X^*_{\Gamma'} \geq_S X^*_{\Gamma}$; i.e., (i), $\mathbf{x} \vee \mathbf{x}' \in X^*_{\Gamma'}$; and (ii) $\mathbf{x} \wedge \mathbf{x}' \in X^*_{\Gamma}$. Note that since $\mathbf{x} \in X^*_{\Gamma} \subseteq \Gamma$ and $\mathbf{x}' \in X^*_{\Gamma'} \subseteq \Gamma'$, that $\Gamma' \geq_S \Gamma$ implies that $\mathbf{x} \vee \mathbf{x}' \in \Gamma'$ and $\mathbf{x} \wedge \mathbf{x}' \in \Gamma$.

(i) As $\mathbf{x}$ is a maximiser in Γ, we must have $f(\mathbf{x}) \geq f(\mathbf{x} \wedge \mathbf{x}')$. By quasi-supermodularity of f, this implies that $f(\mathbf{x} \vee \mathbf{x}') \geq f(\mathbf{x}')$. Since $\mathbf{x}'$ is a maximiser in Γ' and $\mathbf{x} \vee \mathbf{x}' \in \Gamma'$, we must have $f(\mathbf{x} \vee \mathbf{x}') = f(\mathbf{x}')$; i.e., $\mathbf{x} \vee \mathbf{x}' \in X^*_{\Gamma'}$.

(ii) By way of contradiction, suppose that $\mathbf{x} \wedge \mathbf{x}' \notin X^*_{\Gamma}$. Then, $f(\mathbf{x}) > f(\mathbf{x} \wedge \mathbf{x}')$ since $\mathbf{x}$ is a maximiser in Γ and $\mathbf{x} \wedge \mathbf{x}' \in \Gamma$. By quasi-supermodularity of f, this implies that $f(\mathbf{x} \vee \mathbf{x}') > f(\mathbf{x}')$. Since $\mathbf{x} \vee \mathbf{x}' \in \Gamma'$, this contradicts the fact that $\mathbf{x}'$ is a maximiser in Γ'. Thus, $\mathbf{x} \wedge \mathbf{x}' \in X^*_{\Gamma}$. ■

Corollary 3. *Let $(X, \geq)$ be a sublattice in $(\mathbb{R}^d, \geq)$ and $f : X \to \mathbb{R}$ be continuous and quasi-supermodular. Suppose $\Gamma, \Gamma' \subseteq X$ are compact sublattices of $(\mathbb{R}^d, \geq)$ with $\Gamma' \geq_S \Gamma$. Then, X^*_{Γ} and $X^*_{\Gamma'}$ as defined in (2) are nonempty compact sublattices containing their infimum and supremum with*

$$X^*_{\Gamma'} \geq_S X^*_{\Gamma},$$
$$\sup X^*_{\Gamma'} \geq \sup X^*_{\Gamma} \text{ and } \inf X^*_{\Gamma'} \geq \inf X^*_{\Gamma}.$$

Proof. That $X^*_{\Gamma'} \geq_S X^*_{\Gamma}$ follows from Theorem 3. The rest follows from Corollary 2 and Proposition 6. ■

Definition 8. Suppose that $(\Theta, \geq)$ is a partially ordered set. A function $g : \Theta \to \mathbb{R}$ has the *single-crossing property* if, for any $\theta' > \theta$, we have

$$g(\theta') \geq 0\ \forall g(\theta) \geq 0$$

and

$$g(\theta') > 0\ \forall g(\theta) > 0.$$

Remark 11. Any increasing function has the single-crossing property; however, the converse is false (example?).

Definition 9. Suppose $(X, \geq)$ and $(\Theta, \geq)$ are partially ordered sets. A function $f : X \times \Theta \to \mathbb{R}$ has *single-crossing differences in* (x, θ) if, for any $x, x' \in X$ such that $x' > x$, the function $g : \Theta \to \mathbb{R}$ defined as

$$g(\theta) \coloneqq f(x', \theta) - f(x, \theta)$$

has the single-crossing property.

Remark 12. Since increasing differences implies that g is increasing, it follows that a function with increasing differences has single-crossing differences.

Exercise 7 (PS11)**.** Suppose $(X, \geq)$ and $(\Theta, \geq)$ are partially ordered sets and that $f : X \times \Theta \to \mathbb{R}$ has single-crossing differences in (x, θ). Prove that single-crossing property is an ordinal property. Hint: Show that, for any $\phi : \mathbb{R} \times \Theta \to \mathbb{R}$ such that $\phi(\cdot, \theta)$ is strictly increasing for any $\theta \in \Theta$, the function $\tilde{f} : X \times \Theta \to \mathbb{R}$ defined by $\tilde{f}(x, \theta) \coloneqq \phi(f(x, \theta), \theta)$ also has single-crossing differences.

Theorem 4 (Milgrom and Shannon)**.** *Let $(X, \geq)$ be a lattice and $(\Theta, \geq)$ be a partially ordered set. Suppose $f : X \times \Theta \to \mathbb{R}$ and define $X^* : \Theta \times X \rightrightarrows X$ as*

$$X^*_\Gamma(\theta) \coloneqq \arg\max_{x \in \Gamma} f(x, \theta).$$

Then, $X^*_\Gamma(\theta)$ is monotone increasing (i.e., $X^*_\Gamma(\theta') \geq_S X^*_{\Gamma'}(\theta)$ for any $(\theta, \Gamma) \in \Theta \times X$ such that $\theta' \geq \theta$ and $\Gamma' \geq_S \Gamma$) if and only if (i) $f(\cdot, \theta)$ is quasi-supermodular in x for all $\theta \in \Theta$, and (ii) f has single-crossing differences in (x, θ).

Proof. Fix $\theta', \theta \in \Theta$ such that $\theta' \geq \theta$ and $\Gamma' \geq_S \Gamma$. Let $x \in X^*(\theta, \Gamma)$ and $x' \in X^*(\theta', \Gamma')$. Since $\Gamma' \geq_S \Gamma$, $x \wedge x' \in \Gamma$ and $x \vee x' \in \Gamma'$. We first show that $x \vee x' \in X^*(\theta', \Gamma')$. By optimality of x, $f(x \wedge x', \theta) \leq f(x, \theta)$. Because f has single-crossing differences in (x, θ), we must then have $f(x \wedge x', \theta') \leq f(x, \theta')$. By $f(\cdot, \theta)$ is quasi-supermodular, we must have $f(x', \theta') \leq f(x \vee x', \theta')$. This, in turn, implies that $x \vee x' \in X^*(\theta', \Gamma')$. By way of contradiction, suppose that $x \wedge x' \notin X^*(\theta, \Gamma)$; i.e., $f(x, \theta) > f(x \wedge x', \theta)$. By quasi-supermodularity of $f(\cdot, \theta)$, this implies that $f(x \vee x', \theta) > f(x', \theta)$. Since f has single-crossing differences in (x, θ), we obtain that $f(x \vee x', \theta') > f(x', \theta')$. But since $x \vee x' \in \Gamma'$ and $x' \in \Gamma'$, above contradicts the fact that $x' \in X^*(\theta', \Gamma')$. Hence, we must have $x \wedge x' \in X^*(\theta, \Gamma)$.

Conversely, suppose that $X^*_\Gamma(\theta)$ is monotone increasing. To show that $f(\cdot, \theta)$ is quasi-supermodular in x for all $\theta \in \Theta$, fix $x, x' \in X$ and $\theta \in \Theta$. Consider $\Gamma \coloneqq \{x, x \wedge x'\}$ and $\Gamma' \coloneqq \{x', x \vee x'\}$. Observe that $\Gamma' \geq_S \Gamma$. Suppose that $f(x \wedge x', \theta) \leq f(x, \theta)$ so that $x \in X^*_\Gamma(\theta)$. Then, $X^*_{\Gamma'}(\theta) \geq_S X^*_\Gamma(\theta)$ implies that we must have $f(x', \theta) \leq f(x \vee x', \theta)$—if, instead, $f(x', \theta) > f(x \vee x', \theta)$, then $x' \in X^*_{\Gamma'}(\theta)$ and $x \vee x' \notin X^*_{\Gamma'}(\theta)$; but this contradicts the fact that $X^*_{\Gamma'}(\theta) \geq_S X^*_\Gamma(\theta)$ implies that $x \vee x' \in X^*_{\Gamma'}(\theta)$). Similarly, $f(x \wedge x', \theta) < f(x, \theta)$ implies $f(x', \theta) < f(x \wedge x', \theta)$. To show that f has single-crossing differences in (x, θ), fix $x, x' \in X$ and $\theta, \theta' \in \Theta$ such that $x' > x$ and $\theta' > \theta$. Consider $\Gamma \coloneqq \{x, x'\}$. If $f(x, \theta) \leq f(x', \theta)$, then $x' \in X^*_\Gamma(\theta)$ and $X^*_\Gamma(\theta') \geq_S X^*_\Gamma(\theta)$ implies that $f(x, \theta') \leq f(x', \theta')$—if, instead, $f(x, \theta') > f(x', \theta')$, then $x' \notin X^*_\Gamma(\theta')$ but $X^*_\Gamma(\theta') \geq_S X^*_\Gamma(\theta)$ implies that $x \vee x' = x' \in X^*_\Gamma(\theta')$). Similarly, $f(x, \theta) < f(x', \theta)$ implies that $f(x, \theta') < f(x', \theta')$. ■

Remark 13. The theorem above does not require topological assumptions (and so we don't have nonemptiness nor compactness of $X^*(\theta)$).

Remark 14. The monotone comparative statics theorem does not always apply. Indeed, the budget constraint $\Gamma(\mathbf{p}, m) := \{\mathbf{x} \in \mathbb{R}^d : \mathbf{p} \cdot \mathbf{x} \leq m\}$ in utility maximisation problems are not ordered with respect to the strong set order (it is not even a lattice). This means that we cannot conduct demand analysis via the monotone comparative statics theorem. There are generalisations of the result (e.g., C-supermodularity, Quah, 2007) allowing one to handle budget constraints among other things.