ECON 6170 Module 6 and Problem Set 9 Answers

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Exercise 1. Toward a contradiction, suppose that

$$\sup_{x \in \Gamma_1} f(x) > \sup_{x \in \Gamma_2} f(x)$$

Then there exists $x' \in \Gamma_1$ such that

$$f(x') > \sup_{x \in \Gamma_2} f(x)$$

But $x' \in \Gamma_2$ also, so this is a contradiction.

Exercise 2. By definition of sup

$$x^* \in \left\{ x \in X \;\middle|\; f(x) = \sup_{z \in \Gamma} f(z) \right\} \iff f(x^*) \geq f(x) \quad \text{for all } x \in \Gamma$$

and

$$x^* \in \left\{ x \in X \;\middle|\; (g \circ f)(x) = \sup_{z \in \Gamma} (g \circ f)(z) \right\} \iff (g \circ f)(x^*) \geq (g \circ f)(x) \quad \text{for all } x \in \Gamma$$

Because g is strictly increasing, we also have for all x that

$$f(x^*) \geq f(x) \iff (g \circ f)(x^*) \geq (g \circ f)(x)$$

The result follows. If g was a weakly increasing function, we would instead have

$$f(x^*) \geq f(x) \implies (g \circ f)(x^*) \geq (g \circ f)(x)$$

This implies

$$\left\{ x \in X \;\middle|\; f(x) = \sup_{z \in \Gamma} f(z) \right\} \subseteq \left\{ x \in X \;\middle|\; (g \circ f)(x) = \sup_{z \in \Gamma} (g \circ f)(z) \right\}$$

This may hold with inequality: consider the set $\{0, 1\}$ and the functions $f(x) := x$ and $g(x) := 1$.

Exercise 3. By definition, (iii) $\implies$ (ii). If f has a local maximum at x^* then for x sufficiently close to x^*, $f(x^*) \geq f(x)$. Because f is concave and differentiable on the convex set $\operatorname{int} X$, Proposition 14 implies that $\nabla f(x^*)(x - x^*) \leq 0$ for x close to x^*. It follows that $\nabla f(x^*) \leq 0$ and $\nabla f(x^*) \geq 0$,

*Based on solutions provided by Professor Takuma Habu so $\nabla f(x^*) = 0$. Therefore, (ii) $\implies$ (i). It remains to show that (i) $\implies$ (iii). Suppose that $\nabla f(x^*) = 0$. Applying Proposition 4 again,

$$\nabla f\left(x^*\right)\left(x - x^*\right) \geq f\left(x\right) - f\left(x^*\right)$$

for all $x \in \operatorname{int} X$. It follows that $0 \geq f(x) - f(x^*)$ for all such x, or equivalently, $f(x^*) \geq f(x)$. Suppose $x \in \operatorname{bd} X$. Continuity of f (implied by differentiability) tells us that for all $\varepsilon > 0$, there exists x' close enough to x that $f(x) < f(x') + \varepsilon$. This implies that $f(x) < f(x^*) + \varepsilon$ for all $\varepsilon > 0$. That is, $f(x) \leq f(x^*)$. Thus, f has a global maximum at x^*.

Exercise 4. g is clearly C^1 and concave. f is clearly C^1 on $\mathbb{R} \setminus \{0, 1\}$. To see if f is differentiable at 0, we consider left- and right-derivatives,

$$\begin{aligned}
\lim_{h \nearrow 0} \frac{f(h) - f(0)}{h} &= \lim_{h \nearrow 0} \frac{h^3}{h} \\
&= \lim_{h \nearrow 0} h^2 \\
&= 0 \\
&= \frac{0}{h} \\
&= \lim_{h \searrow 0} \frac{f(h) - f(0)}{h}
\end{aligned}$$

and do the same at 1,

$$\begin{aligned}
\lim_{h \searrow 0} \frac{f(1+h) - f(1)}{h} &= \lim_{h \searrow 0} \frac{(1+h-1)^2 - (1-1)^2}{h} \\
&= \lim_{h \searrow 0} \frac{h^2}{h} \\
&= \lim_{h \searrow 0} h \\
&= 0 \\
&= \lim_{h \nearrow 0} \frac{f(1+h) - f(1)}{h}
\end{aligned}$$

Therefore, f is differentiable everywhere, with derivative

$$f'(x) = \begin{cases} 3x^2 & \text{if } x < 0 \\ 0 & \text{if } 0 \leq x \leq 1 \\ 2(x-1) & \text{if } x > 1 \end{cases}$$

It is easy to see that this is continuous: in particular, if $x \to 0$ then $f'(x) \to 0$ and if $x \to 1$ then $f'(x) \to 0$. The derivative of f is everywhere nonnegative, so f is monotone and thus quasiconcave. f is not concave because

$$f\left(\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 2\right) = f\left(1\right) = 0 < \frac{1}{2} = \frac{1}{2} \cdot 1^2 = \frac{1}{2} f(0) + \frac{1}{2} f(2)$$

Let $x^* \in [0, 1]$. Then $f'(x^*) = 0$ and $g(x^*) \geq 0$, so the KKT conditions are satisfied by $\lambda^* = 0$. But $f(x^*) = 0 < f(x)$ for any $x > 1$. This suggests that we cannot remove condition (14) from Theorem 5: if $\nabla f(x^*) = 0$ and f is not concave, then x^* may not be a maximum.

PS 9 Additional Exercises

Exercise 1.

(i) If problem (1) attains a global maximum at x^* and the constraint qualification holds, then by the Theorem of Lagrange, there exists $\mu^* \in \mathbb{R}^K$ such that

$$\nabla f(x^*) + \sum_{k=1}^{K} \mu_k^* \cdot \nabla h_k(x^*) = 0$$

But the left-hand side is just $\nabla_x \mathcal{L}(x^*, \mu^*)$. Moreover, the constraints imply $\nabla_\mu \mathcal{L}(x^*, \mu^*) = h(x^*) = 0$. Taken together, we have

$$\nabla \mathcal{L}(x^*, \mu^*) = 0$$

So $(x^*, \mu^*) \in S$ and $x^* \in S_X$. It follows that $f(x^\circ) \geq f(x^*)$. Moreover, $x^\circ \in S_X$ implies that there exists μ° such that $\nabla \mathcal{L}(x^\circ, \mu^\circ) = 0$. But $\nabla_\mu \mathcal{L}(x^\circ, \mu^\circ) = 0$ implies that x° satisfies the constraints. Therefore, x° is also a global maximiser for problem (1).

(ii) *(Note: This problem should have had the the additional hypothesis: "Suppose (4) has a solution.")* Let (x', μ') solve (4). Then Proposition 1 on unconstrained optimisation implies that $(x', \mu') \in S$, so $x' \in S_X$. Moreover, by definition of (x', μ'),

$$\mathcal{L}(x', \mu') = f(x') + \sum \mu_k' h_k(x') \geq f(x^\circ) + \sum \mu_k^\circ h_k(x^\circ) = \mathcal{L}(x^\circ, \mu^\circ)$$

where x° maximises (3). But $x', x^\circ \in S_X$ implies $h(x') = h(x^\circ) = 0$.It follows that

$$f(x') \geq f(x^\circ)$$

so x' is also a solution to (3). Conversely, $x' \in S_X$ and the definition of x° imply

$$f(x^\circ) \geq f(x')$$

Moreover, we know that $h(x^\circ) = h(x') = 0$ so

$$\mathcal{L}(x^\circ, \mu^\circ) = f(x^\circ) + \sum \mu_k^\circ h_k(x^\circ) \geq f(x') + \sum \mu_k' h_k(x') = \mathcal{L}(x', \mu')$$

for any μ°. It follows that (x°, μ°) solves (4).

If problem (1) attains a global maximum at x^* and the constraint qualification holds, then by the Theorem of Lagrange, there exists $\mu^* \in \mathbb{R}^K$ such that

$$\nabla f(x^*) + \sum_{k=1}^{K} \mu_k^* \cdot \nabla h_k(x^*) = 0$$

But the left-hand side is just $\nabla_x \mathcal{L}(x^*, \mu^*)$. Moreover, the constraints imply $\nabla \mu \mathcal{L}(x^*, \mu^*) = h(x^*) = 0$. Taken together, we have

$$\nabla \mathcal{L}(x^*, \mu^*) = 0$$

So $(x^*, \mu^*) \in S$ and $x^* \in S_X$. It follows that $f(x^\circ) \geq f(x^*)$. Moreover, $x^\circ \in S_X$ implies that there exists μ° such that $\nabla \mathcal{L}(x^\circ, \mu^\circ) = 0$. But $\nabla_\mu \mathcal{L}(x^\circ, \mu^\circ) = 0$ implies that x° satisfies the constraints. Therefore, x° is also a global maximiser for problem (1).

(iii) Let (x', μ') solve (4). Then Proposition 1 on unconstrained optimisation implies that $(x', \mu') \in S$, so $x' \in S_X$. Moreover,

$$f(x') + \sum \mu'_k h_k(x') \geq f(x^\circ) + \sum \mu^\circ_k h_k(x^\circ)$$

where x° maximises (3). But part (i) tells us that $h(x^\circ) = 0$ and Proposition 1 tells us that $h(x') = 0$. it follows that

$$f(x') \geq f(x^\circ)$$

so x' is also a solution to (3). Conversely, suppose x° is a solution to (3). Then, the previous direction implies

$$f(x^\circ) \geq f(x')$$

for any x' such that (x', μ') solves (4). Moreover, we know that $h(x^\circ) = h(x') = 0$ so

$$\mathcal{L}(x^\circ, \mu^\circ) = f(x^\circ) + \sum \mu^\circ_k h_k(x^\circ) \geq f(x') + \sum \mu'_k h_k(x') = \mathcal{L}(x', \mu')$$

for some μ°. It follows that (x°, μ°) solves (4).

Exercise 2. $y^3 - x^2 = 0$ is equivalent to $y^3 = x^2$. In particular, this implies that $y \geq 0$. Maximising $-y$ is equivalent to minimising y, which is achieved by choosing $y = 0$. The constraint then implies that $x = \sqrt{y} = 0$.

$$Dh(0,0) = \begin{bmatrix} \frac{\partial h(0,0)}{\partial x} & \frac{\partial h(0,0)}{\partial y} \end{bmatrix} = \begin{bmatrix} -2 \cdot 0 & 3 \cdot 0^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

and the constraint qualification is that $\operatorname{rank} Dh(x, y) = 1$. The rank of a matrix is the maximal number of its rows (or columns) that can comprise a linearly independent set. Here, we have one row, which is a zero vector, and the set $\{0\}$ is not linearly independent. Therefore $\operatorname{rank} Dh(x, y) = 0$, violating the constraint qualification.

Note also that for any $\mu \in \mathbb{R}$,

$$\nabla f(0,0) + \mu \nabla h(0,0) = \begin{bmatrix} 0 & -1 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Exercise 3. The solution to (1) can be obtained by plugging the constraint $y = x$ into the objective function to get $\frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x$. This function does not attain a maximum, as $\lim_{x \to \infty} \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x = \lim_{x \to \infty} x^3 \cdot (\frac{1}{3} - \frac{3}{2x} + \frac{2}{x^2}) = \infty \cdot \frac{1}{3} = \infty$.

$Dh(x, y) = \begin{bmatrix} 1 & -1 \end{bmatrix}$ which has rank 1, as required by the constraint qualification.

The Lagrangian is

$$\mathcal{L}(x, y, \mu) = \frac{1}{3}x^3 - \frac{3}{2}y^2 + 2x + \mu \cdot (x - y)$$

Again, by choosing $x = y \to \infty$, we can make this expression arbitrarily large, so (4) doesn't have a solution.

Exercise 4. Exercise 2 tells us that the constraint qualification is necessary in the Theorem of Lagrange. Exercise 3 tells us that the constraint qualification and $\nabla \mathcal{L}(x, \mu) = 0$ are not *sufficient* conditions for the existence of an equality-constrained maximum.