# Problem Set 7

Due: TA Discussion, 18 October 2024.

## 1 Exercises from class notes

All from "5. Differentiation.pdf".

**Exercise 1.** TFU: If $f : X \subseteq \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in \text{int}(X)$, then $f$ is differentiable at $x_0$.

**Exercise 3.** Prove the Chain Rule: Suppose $f : X \subseteq \mathbb{R} \to \mathbb{R}$ is differentiable at $x_0 \in \text{int}(X)$ and that $g : Y \to \mathbb{R}$, where $f(X) \subseteq Y$, and $g$ is differentiable at $f(x_0)$. Then, $g \circ f$ is differentiable at $x_0$ and

$$(g \circ f)'(x_0) = (g' \circ f)(x_0) \cdot f'(x_0).$$

**Exercise 4.** Prove the following: Suppose $f : (a, b) \subseteq \mathbb{R} \to \mathbb{R}$ and $f$ is strictly increasing and differentiable on $(a, b)$. Then,

$$\left(f^{-1}\right)'(f(x)) = \frac{1}{f'(x)} \; \forall x \in (a, b).$$

**Exercise 5.** Prove the following: Let $[a, b]$ be an a closed and bounded interval in $\mathbb{R}$ and suppose $f : [a, b] \to \mathbb{R}$ is continuous and differentiable on $(a, b)$. If $f'(x) = 0$ for all $x \in (a, b)$, then $f$ is constant.

**Exercise 6.** Prove the following: Suppose $f : (a, b) \subseteq \mathbb{R} \to \mathbb{R}$, $f \in \mathbf{C}^k$ and that $f'(x_0) = f''(x_0) = \ldots = f^{(k-1)}(x_0) = 0$ and $f^{(k)}(x_0) \neq 0$. Then, if $k$ is even and $f^{(k)}(x_0) > 0$, then f has a local minimum at $x_0$. *Hint:* If $g$ is continuous and $g(x_0) > 0$, then $g > 0$ in some neighbourhood of $x_0$.

## 2 Additional Exercises

**Theorem 1** (Cauchy-Schwarz Inequality)**.** *For any* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$*,*

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \, \|\mathbf{y}\|.$$

**Exercise 1.** Prove the Cauchy-Schwarz Inequality.