About TA sections:

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Our plan for today:[1]

[1] Materials adapted from notes provided by a previous Teaching Assistant, Zhuoheng Xu.

1 Envelope Condition

1.1 Motivation

In optimization problems, we often need to assess how the objective function value responds to marginal changes in one or more exogenous variables. It can be interpreted as how the objective function value changes if the 'environment' of the model changes.

Example from microeconomics: Suppose there is one consumer with two goods: x_1 and x_2. Prices of goods are denoted by p_1 and p_2, and the consumer is also endowed with income m. In this model, the set of exogenous variables: $\{p_1, p_2, m\}$. In this setup, we may want to know the impact of a marginal increase in income m on the utility of the consumer.

The envelope theorem tells us that, holding the optimal choices x_1^*, x_2^* fixed, the change in the value of the objective function (here, the utility) with respect to a marginal change in m is given by the partial derivative of the Lagrangian with respect to m:

$$\frac{dV}{dm} = \frac{\partial \mathcal{L}}{\partial m} = \lambda$$

The Envelope theorem tells us that the rate at which the consumer's utility increases with a marginal increase in income m is equal to λ, the Lagrange multiplier.

1.2 Application in a Simple Growth Model

In the context of the Bellman Equation, consider the simple neoclassical model. The Bellman Equation of this model is

$$v(k) = \max_{0 \leq k' \leq k^\alpha} \{\log(k^\alpha - k') + \beta v(k')\}$$

To derive the optimality condition, we need to find the FOC first. In this simple model, the control variable is k', and the FOC is

$$\frac{\partial v(k)}{\partial k'} = -\frac{1}{k^\alpha - k'} + \beta \frac{\partial v(k')}{\partial k'} = 0$$

In this FOC, two distinct terms appear. The interpretation of the first term is the marginal utility on consumption if the capital in the next period increases by one unit. This term is negative because if we want to have one more unit of capital in the next period, we need to increase investment today by one unit, so that consumption has to decrease by one unit, leading to a loss of utility.

The second term demonstrates that higher capital in the next period also impacts the future value. *From the perspective of the next period, capital in the next period serves as the state variable.* Essentially, it illustrates the effect on the objective function when the 'environment' undergoes marginal changes.

Rather than deriving the expression for $\frac{\partial v(k')}{\partial k'}$ directly, we can find $\frac{\partial v(k)}{\partial k}$ and lead one period forward. Recall that the policy function is a mapping from the state variable to the control variable, we can write k' as $k'(k)$, i.e., k' as a function as k. The Bellman Equation becomes

$$v(k) = \max_{0 \le k' \le k^\alpha} \{\log(k^\alpha - k'(k)) + \beta v(k'(k))\}$$

Taking the derivative w.r.t. k, we obtain

$$\frac{\partial v(k)}{\partial k} = \frac{1}{k^\alpha - k'(k)}\left(\alpha k^{\alpha-1} - \frac{\partial k'(k)}{\partial k}\right) + \beta \frac{\partial v(k'(k))}{\partial k'}\frac{\partial k'(k)}{\partial k} =$$

$$= \underbrace{\frac{\alpha k^{\alpha-1}}{k^\alpha - k'}}_{\text{Direct Effect}} + \underbrace{\left(-\frac{1}{k^\alpha - k'} + \beta\frac{\partial v(k'(k))}{\partial k'}\right)\frac{\partial k'(k)}{\partial k}}_{\substack{\text{Indirect Effect} \\ = 0 \text{ by FOC}}} = \frac{\alpha k^{\alpha-1}}{k^\alpha - k'}$$

Interpretation of the result: There are two effects induced by the change in capital k

- **Direct Effect:** Higher capital k leads to higher output k^α, leading to higher consumption and higher utility. Direct Effect = Marginal Utility of Consumption $\times$ Marginal Product of Capital (i.e., $\frac{1}{k^\alpha - k'} \times \alpha k^{\alpha-1}$).

- **Indirect Effect:** The change in the state variable k leads to changes in the control variable k', which in turn has impacts on the value of the objective function. The optimality condition shows that the Indirect Effect is equal to 0.

Trick: The derivative of the value function with respect to an endogenous state can be obtained by taking the derivative of the objective function with respect to the endogenous state variable *as if the control variable does not depend on the state variable*. That is, we can treat the control variable as a constant.

Lead one period forward

$$\frac{\partial v(k')}{\partial k'} = \frac{\alpha k'^{\alpha-1}}{k'^{\alpha} - k''}$$

Substitute into the FOC

$$\frac{1}{k^{\alpha} - k'} = \beta \frac{\alpha k'^{\alpha-1}}{k'^{\alpha} - k''}$$

Which is equivalent to

$$u'(c) = \beta \alpha k'^{\alpha-1} u'(c')$$

i.e., the Euler Equation.

1.3 Practice Question 1 (Midterm 2020)

An infinitely-lived logger owns a tree and must decide how much to chop down each day. The tree starts out at height x_0 on day 0. Let x_t denote the height of the tree on the morning of day t and y_t denote the height on the evening of day t. If the logger chops down c_t feet of the tree on day t, then

$$y_t = x_t - c_t$$

The remaining part of the tree grows overnight according to a function $h(\cdot)$. Specifically, the tree's height the next morning is

$$x_{t+1} = h(y_t) = h(x_t - c_t)$$

where $h(0) = 0$ (dead trees don't grow) and $h' \geq 0$. Once the wood is chopped down, it cannot be stored and must be eaten right away. The logger likes to eat wood and maximizes

$$U(\{c_t\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

where $\beta \in (0, 1)$. Consumption has to satisfy $0 \leq c_t \leq x_t$ for all t.

1. Write down the logger's sequence problem including all relevant constraints.

2. Let $V(x)$ be the value to the logger of waking up with a tree of height x. Write down the logger's Bellman equation including all relevant constraints.

3. From now on, assume $h(y) = (1+r)y$ and $u(c) = log(c)$. Find the Euler equation that characterizes the optimal tree-cutting policy. Show that, under the optimal policy, $c_t = \alpha^t c_0$ for some constant α, and find the value of α. (Hint: you can assume that the Benveniste-Scheinkman theorem works here.)

2 Competitive Equilibriums in Growth Model: Recursive Competitive Equilibrium

2.1 Characterization

Take the simple neoclassical model as the example. Let's consider the household's problem first. More specifically, consider a tiny household within the population. This assumption implies that the decisions made by individual households have no influence on aggregate-level outcomes. It is analogous to the situation in a perfectly competitive market where each producer is a price taker. In general, we normalize the mass of households to 1. For each household, the time endowment in each period is 1.

The household's utility maximization problem (ignoring non-negativity constraints) is

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

subject to

$$c_t + k_{t+1} = w(K_t) + (1 + r(K_t) - \delta)\, k_t$$

where k_t represents the capital of the household, and K_t represents the aggregate capital.

Remark 1: Since the household is tiny, the wage rate w and the rental rate r are determined by the aggregate capital K_t rather than the household capital k_t. **The household takes the aggregate capital K_t as given. Moreover, the household takes all aggregate variables as given.**

Remark 2: The budget constraint is not sufficient to solve the problem. To make the consumption-investment decision, the household also needs to know the rental rate in the next period r_{t+1}. Since the rental rate is a function of aggregate capital in the next period K_{t+1}, the household has to know the path of aggregate capital. Suppose the household believes the aggregate capital evolves according to the mapping H, that is, $K_{t+1} = H(K_t)$.

Remark 3: The interpretation of the mapping H: knowing the aggregate capital today K_t enables the household to project the aggregate capital path into the future aggregate capital and the path for prices.

Remark 4: State variables are (k, K), rather than just k. Control variables are (c, k').

Since we have identified all state variables, we can write the household's utility maximization problem into the recursive form

$$v(k, K) = \max_{c,k'} U(c) + \beta v(k', K')$$

subject to

$$c + k' = w(K) + (1 + r(K) - \delta)\, k$$
$$K' = \mathcal{H}(K)$$

Solution: value function v and policy functions $c = C(k, K)$ and $k' = G(k, K)$.

Next, consider the firm's profit maximization problem. FOCs are

$$w(K) = F_n(K, 1)$$

$$r(K) = F_k(K, 1)$$

The third condition, which is the distinctive feature of the recursive formulation of competitive equilibrium, is the **Consistency Condition**

$$K' = G(K, K)$$

Interpretation of the Consistency Condition: Suppose the household possesses a quantity of capital equal to the aggregate capital, then the household's individual behavior in equilibrium will be exactly the same as the aggregate behavior. That is, the aggregate law of motion perceived by the agent (i.e., $\mathcal{H}(K)$) must be consistent with the actual behavior of individuals (i.e., $G(K, K)$). This links individual decisions to the aggregate outcome, ensuring equilibrium.

Finally, we need to have the market clearing condition

$$C(K, K) + G(K, K) = F(K, 1) + (1 - \delta)K$$

where $C(K, K)$ and $G(K, K)$ are aggregate consumption today and aggregate capital tomorrow since the first argument in these two policy functions is K rather than k.

Finally, combining all these equations, we have the following definition of RCE:

An **RCE** is a value function $v : \mathbb{R}_+^2 \to \mathbb{R}$ and policy functions $C, G : \mathbb{R}_+^2 \to \mathbb{R}_+$ for the representative household, pricing functions $w, r : \mathbb{R}_+ \to \mathbb{R}_+$ and an aggregate law of motion $H : \mathbb{R}_+ \to \mathbb{R}_+$ such that:

1. Given w, r and H, v solves the Bellman equation and C, G are the associated policy functions.
2. The pricing functions satisfy the firms FOC.
3. Consistency: $\mathcal{H}(K) = G(K, K)$.
4. For all $K \in \mathbb{R}_+$:

$$C(K, K) + G(K, K) = F(K, 1) + (1 - \delta)K$$

Remark 5: If we assume households are identical, then in equilibrium, each household will have the same level of consumption, that is, $c_{it} = c_t$ for all t, then we can conclude

$$\begin{aligned} C_t &\equiv \int_0^1 c_{it}\, di \text{ (by definition)} \\ &= \int_0^1 c_t\, di \text{ (by } c_{it} = c_t\text{)} \\ &= c_t \int_0^1 1\, di \text{ (by } c_t \text{ independent of } i\text{)} \\ &= c_t \end{aligned}$$

Moreover, since the mass of the households is equal to 1, we can calculate the average consumption as

$$\bar{C}_t = \frac{C_t}{1} = C_t$$

Hence we have the following relationship:

Household Consumption(c_{it} or c_t) = Aggregate Consumption(C_t) = Average Consumption($\bar{C}_t$)

The same relationship also holds for other variables, e.g., Household Capital = Aggregate Capital = Average Capital.

2.2 Practice Question 2 (Economy with Labor Externality)

Suppose there is a unit measure of identical households. The preference of the household can be represented by the lifetime utility

$$\sum_{t=0}^{\infty} \beta^t u(c_t, h_t)$$

where $0 < \beta < 1$. u is a function defined over the household's consumption c_t and the household's labor supply relative to the economy-wide labor supply. That is, $h_t = l_t/\bar{L}_t$, where l_t is the household's labor supply and $\bar{L}_t$ is the average labor supply across all households. The function u has the following properties: $u_c > 0$, $u_{cc} < 0$, $u_h < 0$, and $u_{hh} < 0$. This means that the household likes consumption c_t with diminishing marginal utility, and the household dislikes work, but dislikes more if the household works more than other households. Inada conditions hold for both two variables c_t and l_t. Time is normalized to 1, so that $0 \leq H_t \leq 1$. The output is produced by a single competitive firm, with the production function

$$Y_t = F(K_t^d, L_t^d)$$

where K^d is the firm's demand for capital, and L^d is the firm's demand for labor. The production function F has the property of constant returns to scale.

Each household is endowed with k_0 units of capital in the first period, and the aggregate resource constraint is given by

$$C_t + K_{t+1} = F(K_t, L_t) + (1 - \delta)K_t$$

(a) Find the Bellman equation describing the household's problem and derive the household's optimality conditions.

(b) Derive the firm's optimality conditions

(c) Define and characterize the recursive competitive equilibrium (Hint: characterize by the EE, the consumption-labor equation, and the resource constraint)

(d) Characterize the social planner's problem using the recursive approach and derive the optimality conditions. Compare answers to your answers in part (c).