ECON 6170

Problem Set 3

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Exercise 2

1. Prove that the arbitrary union of open sets is open

   Proof. Define a (not necessarily finite) collection of open sets $\mathcal{O}$. We want to show that $\bigcup_{O_i \in \mathcal{O}} O_i$ is open. Take some $x \in \bigcup_{O_i \in \mathcal{O}} O_i$. $\exists$ i s.t. $x \in O_i$. Since O_i is open, $\exists$ ε s.t. $B_\varepsilon(x) \subseteq O_i$. Then $B_\varepsilon(x) \subseteq \bigcup_{O_i \in \mathcal{O}} O_i$. Thus, $\bigcup_{O_i \in \mathcal{O}} O_i$ is open. $\square$

2. Prove that the intersection of finitely many open sets is open

   Proof. Define a finite collection of open sets $\mathcal{O}$, where $|\mathcal{O}| = n < \infty$. We want to show that $\bigcap_{i=1}^n O_i$ is open. Take some $x \in \bigcap_{i=1}^n O_i$. We have that $x \in O_i$ $\forall$ $i = 1, \dots, n$. Since each O_i is open, for each O_i $\exists$ ε_i s.t. $B_{\varepsilon_i}(x) \subseteq O_i$. Define $\varepsilon^\star = \min\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$. Then $B_{\varepsilon^\star}(x) \subseteq B_{\varepsilon_i}(x)$ $\forall$ i. Thus, $B_{\varepsilon^\star}(x) \subseteq O_i$ $\forall$ i, so $B_{\varepsilon^\star}(x) \subseteq \bigcap_{i=1}^n O_i$, and $\bigcap_{i=1}^n O_i$ is open. $\square$

3. What about arbitrary intersections of open sets? Nope!

   Disproof. Consider as an example $\mathcal{O}$, where $O_i \in \mathcal{O}$ is defined as the interval $(-1, \frac{1}{i})$, and O_i is defined as such for all $i \in \mathbb{N}$. Then $\bigcap_{O_i \in \mathcal{O}} O_i = (-1, 0]$. Taking $x = 0$, any ε admits $B_\varepsilon(x)$ that contains some element of the positive reals. Thus, $B_\varepsilon(x) \not\subseteq (-1, 0]$ for all $\varepsilon > 0$, and $\bigcap_{O_i \in \mathcal{O}} O_i$ is not open. $\square$

Exercise 3 Prove that the interval $[a, b]$ is closed.

Proof. Note first that $\forall$ $x \in [a, b]$, $x \geq a$ and $x \leq b$. This means that $\sup[a, b] = b$, and $\inf[a, b] = a$, *i.e.*, the interval $[a, b]$ contains its suprema. Towards a contradiction, assume that there exists some sequence $\{x_n\} \subseteq [a, b]$ such that $x_n \to x$ for some $x \notin [a, b]$. This admits two cases. First, assume that $x > b$, meaning that $\exists$ $\varepsilon' > 0$ s.t. $x = b + \varepsilon'$. Choosing $\varepsilon < \varepsilon'$, we have that $\exists$ $N \in \mathbb{N}$ s.t. $|x_n - x| < \varepsilon$ $\forall$ $n > N$. This means that $\exists$ $x_n \in B_\varepsilon(x)$. However, since we assumed that $x - b = \varepsilon' > \varepsilon$, this would mean that $x_n > b$, contradicting the fact that $b = \sup[a, b] \geq x_n$. The second case assumes that $x < a$, which means that $\exists$ $\varepsilon' > 0$ s.t. $x = a - \varepsilon'$. Choosing $\varepsilon < \varepsilon'$, we have that $\exists$ $N \in \mathbb{N}$ s.t. $|x_n - x| < \varepsilon$ $\forall$ $n > N$. This means that $\exists$ $x_n \in B_\varepsilon(x)$. However, since we assumed that $a - x = \varepsilon' > \varepsilon$, this would mean that $x_n < a$, contradicting the fact that $a = \inf[a, b] \leq x_n$. Thus, we have found a contradiction, so $[a, b]$ contains its limit points and is closed. $\square$

A second, topological proof:

Proof. Define an open cover of $[a, b]$ as $\mathcal{O}$, a (not necessarily finite) collection of open sets such that $[a, b] \subseteq \bigcup_{O_i \in \mathcal{O}} O_i$. Define $S \coloneqq \{x \mid [a, x] \text{ is covered by finitely many elements of } \mathcal{O}\}$. Note that S is nonempty because taking $x = a$, since $a \in \bigcup_{O_i \in \mathcal{O}} O_i$, there exists at least one i such that $a \in O_i$, and $\{a\}$ is covered by finitely many (namely, one) elements of $\mathcal{O}$. It remains to show that $\sup S = b$. First, note that b is an upper bound of S trivially, because $x \leq b$ $\forall$ $x \in [a, b]$. Define $x_0 = \sup S$. Towards a contradiction, assume that $x_0 < b$. First note that $x_0 > a$, because $a \in O_i$, so $\exists$ $\varepsilon > 0$ s.t. $B_\varepsilon(a) \subseteq O_i$, which means that $a + \varepsilon > a$ can be covered by finitely many $O_i \in \mathcal{O}$, so $x_0 > a$. Note that since x_0 is also in an open set (call it O_0), $\exists$ $\varepsilon > 0$ s.t. $B_\varepsilon(x_0) \subseteq O_0$. By the density of the reals, $\exists$ $y \in [a, b]$ s.t. $x_0 < y < x_0 + \varepsilon$. Thus, $[a, y]$ can be covered by a finite union of open sets, taking the union of the finite collection of open sets that cover

$[a, x_0]$ and U_0. This contradicts the earlier assumption that $x_0 = \sup S$, so $b = \sup S$. Thus, $[a, b]$ can be covered by a finite subcover of $\mathcal{O}$, and is compact. By Heine-Borel, it is closed. $\square$

Exercise 4

1. Prove that the arbitrary intersection of closed sets is closed

   Proof. Define a (not necessarily finite) collection of closed sets $\mathcal{C}$. We aim to show that $\bigcap_{C_i \in \mathcal{C}} C_i$ is closed. Take some sequence $\{x_n\} \subseteq \bigcap_{C_i \in \mathcal{C}} C_i$, where $x_n \to x$. We have that $\{x_n\} \in C_i \;\forall\; C_i \in \mathcal{C}$. Since each C_i is closed, by the limit definition of closed sets, $x \in C_i \;\forall\; C_i \in \mathcal{C}$. Thus, $x \in \bigcap_{C_i \in \mathcal{C}} C_i$, and $\bigcap_{C_i \in \mathcal{C}} C_i$ is closed. $\square$

2. Prove that the finite union of closed sets is closed

   Proof. Define a finite collection of closed sets $\mathcal{C}$, where $|\mathcal{C}| = n < \infty$. We aim to show that $\bigcup_{i=1}^n C_i$ is closed. Take some sequence $\{x_n\} \subseteq \bigcup_{i=1}^n C_i$, where $x_n \to x$. We have that at least one $C_j \in \bigcup_{i=1}^n C_i$ contains infinite elements of $\{x_n\}$, as $\{x_n\}$ is an infinite sequence so it cannot be the case that only finite elements of it are contained in a finite union of sets. Define the subsequence $\{x_{n_k}\}$, where $n_k \in \{n \mid x_n \in C_j\}$. From Exercise 26 in Problem Set 2, $\{x_{n_k}\} \to x$. Since C_j is closed, $x \in C_j$, which means that $x \in \bigcup_{i=1}^n C_i$, so $\bigcup_{i=1}^n C_i$ is closed. $\square$

3. What about arbitrary unions of closed sets? Nope!

   Disproof. Define an infinite collection of closed sets $\mathcal{C}$ as follows: $C_n \in \mathcal{C}$ is such that $C_n = \{\frac{1}{n}\}$, for all $n \in \mathbb{N}$. Each set is closed because finite sets are closed. However, taking $\{x_n\}$ where $x_n = \frac{1}{n}$, we have that $x_i \in C_i \;\forall\; i \in \mathbb{N}$. However, $x_n \to 0$, and $0 \notin \bigcup_{C_n \in \mathcal{C}} C_n$, so an infinite union of closed sets is not necessarily closed. $\square$

Additional Exercise 1 Extend Bolzano-Weierstrass into $\mathbb{R}^d$. The statement is:
Theorem 1. *Every bounded sequence $\{x_n\}_n \in \mathbb{R}^d$ has a convergent subsequence $\{x_{n_k}\}_k \in \mathbb{R}^d$.*

Proof. Induction on d. The base case $\mathbb{R}^1$ is the exact statement of Bolzano-Weierstrass we had in class. Assume that this theorem holds for each bounded sequence in $\mathbb{R}^k$. It remains to show that it holds for bounded sequences in $\mathbb{R}^{k+1}$. Take some sequence $\{x_n\}_n \in \mathbb{R}^{k+1}$ which is bounded, meaning that there exists $b \in \mathbb{R}$ such that $|x_{i,j}| < b \;\forall\; i, j$. Take the sequence $\{x_n\}_n \setminus x_{k+1} \in \mathbb{R}^k$, *i.e.*, the above bounded sequence less its final element. Since this sequence is bounded, there exists a subsequence $\{x_{n_i}\} \in \mathbb{R}^k$ such that $x_{n_i} \to x \in \mathbb{R}^k$. This means, from class, that $x_{n_i,j} \to x_j \;\forall\; j = 1, \ldots, k$. Finally, consider the sequence $\{x_{n,k+1}\}_n \in \mathbb{R}^1$. We have that this sequence is bounded, so by Bolzano-Weierstrass it has a convergent subsequence, which we denote as $\{x_{n_l,k+1}\}_l \in \mathbb{R}$, where $x_{n_l,k+1} \to x_{k+1} \in \mathbb{R}$. Construct a subsequence of the (original) $\{x_n\}_n \in \mathbb{R}^{k+1}$ as follows. $\{x_{n_m}\}_m$, where $m \in \{m \mid \exists\, x_{n_i} \in \{x_{n_i}\} \text{ s.t. } m = n_i, \exists\, x_{n_l} \in \{x_{n_l}\} \text{ s.t. } m = n_l\}$. Since the two other respective subsequences are infinite, $\{x_{n_m}\}$ has infinite elements, and since $x_{n_m,j} \to x_j \;\forall\; j \in 1, \ldots, k+1$ by definition, $x_{n_m} \to x$. Thus, an arbitrary bounded sequence $\{x_n\}_n \in \mathbb{R}^{k+1}$ has a convergent subsequence. $\square$

Additional Exercise 2 Prove the following:
Theorem 2. *A set $S \subseteq \mathbb{R}^d$ is sequentially compact if and only if it is closed and bounded.*

Proof. *(closed and bounded $\Rightarrow$ sequentially compact):* We have that S is closed and bounded. Take some sequence $\{x_n\}_n \in S$. Since S is bounded, $\{x_n\}_n$ is bounded and has a convergent subsequence $\{x_{n_k}\}_k \to x$ by Bolzano-Weierstrass. Since $\{x_{n_k}\}_k \in S$ and S is closed, $x \in S$ by the limit definition of closed sets. Thus, S is sequentially compact.

(sequentially compact $\Rightarrow$ closed): We have that S is sequentially compact. Take some convergent sequence $\{x_n\}_n \in S$ where $x_n \to x$. Since S is sequentially compact, $\{x_n\}_n$ has a convergent subsequence $\{x_{n_k}\}_k$

where $x_{n_k} \to x' \in S$. Since all subsequences of a convergent sequence converge to the same limit, $x' = x$, so $x \in S$. Thus, S contains its limit points and is closed.

(sequentially compact $\Rightarrow$ bounded): Towards a contradiction, assume that S is sequentially compact but not bounded. WLOG, assume that it is not bounded above, meaning that $\forall\, b \in \mathbb{R}^d$, $\exists\, x \in S$ s.t. $x > b$. We will construct a sequence as follows. Take some $x_1 \in S$. Since S is not bounded above, $\exists\, x_2 \in S$ s.t. $x_2 > y \;\forall\, y \in B_1(x_1)$, where $B_1(x_1)$ is the ε-ball of length 1 about x_1. If x_2 did not exist in S, x_2 would be an upper bound for S. Thus, $x_2 \in S$. Define x_3, where $x_3 \in S$ where $x_3 > y \;\forall\, y \in B_1(x_2)$, and do so to construct $x_n \;\forall\, n \in \mathbb{N}$. $\{x_n\}_n$ is a sequence in S. It remains to show that it has no convergent subsequence. Note that $\{x_n\}_n$ is not Cauchy, since taking $\varepsilon < 1$, $|x_n - x_m| > 1 > \varepsilon \;\forall\, n, m \in \mathbb{N}$. Thus, $\{x_n\}_n$ is not convergent. Additionally, this condition holds for every subsequence $\{x_{n_k}\}_k$, since $\forall\, n_k, m_k \in \mathbb{N}, |x_{n_k} - x_{m_k}| > 1$, so every subsequence is not convergent, contradicting the definition of sequential compactness. Note that we assumed S was not bounded above, but the same proof with signs flipped suffices if S is not bounded below. Thus, S is bounded. $\square$