ECON 6170 Module 3 Answers

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Exercise 1. The base cases $k = 1, 2$ hold by definition of a convex set. Suppose that the proposition is true for the case k. That is, if $\{\alpha_1, \ldots, \alpha_k\} \subseteq [0, 1]$ satisfy $\sum_{i=1}^{k} \alpha_i$ and $\{x_1, \ldots, x_k\} \subseteq S$ then

$$\sum_{i=1}^{k} \alpha_i x_i \in S$$

Suppose $\{\lambda_1, \ldots, \lambda_{k+1}\} \subseteq [0, 1]$ sum to 1 and $\{x_1, \ldots, x_{k+1}\} \subseteq S$. If $\lambda_j = 1$ then we are done. Otherwise, define

$$\alpha_i := \frac{\lambda_i}{\sum_{j=1}^{k} \lambda_j}$$

for $i = 1, \ldots, k$. Then $\{\alpha_1, \ldots, \alpha_k\} \subseteq [0, 1]$ satisfy $\sum_{i=1}^{k} \alpha_i$ and so

$$x := \sum_{i=1}^{k} \frac{\lambda_i}{\sum_{j=1}^{k} \lambda_j} x_i \in S$$

by the induction hypothesis. Let $\gamma = \sum_{j=1}^{k} \lambda_j$. Then $\gamma \in [0, 1]$. By the base case,

$$\sum_{i=1}^{k+1} \lambda_i x_i = \gamma x + (1 - \gamma) x_{k+1} \in S$$

Exercise 2. Let $x, y \in \bigcap_{S \in \mathcal{C}} S$ and $\alpha \in [0, 1]$. Then $x, y \in S$ for all $S \in \mathcal{C}$ and so, by convexity of each, $\alpha x + (1 - \alpha) y \in S$ for all $S \in \mathcal{C}$. It follows that $\alpha x + (1 - \alpha) y \in \bigcap_{S \in \mathcal{C}} S$.

Exercise 3. Let C denote the set of all finite convex combinations of elements of S.

By Proposition 1, we know that for every convex set T that contains S, every finite convex combination of elements of S is an element of T. Since that holds for all T that contain S, every finite convex combination of elements of S is in the intersection of all T that contain S, which is $co(S)$. C is, therefore, a subset of $co(S)$.

Let $x = \sum_i^n \alpha_i x_i$ and $y = \sum_j^m \beta_j y_j$ be convex combinations of elements in S. Then

$$\lambda x + (1 - \lambda) y = \sum_i^n \lambda \alpha_i x_i + \sum_j^m (1 - \lambda) \beta_j y_j$$

is also a convex combination of elements of S, so C is convex. Clearly C contains S, so $co(S) \subseteq C$.

Exercise 4. True. This holds trivially for the empty set. Suppose, then, that S is nonempty and open. Let $z \in \operatorname{co}(S)$. Then we can write

$$z = \sum_{i=1}^{n} \alpha_i x_i$$

for some $x_i \in S$ and $\alpha_i \in [0,1]$ that sum to 1. Openness of S implies that for each x_i, there exists ε_i such that $B_{\varepsilon_i}(x_i) \subseteq S$. Let $\varepsilon = \min\{\varepsilon_i \mid i = 1, \dots, n\}$. Then we can write

$$B_\varepsilon(x_i) \subseteq S$$

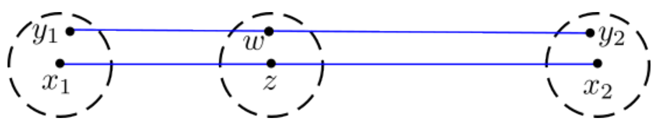
for all i. Take $w \in B_\varepsilon(z)$. We want to show that $w \in \operatorname{co}(S)$, which would imply $B_\varepsilon(z) \subseteq \operatorname{co}(S)$. This, in turn, would be sufficient to prove openness of $\operatorname{co}(S)$. Write

$$w = z + w - z = \sum_{i=1}^{n} \alpha_i x_i + w - z = \sum_{i=1}^{n} \alpha_i (x_i + w - z) =: \sum_{i=1}^{n} \alpha_i y_i$$

where $y_i := x_i + w - z$ for all i. Thus w is a convex combination of $y_1, \dots, y_n$, so if $y_1, \dots, y_n \in S$, we would have $w \in \operatorname{co}(S)$. But for all i,

$$\|y_i - x_i\| = \|x_i + w - z - x_i\| = \|w - z\| < \varepsilon$$

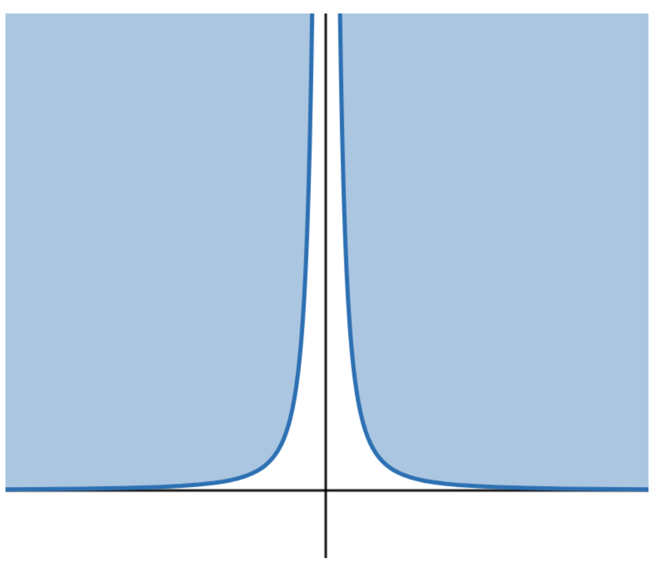
so $y_i \in B_\varepsilon(x_i) \subseteq S$.



Exercise 5. False. Take the set

$$\{(x, y) \in \mathbb{R}^2 \mid x \neq 0 \text{ and } y \geq 1/x^2\}$$

This set is closed, but its convex hull is $\mathbb{R} \times \mathbb{R}_{++}$ which is not closed.



Exercise 6. To show that $co(X)$ is bounded, fix a point $y \in X$. Let $M := \sup\{\|x\| \mid x \in X\}$. Let z be a point in $co(X)$. For some $m \in \mathbb{N}$, z can be written as a convex combination of m points in X. Write

$$z = \sum_{i=1}^{m} \alpha_i x_i$$

where $\sum_{i=1}^{m} \alpha_i = 1$ and $0 \leq \alpha_i \leq 1$ for $i = 1, 2, \ldots, m$. Then

$$\|z\| = \left\| \sum_{i=1}^{m} \alpha_i x_i \right\| \leq \sum_{i=1}^{m} \alpha_i \|x_i\| \leq \sum_{i=1}^{m} \alpha_i M = M$$

To show that $co(X)$ is closed, fix a sequence in $co(X)$, (x_i), converging to $x \in \mathbb{R}^n$. Using Carathéodory's Theorem, for all $i \in \mathbb{N}$ we can write

$$x_i = \alpha_{i,1} x_{i,1} + \ldots + \alpha_{i,n+1} x_{i,n+1}$$

where the $\alpha_{i,k}$ lie in $[0,1]$ and sum to 1, and the $x_{i,k}$ are points in X. By the compactness of X, each sequence $(x_{i,k})_{i \in \mathbb{N}}$ has a subsequence that converges to a point, $x_k^* \in X$. Moreover, by the compactness of $[0,1]$, each $(\alpha_{i,k})_{i \in \mathbb{N}}$ has a subsequence that converges to a number, α_k^*, between 0 and 1. Passing to the subsequences,

$$\lim_i x_i^s = \lim_i (\alpha_{i,1} x_{i,1} + \ldots + \alpha_{i,n+1} x_{i,n+1}) = \alpha_1^* x_1^* + \ldots + \alpha_{n+1}^* x_{n+1}^*$$

Let $x^* := \alpha_1^* x_1^* + \ldots + \alpha_{n+1}^* x_{n+1}^*$. Then (the subsequence) $x_i^s \to x^*$. We want to show that $x^* \in co(X)$. It suffices to show that $\sum_{k=1}^{n+1} \alpha_k^* = 1$. But this is just the limit of the sequence $(\sum_{k=1}^{n+1} \alpha_{i,k})_{i \in \mathbb{N}} = (1)_{i \in \mathbb{N}}$, which is 1. Therefore, the sequence (x_i) has a subsequence (x_i^s) converging to $x^* \in co(X)$. But $x_i \to x$ so $x = x^* \in co(X)$. Since (x_i) is an arbitrary convergent sequence in X, X is closed.

Exercise 7. $\overline{co}(S)$ is a convex set containing S, so $co(S) \subseteq \overline{co}(S)$. It is therefore a closed set containing $co(S)$, so $cl(co(S)) \subseteq \overline{co}(S)$.

Conversely, to show that $\overline{co}(S) \subseteq cl(co(S))$, we need to show that the closure of a convex set is itself convex. Then $cl(co(S))$ will be a closed, convex set containing $co(S)$.

Let x and y be elements of $cl(C)$, where C is some convex set. Then there exist sequences of elements of C such that $x_n \to x$ and $y_n \to y$. But then $\alpha x_n + (1-\alpha) y_n$ defines a sequence of elements that are also in our convex set C. Moreover, $\alpha x_n + (1-\alpha) y_n \to \alpha x + (1-\alpha) y$, so the latter is also in $cl(C)$, implying $cl(C)$ is convex.

Exercise 8. Take the example of

$$\{(x,y) \in \mathbb{R}^2 \mid x < 0 \text{ and } y \geq 1/x^2\}$$

and

$$\{(x,y) \in \mathbb{R}^2 \mid x > 0 \text{ and } y \geq 1/x^2\}$$

(compare Exercise 5). Both sets are closed. Any closed halfspace containing either must also include the y-axis. Therefore, they are not strongly separated.

Exercise 9. Apply the Strong Separating Hyperplane Theorem, noting that $\{x\}$ is a compact and convex set disjoint from Y.

Exercise 10. Let $X := \mathbb{R} \times \mathbb{R}_{++} \cup \{(1,0)\}$ and $Y := \mathbb{R} \times \mathbb{R}_{--} \cup \{(-1,0)\}$. Both sets are nonempty and convex and they are disjoint from each another. The unique separating hyperplane is the x-axis, $(0,1) \cdot (x,y) = 0$. But $(0,1) \cdot (1,0) = 0$ and $(0,1) \cdot (-1,0) = 0$.

Exercise 11. Suppose $f : S \to \mathbb{R}$ is concave. Let (x,y) and (x',y') be elements of the subgraph of f. Then $y \le f(x)$ and $y' \le f(x')$, so

$$\alpha y + (1-\alpha)y' \le \alpha f(x) + (1-\alpha) f(x') \le f(\alpha x + (1-\alpha)x')$$

where the last inequality uses concavity of f. It follows that

$$\alpha(x,y) + (1-\alpha)(x',y') = (\alpha x + (1-\alpha)x', \alpha y + (1-\alpha)y') \in \operatorname{sub} S$$

Therefore, the subgraph of f is convex.

Conversely, if the subgraph of f is convex, then it contains the convex combination

$$\alpha(x, f(x)) + (1-\alpha)(x', f(x')) = (\alpha x + (1-\alpha)x', \alpha f(x) + (1-\alpha) f(x'))$$

But this implies $f(\alpha x + (1-\alpha)x') \ge \alpha f(x) + (1-\alpha) f(x')$, so f is concave.

The proof of the corresponding result for the epigraph of a convex function is analogous.

Exercise 12.

$$\begin{aligned} f(\lambda x + (1-\lambda)x') &= a \cdot (\lambda x + (1-\lambda)x') + b \\ &= \lambda(a \cdot x + b) + (1-\lambda)(a \cdot x' + b) \\ &= \lambda f(x) + (1-\lambda) f(x') \end{aligned}$$

Exercise 13. Suppose f is quasiconcave. Fix $y \in \mathbb{R}$ and consider x, x' such that $f(x), f(x') \ge y$. Then $f(\alpha x + (1-\alpha)x') \ge \min\{f(x), f(x')\} \ge y$. This implies convexity of the upper contour sets of f.

Conversely, suppose the upper contour sets of f are all convex. Fix $x, x' \in X$. WLOG, $f(x) \ge f(x')$ so that both x and x' both lie in the upper contour set of f with bound $y = f(x')$. Then their convex combination $\alpha x + (1-\alpha)x'$ also lies in this upper contour set, so $f(\alpha x + (1-\alpha)x') \ge f(x') = \min\{f(x), f(x')\}$. It follows that f is quasiconcave.

f is quasiconvex $\iff$ $-f$ is quasiconcave $\iff$ the upper contour sets of $-f$ are convex. But the upper contour sets of $-f$ are clearly just the lower contour sets of f.

Exercise 14. True.

$$\begin{aligned} (h \circ f)(\alpha x + (1-\alpha)x') &\ge h(\min\{f(x), f(x')\}) \\ &= \min\{(h \circ f)(x), (h \circ f)(x')\} \end{aligned}$$

The analogous result does not hold for concave functions. For example, $f(x) := x$ is concave and $h(x) := \exp(x)$ is strictly increasing, but $(\exp \circ f)(x) = \exp(x)$ is strictly convex.

Exercise 15. False. Consider the piecewise function given by

$$f(x) := \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ x & \text{if } x < 0 \text{ or } x > 1 \end{cases}$$

This function has a local maximum at $x = 1/2$, but no global maximum.

Exercise 1 (Additional exercise on PS 5)**.**

(i) True.

$$\begin{aligned} (f+g)(\alpha x + (1-\alpha)x') &= f(\alpha x + (1-\alpha)x') + g(\alpha x + (1-\alpha)x') \\ &\leq \alpha f(x) + (1-\alpha) f(x') + \alpha g(x) + (1-\alpha) g(x') \\ &= \alpha (f+g)(x) + (1-\alpha)(f+g)(x') \end{aligned}$$

(ii) False. For example, $f(x) := -\mathrm{e}^x$ and $g(x) := -\mathrm{e}^{-x}$ are both monotone (and hence quasiconvex). But $(f+g)(x) = -\mathrm{e}^x - \mathrm{e}^{-x}$ is not quasiconvex, as for $x = \log 2$, for example,

$$\begin{aligned} (f+g)\left(\frac{1}{2}x + \frac{1}{2}(-x)\right) &= (f+g)(0) = -2 > -2 - \frac{1}{2} \\ &= -\mathrm{e}^{\log 2} - \frac{1}{\mathrm{e}^{\log 2}} \\ &= (f+g)(\log 2) \\ &= (f+g)(x) \\ &= \max\{(f+g)(x), (f+g)(-x)\} \end{aligned}$$

(iii) True. This follows from $\alpha f(x) + (1-\alpha) f(x') \geq \min\{f(x), f(x')\}$ for $\alpha \in [0,1]$.

(iv) True. Follows as (iii) but with $x \neq x'$ and strict inequality.