ECON 6170
Problem Set 11
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1 Exercises From Class Notes

Exercise 1. We have that the Lagrangian is

$$\mathcal{L}(x,\lambda) = f(x,\theta) + \sum_{i=1}^{K} \lambda_k h_k(x,\theta)$$

the first order condition is that

$$\nabla_x f(x,\theta) + \sum_{i=1}^{K} \lambda_k \nabla_x h_k(x,\theta) = 0$$

Fix some $\theta^\star \in \Theta$. Then there is a unique $x^\star \in \mathbb{R}^d$ that solves the equation. We have that the first order condition holds if and only if θ, x is a solution to the maximization problem. That means we can define a function $F(x,\theta) = \nabla_x f(x,\theta) + \sum_{i=1}^{K} \lambda_k \nabla_x h_k(x,\theta)$ where $F(x,\theta) = 0$ if and only if x, θ solves the problem. Using the Implicit Function Theorem, which holds since f, h_k are $\mathbf{C}^2$ meaning that F is $\mathbf{C}^1$, and the constraint qualification holding implies that $\nabla_\theta F(x,\theta)$ is invertible, we can define $g : \mathbb{R}^m \to \mathbb{R}^d$ with the property that

$$F(g(\theta),\theta) = 0$$

Thus, the solution varies with θ according to

$$Dg(\theta) = -(D_y F(g(\theta),\theta))^{-1} D_x F(g(\theta),\theta)$$

which is equivalent to

$$Dg(\theta) = -\left(\nabla_\theta \nabla_x f(x,\theta) + \sum_{i=1}^{K} \lambda_k \nabla_\theta \nabla_x h_k(x,\theta)\right)^{-1} \left(\nabla_x \nabla_x f(x,\theta) + \sum_{i=1}^{K} \lambda_k \nabla_x \nabla_x h_k(x,\theta)\right)$$

For this to hold with inequality constraints as well, we must have that the constraint qualification holds for all binding inequality constraints, and that the conditions for the KKT theorem also hold.

Exercise 2. The statement of the Envelope Theorem is:
Theorem 1. *Assuming that the endogenous assumptions from the Envelope Theorem hold, that the Lagrange multipliers are unique, and defining $x^\star(\theta)$ and $\lambda^\star(\theta)$ as the unique solution to the first order conditions, we have that the value function*

$$V(\theta) = \max_{x\in X} f(x,\theta) \text{ s.t. } h_k(x,\theta) = 0 \ \forall\ k$$

satisfies

$$\frac{\partial V(\theta)}{\partial \theta} = \frac{\partial f(x^\star,\theta)}{\partial \theta} + \sum_{k=1}^{K} \lambda_k^\star \frac{\partial h_k(x^\star,\theta)}{\partial \theta}$$

Proof. From Exercise 1, we know that we can define $x^\star(\theta) = g(\theta)$. The value function is thus $V(\theta) = f(x^\star(\theta), \theta)$. Differentiating with respect to θ while using the chain rule, we get that

$$\frac{\partial V(\theta)}{\partial \theta} = \frac{\partial f(x^\star, \theta)}{\partial \theta} + \nabla_x f(x^\star, \theta) \frac{\partial x^\star}{\partial \theta}$$

From the first order condition, we have that

$$\nabla_x f(x^\star, \theta) = -\sum_{k=1}^{K} \lambda_k^\star \nabla_x h_k(x^\star, \theta)$$

Since we assumed that $x^\star$ and $\lambda^\star$ were endogenous functions of θ, the intermediate partial derivatives (and gradients) fall out, and we get

$$\frac{\partial V(\theta)}{\partial \theta} = \frac{\partial f(x^\star, \theta)}{\partial \theta} + \sum_{k=1}^{K} \lambda_k^\star \frac{\partial h_k(x^\star, \theta)}{\partial \theta}$$

$\square$

If we were to replace the endogenous assumptions that $x^\star(\theta)$ and $\lambda^\star(\theta)$ exist and are unique, we would need that the Lagrangian is $\mathbf{C}^2$, that the constraint qualification holds, and that f, h_k are concave in x.

Exercise 6. Suppose that X and Θ are open sublattices of $\mathbb{R}^d$ and $\mathbb{R}^m$ respectively. Prove that $f : X \times \Theta \to \mathbb{R}$ that is $\mathbf{C}^2$ has increasing differences in $(x, \theta) \in X \times \Theta$ if and only if

$$\frac{\partial^2 f}{\partial x_i \partial \theta_j}(x, \theta) \geq 0 \; \forall \; (i, j) \in \{1, \ldots, d\} \times \{1, \ldots, m\}$$

Proof. Define $Z \coloneqq X \times \Theta$. It suffices to show that f is supermodular on Z if and only if

$$\frac{\partial^2 f}{\partial z_i \partial z_j}(z) \geq 0 \; \forall \; i \neq j$$

because by Lemma 1 in the notes, a function is supermodular if and only if it has increasing differences. We will mimic the proof to Proposition 3. We have that f is supermodular if and only if, for distinct i, j and some $\delta, \varepsilon > 0$, we have that

$$f(z_i + \varepsilon, z_j + \delta) - f(z_i + \varepsilon, z_j) \geq f(z_i, z_j + \delta) - f(z_i, z_j)$$

Dividing both sides by δ and letting $\delta \to 0$, this becomes

$$\lim_{\delta \to 0} \frac{f(z_i + \varepsilon, z_j + \delta) - f(z_i + \varepsilon, z_j)}{\delta} \geq \lim_{\delta \to 0} \frac{f(z_i, z_j + \delta) - f(z_i, z_j)}{\delta}$$

which is equivalent to

$$\frac{\partial f}{\partial z_j}(z_i + \varepsilon, z_j) \geq \frac{\partial f}{\partial x_j}(z_i, z_j)$$

Subtracting the right side from the left, dividing by ε, and letting $\varepsilon \to 0$, this becomes

$$\lim_{\varepsilon \to 0} \frac{\frac{\partial f}{\partial z_j}(z_i + \varepsilon, z_j) - \frac{\partial f}{\partial x_j}(z_i, z_j)}{\varepsilon} \geq 0$$

which is equivalent to

$$\frac{\partial^2 f}{\partial z_i \partial z_j}(z_i, z_j) \geq 0$$

Thus, f is supermodular if and only if $\frac{\partial^2 f}{\partial z_i \partial z_j}(z_i, z_j) \geq 0$, which is equivalent to saying that f has increasing differences in (x, θ) if and only if

$$\frac{\partial^2 f}{\partial x_i \partial \theta_j}(x, \theta) \geq 0 \; \forall \; (i, j) \in \{1, \ldots, d\} \times \{1, \ldots, m\}$$

$\square$

Exercise 7. Suppose that $(X, \geq)$ and $(\Theta, \geq)$ are partially ordered sets and that $f : X \times \Theta \to \mathbb{R}$ has single-crossing differences in (x, θ). Prove that single-crossing property is an ordinal property.

Proof. Following the hint, we will show that for any $\phi : \mathbb{R} \times \Theta \to \mathbb{R}$ such that $\phi(\cdot, \theta)$ is strictly increasing for any $\theta \in \Theta$, then the function $\tilde{f} : X \times \Theta \to \mathbb{R}$ defined by $\tilde{f}(x, \theta) \coloneqq \phi(f(x, \theta), \theta)$ has single-crossing differences. This will suffice to show that the single-crossing property is ordinal.

Defining $g(\theta) \coloneqq f(x', \theta) - f(x, \theta)$ for any $x' > x$, and defining $\tilde{g}(\theta) \coloneqq \tilde{f}(x', \theta) - \tilde{f}(x, \theta)$, we have that $\tilde{f}$ has single-crossing differences if and only if $\tilde{g}$ has the single-crossing property, that is, if for any $\theta' > \theta$ we have that

$$\tilde{g}(\theta') \geq 0 \; \forall \; \tilde{g}(\theta) \geq 0 \equiv \phi(f(x', \theta'), \theta') \geq \phi(f(x, \theta'), \theta') \; \forall \; \phi(f(x', \theta), \theta) \geq \phi(f(x, \theta), \theta)$$

and

$$\tilde{g}(\theta') > 0 \; \forall \; \tilde{g}(\theta) > 0 \equiv \phi(f(x', \theta'), \theta') > \phi(f(x, \theta'), \theta') \; \forall \; \phi(f(x', \theta), \theta) > \phi(f(x, \theta), \theta)$$

Take some $\theta' > \theta$, and assume that $\tilde{g}(\theta) \geq 0$, meaning that

$$\phi(f(x', \theta), \theta) \geq \phi(f(x, \theta), \theta)$$

Then, since ϕ is strictly increasing, we have that $f(x', \theta) \geq f(x, \theta)$, meaning that $g(\theta) \geq 0$. That implies that $g(\theta') \geq 0$, since f has single-crossing differences, meaning that $f(x', \theta') \geq f(x, \theta')$. Since ϕ is strictly increasing, this implies that

$$\phi(f(x', \theta'), \theta') \geq \phi(f(x, \theta'), \theta') \implies \tilde{g}(\theta') \geq 0$$

An analogous proof holds when assuming that $\tilde{g}(\theta) > 0$, replacing all weak inequalities with strict ones. Thus, $\tilde{f}$ has single-crossing differences, meaning that the single-crossing property is ordinal. $\square$

2 Additional Exercises

Exercise 2. Prove that if f is log-supermodular then f is quasi-supermodular.

Proof. We have that $\ln f$ is supermodular, meaning that for any $x, x' \in X$, $\ln f(x) + \ln f(x') \leq \ln f(x \vee x') + \ln f(x \wedge x')$. Combining, this implies that $\ln f(x) f(x') \leq \ln f(x \vee x') f(x \wedge x')$, which implies that $f(x) f(x') \leq f(x \vee x') f(x \wedge x')$. Note that the log transform means that all involved values are strictly positive. Assume that $f(x) \geq f(x \wedge x')$. This means that

$$f(x') \cdot \underbrace{\frac{f(x)}{f(x \wedge x')}}_{\geq 1} \leq f(x \vee x') \implies f(x \vee x') \geq f(x')$$

A similar proof holds when assuming that $f(x) > f(x \wedge x')$, replacing the weak inequalities with strict. Thus, f is quasi-supermodular. $\square$

Exercise 3. Let $f : \mathbb{R}^d \to \mathbb{R}$ denote the firm's production function and consider the firm's profit maximization problem

$$\max_{y \in \mathbb{R}^d_+} pf(y) - q \cdot y$$

where p and q are output and input prices respectively. Suppose that f is nondecreasing and supermodular. Prove that if the price of the firm's output increases and/or the price of any of its inputs decreases, then the firm increases the use of all of its inputs.

Proof. Define the optimal input correspondance as follows:

$$Y^\star(p, q) = \operatorname*{argmax}_{y \in \mathbb{R}^d_+} pf(y) - qy$$

Since $\mathbb{R}^d_+$ is a lattice with a full order, we have by Milgrom and Shannon that if f is quasi-supermodular in $x \in \{p, -q\}$ and has single-crossing differences in $x \in \{p, -q\}$ and y, then $Y^\star$ is monotone increasing in the strong set order. Thus, as p and $-q$ increase (*i.e.*, as p increases and q decreases), $Y^\star(p, q)$ will increase in the strong set order, meaning that the optimal y will (weakly) increase in all elements. $\square$

Exercise 4. Let $f : \mathbb{R}^d \to \mathbb{R}$, and define $c : \mathbb{R} \to \mathbb{R}$ as

$$c(x) \coloneqq \min_{y \in \mathbb{R}^d_+ : f(y) \geq x} q \cdot y$$

Show that

$$X^\star(p) \in \operatorname*{argmax}_{x \in \mathbb{R}} px - c(x)$$

is increasing in the strong set order.

Proof. Note first that $c(x)$ is non-decreasing in x, as feasibility requires a y such that $f(y) \geq x$, and decreasing the feasible set over a minimization problem can never decrease $c(x)$. From Berge's Theorem, $c(x)$ is a convex and differentiable function. Thus, $x' \in X^\star(p)$ if and only if the first order conditions hold, so when $p - c'(x) = 0 \implies c'(x') = p$. Since c is convex, $c'(x)$ is increasing in x. Taking some $p < p'$, we set x, x' such that $c'(x) = p$ and $c'(x') = p'$. Since c' is increasing in x, $p' > p \implies x' > x$. $\square$

Exercise 5. Let Θ be a poset. The function $F : \mathbb{R}^{d_1}_{++} \times \mathbb{R}^{d_2}_{++} \times \Theta \to \mathbb{R}$ has increasing differences in $((x, y); \theta)$ and, for each $\theta \in \Theta$, $F(\cdot, \theta)$ is a supermodular function. Suppose that

$$(x', y') \in \operatorname*{argmax}_{(x,y) \in \mathbb{R}^{d_1}_{++} \times \mathbb{R}^{d_2}_{++}} F(x, y, \theta')$$

Let $\theta'' > \theta'$ and assume that the sets

$$Z^\star(y', \theta'') \coloneqq \operatorname*{argmax}_{x \in \mathbb{R}^{d_1}_{++}} F(x, y', \theta'')$$
$$Z^{\star\star}(\theta'') \coloneqq \operatorname*{argmax}_{(x,y) \in \mathbb{R}^{d_1}_{++} \times \mathbb{R}^{d_2}_{++}} F(x, y, \theta'')$$

are nonempty and compact. Show that there is an $x^\star$ and $(x^{\star\star}, y^{\star\star})$ with the following properties:

(i) $x^\star \in Z^\star(y', \theta'')$

(ii) $(x^{\star\star}, y^{\star\star}) \in Z^{\star\star}(\theta'')$

(iii) $x' \leq x^\star \leq x^{\star\star}$

(iv) $y' \leq y^{\star\star}$

Proof. Define $(x^{\star\star}, y^{\star\star}) = \sup Z^{\star\star}(\theta'')$. Note first that $F(x', y', \theta') \geq F(x, y, \theta')$ for all x, y feasible. Since $\theta'' > \theta'$ and F has increasing differences in $((x, y); \theta)$, it follows that $F(x', y', \theta'') \geq F(x, y', \theta'')$ for all $x \in \mathbb{R}^{d_1}_{++}$, meaning that $x \in Z^{\star}(y', \theta'')$. Additionally, since $(x^{\star\star}, y^{\star\star}) = \sup Z^{\star\star}(\theta'')$, it must be the case that $F(x^{\star\star}, y^{\star\star}, \theta'') \geq F(x', y', \theta'')$. Since F is supermodular in (x, y), it is increasing in y, meaning that $y^{\star\star} \geq y'$.

Define $x^{\star} = \sup Z^{\star}(y', \theta'')$. Since from above we have that $x' \in Z^{\star}(y', \theta'')$, we have that $x^{\star} \geq x'$. Finally, since we have that $(x^{\star\star}, y^{\star\star}) \in Z^{\star\star}(\theta'')$, we have that $F(x^{\star\star}, y^{\star\star}, \theta'') \geq F(x^{\star}, y^{\star\star}, \theta'')$. Since F is supermodular, it is increasing in x, so we must have that $x^{\star\star} \geq x^{\star}$. $\square$