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ECON 6130 Notes:
A Search and Matching Model

The problem sets in this course will focus on a "simple" Neoclassic model with search and matching in the labor market. Intellectually, search is an important feature for our models to be able to speak to unemployment and, depending on the wage-setting design, can do a better job matching macro data than a model with frictionless labor markets. Computationally, search makes employment a state variable, which makes our job as computations macroeconomists a little more interesting. The following notes present the model we will be using, in two different formulations.

1 Planner's Economy

We first introduce the planner's version of our model economy. This will make our derivation of the equilibrium conditions much easier. But it also eliminates some interesting externalities that play a crucial role in many analyses of model with search.

In the planner economy, consumer welfare is given by

$$U(\{C_t\}) \equiv \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma} \tag{1}$$

The household supplies a unit of labor inelastically, $S_t = 1$.

The production side of the economy is specified by a production function

$$Y_t = A_t K_t^{\alpha} N_t^{1-\alpha}, \tag{2}$$

a law of motion that describes how capital evolves,

$$K_{t+1} = (1-\delta_k)K_t + I_t, \tag{3}$$

and a law of motion that describes how workers match and separate with firms,

$$N_t = (1-\delta_n)N_{t-1} + M(V_t, S_t). \tag{4}$$

In the above, $M(V_t, S_t)$ is an aggregate matching function that determines how many new workers the firm can hire as function a function of the number of vacancies posted in the economy, V_t, and the number of workers searching for positions $S_t = 1$. This function is assumed to be a constant returns to scale function. The parameters δ_k measures how quickly capital depreciates, and δ_n is the exogenous rate of separation of workers who leave their jobs in a period.

The aggregate resource constraint is

$$Y_t = C_t + I_t + \phi_n V_t. \tag{5}$$

The planner problem is therefore

$$\max_{\{C_t, I_t, V_t, Y_t, N_t, K_{t+1}\}} E_0 \left[U(\{C_t\}) \right]$$

subject to (2) - (5). We could eliminate variables and constraints from this problem, but we'll wait until later to do that.

We'll now derive necessary optimality conditions using two different techniques. Eventually, we'll see that the two approach suggest different methods for solving the model.

1.1 Solution Method 1: The Lagrangian

The Lagrangian of the model is:

$$\begin{aligned} \max_{\{C_t, I_t, V_t, Y_t, N_t, K_{t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ & \frac{C_t^{1-\sigma}}{1-\sigma} + \lambda_{1,t} \left(A_t K_t^\alpha N_t^{1-\alpha} - Y_t \right) \\ & + \lambda_{2,t} \left((1-\delta_k) K_t + I_t - K_{t+1} \right) \\ & + \lambda_{3,t} \left((1-\delta_n) N_{t-1} + M(V_t, S_t) - N_t \right) \\ & + \lambda_{4,t} \left(Y_t - C_t - I_t - \phi_n V_t \right) \Bigg\} \end{aligned}$$

The model's necessary first order conditions – without any simplification – are:

$$\begin{aligned} 0 &= C_t^{-\sigma} - \lambda_{4,t} && \text{(C)} \\ 0 &= \lambda_{2,t} - \lambda_{4,t} && \text{(I)} \\ 0 &= \lambda_{3,t} M_v(V_t, S_t) - \phi_n \lambda_{4,t} && \text{(V)} \\ 0 &= -\lambda_{1,t} + \lambda_{4,t} && \text{(Y)} \\ 0 &= \lambda_{1,t} A_t K_t^\alpha (1-\alpha) N_t^{-\alpha} - \lambda_{3,t} + \beta E_t \left[\lambda_{3,t+1} (1-\delta_n) \right] && \text{(N)} \\ 0 &= -\lambda_{2,t} + \beta E_t \left[\lambda_{1,t+1} A_{t+1} \alpha K_{t+1}^{\alpha-1} N_{t+1}^{1-\alpha} + \lambda_{2,t+1} (1-\delta_k) \right]. && \text{(K)} \end{aligned}$$

Notice that I've been careful to state my constraints so that all Lagrange multipliers turn out to be positive. The equations above imply that $\lambda_{1,t} = \lambda_{2,t} = \lambda_{4,t} = C_t^{-\sigma}$ and $\lambda_{3,t} = C_t^{-\sigma} \phi_n / M_v(V_t, S_t)$, so we eliminate these variables from everywhere. The simplified first order conditions are

$$1 = \beta E_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \left(A_{t+1} \alpha (K_{t+1}/N_{t+1})^{\alpha-1} + 1 - \delta \right) \right] \tag{6}$$

$$\frac{\phi_n}{M_v(V_t, S_t)} = A_t (1-\alpha) (K_t/N_t)^\alpha + \beta E_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{\phi_n}{M_v(V_{t+1}, S_{t+1})} (1-\delta_n) \right], \tag{7}$$

plus all of constraints (2) - (5) again.

Equation (6) is often called the "capital Euler Equation", or sometimes the "intertemporal Euler equation" although obviously there is more than one intertemporal optimality condition in this model. This equation equates the cost of purchasing one unit of investment (the consumptions-investment exchange rate is just unity) and the benefits that it brings. These benefits only occur the following period, when the capital earns its marginal product and a fraction of it continue on into the second period.

Equation (7) is the "labor Euler equation" or, sometimes, the "vacancy posting condition", since it equates the cost of hiring an additional work (equal to the cost of posting a position ϕ, divided the by the marginal increase in the number of jobs created by that posting, $M_v(V_t, S_t)$) with the returns of having that worker. Notice that since we assume the work starts producing in the same period, their marginal product is in terms of today's technology A_t. The second term on the right-hand-side of (7) is the value of having a worker (equivalently, the cost of replacing a worker) times the probability that the worker hired today does not separate between periods.

In both (6) and (7), returns that happen in the future are discounted to be in terms of units of consumption today by $M_{t+1} \equiv \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}$. This object M_{t+1} is often referred to as the "stochastic discount factor" because it converts future returns which may be uncertain into certainty-equivalent units of consumption today. It has a very important role in macro-finance, asset pricing, as well as macroeconomics models where risk plays an important role. Please take note that M_{t+1} is not related with the matching function $M(\cdot)$ or its derivative $M_v(\cdot)$: I've chosen this confusing notation because these objects often appear in this literature with similar notation. The fact that this notation is fairly standard give you a hint that asset pricing and labor search models are not studied together all that often.

The Matching Function

In the above, we've left the matching function $M(V_t, S_t)$ as a generic function. Most of the literature following Mortensen and Pissarides (1994) has assumed that the matching function satisfies the standard assumption of a Neoclassical production function. For now we will assume

$$M(V_t, S_t) = \chi V_t^{\varepsilon} S_t^{1-\varepsilon}. \tag{8}$$

This function has lots of advantages, but also a big disadvantage: it doesn't guarantee that the probability that a vacancy posted is filled by a worker actually falls in $[0, 1]$. Or, relatedly, that the number of people hired actually is less than the number of people looking for jobs. You can look at den Haan et al. (2000) for some ideas about other possible matching functions that are more micro-founded, though they can also be a bit harder to work with.

Let now impose this form and our assumption that $S_t = 1$, to get

$$M_v(V_t, S_t) = \varepsilon \chi V_t^{\varepsilon-1}. \tag{9}$$

Exogenous Processes

The final step to closing our model is to make some assumption about the exogenous processes in the model economy. We could assume that any of the parameters we've named are time-varying, and indeed some of these "shocks" have appeared in all kinds of DSGE models.

Instead, for now, we'll assume that technology is the only thing that is moving exogenously. In particular, we'll assume

$$\log(A_{t+1}) = \rho \log(A_t) + \sigma_a \epsilon_{t+1}. \tag{10}$$

Steady State

Before we approach our model numerically, we are almost always going to want to find the value of endogenous variables at the models steady state. (If the model has a trend in productivity, then we are going to look for the balanced growth path, which can be surprising tricky!). To find the steady state, we can write necessary equilibrium conditions (2) - (7) again but dropping time subscripts everywhere. Our assumptions about productivity imply that $A = 1$ in a non-stochastic steady state.

$$Y = K^\alpha N^{1-\alpha} \tag{11}$$
$$I = \delta_k K \tag{12}$$
$$N = \frac{\chi}{\delta_n} V^\varepsilon \tag{13}$$
$$Y = C + I + \phi_n V \tag{14}$$
$$1 = \beta\left(\alpha (K/N)^{\alpha-1} + 1 - \delta\right) \tag{K^{ss}}$$
$$\frac{\phi_n}{\varepsilon \chi V^{\varepsilon-1}} = (1-\alpha)(K/N)^\alpha + \beta \frac{\phi_n}{\varepsilon \chi V^{\varepsilon-1}}(1-\delta_n) \tag{V^{ss}}$$

As is often the case in models based on the Neoclassical growth model, equation (K^{ss}) is going to be a good place to start to "unwind" these equations. We find that the steady-state capital to labor ratio is:

$$\frac{K}{N} = \left(\frac{\beta^{-1} - 1 + \delta}{\alpha}\right)^{\frac{1}{\alpha-1}}. \tag{15}$$

Often in models like this it easy to find K/N and hard to find their levels. But the search side of this economy actually helps us in this case. Using (V^{ss}), we can do some rearranging to find

$$V = \left(\frac{\varepsilon \chi}{\phi_n} \frac{(1-\alpha)}{1 - \beta(1-\delta_n)} (K/N)^\alpha\right)^{\frac{1}{1-\varepsilon}}. \tag{16}$$

From here, equation (13) lets us back out N. We can can then get the level of K from the ratio. The levels Y and I come from equations (11) and (12). Finally, we can back out C from (14).

Whenever we want to solve a model, its a very good idea to have this steady state worked out. First, in doing so, we'll often catch mistakes with out dynamic equations. Second, we're able to see peculiarities about our model that are hard to see otherwise. Here, for example, its clear could find parameters that lead to negative consumption in steady-state. That's not a great implication for a model to have and have to decide if we think this is a sign of fundamental problem with the coherence of the model, or just a situation that we can ignore because, for example, it only occurs at really strange values of the parameters.

1.2 Solution Method 2: The Value function

To solve the planner's problem with the value function approach, we first set up our Bellman Equation:

$$V\left(K_t, N_{t-1}, A_t\right) = \max_{K_{t+1}, N_t} U_t\left(C_t\right) + \beta E_t V\left(K_{t+1}, N_t, A_{t+1}\right) \tag{17}$$

subject to

$$\begin{aligned} C_t &= Y_t + I_t + \phi V_t & (18) \\ &= A_t K_t^{\alpha} N_t^{1-\alpha} - \left(K_{t+1} - \left(1-\delta_k\right) K_t\right) - \phi_n \left(\frac{N_t - \left(1-\delta_n\right) N_{t-1}}{\chi}\right)^{\frac{1}{\varepsilon}}. & (19) \end{aligned}$$

The second equality uses $Y_t = A_t K_t^{\alpha} N_t^{1-\alpha}$, $I_t = K_{t+1} - \left(1-\delta_k\right) K_t$. Also, we know that

$$V_t = \left(\frac{N_t - \left(1-\delta_n\right) N_{t-1}}{\chi}\right)^{\frac{1}{\varepsilon}} \tag{20}$$

from rearranging $N_t = \left(1-\delta_n\right) N_{t-1} + \chi V_t^{1-\varepsilon} S_t^{\varepsilon}$ and imposing that $S_t = 1$.

Imposing these constraints, we have:

$$V\left(K_t, N_{t-1}, A_t\right) = \max_{K_{t+1}, N_t} U\left(A_t K_t^{\alpha} N_t^{1-\alpha} - \left(K_{t+1} - \left(1-\delta_t\right) K_t\right) - \phi_n \left(\frac{N_t - \left(1-\delta_n\right) N_{t-1}}{\chi}\right)^{\frac{1}{\varepsilon}}\right) + \beta E_t V\left(K_{t+1}, N_t, A_{t+1}\right). \tag{21}$$

The first-order conditions are:

$$-U'\left(C_t\right) + \beta E_t V_K\left(K_{t+1}, N_t, A_{t+1}\right) = 0 \qquad \left(K_{t+1}\right)$$

$$U'\left(C_t\right)\left[-\frac{\phi_n}{\varepsilon\chi}\left(\frac{N_t - \left(1-\delta_n\right) N_{t-1}}{\chi}\right)^{\frac{1}{\varepsilon}-1} + A_t\left(\frac{K_t}{N_t}\right)^{\alpha}\left(1-\alpha\right)\right] + \beta E_t V_N\left(K_{t+1}, N_t, A_{t+1}\right) = 0 \qquad \left(N_t\right)$$

Side bar: envelope theorem

To proceed, we need to compute the derivatives $V_K(\cdot)$ and $V_N(\cdot)$. Basically, the envelop theorem says that we ignore the "chain rule" terms in taking these derivatives, and just focus on the "obvious" appearances of the states K_t and N_{t-1}. Let us do a quick proof of the result we need for $V_K(\cdot)$.

Suppose that we already knew the optimal policy functions $K_{t+1} = \mathbf{K}(K_t, N_{t-1}, A_t)$ and $N_t = \mathbf{N}(K_t, N_{t-1}, A_t)$. Then we could write the value function as

$$\begin{aligned} V\left(K_t, N_{t-1}, A_t\right) = U\Bigg(A_t K_t^{\alpha} N_t^{1-\alpha} - \left(\mathbf{K}(K_t, N_{t-1}, A_t) - \left(1-\delta_t\right) K_t\right) - \phi_n \left(\frac{\mathbf{N}(K_t, N_{t-1}, A_t) - \left(1-\delta_n\right) N_{t-1}}{\chi}\right)^{\frac{1}{\varepsilon}}\Bigg) + \\ \beta E_t V\left(\mathbf{K}(K_t, N_{t-1}, A_t), \mathbf{N}(K_t, N_{t-1}, A_t), A_{t+1}\right) \end{aligned}$$

<u>without the max operator.</u> Now let's take the derivative for K_t, using all the appropriate chain rules:

$$\begin{aligned} V_K\left(K_t, N_{t-1}, A_t\right) = \quad & U'\left(C_t\right) \cdot \left[A_t\left(\frac{K_t}{N_t}\right)^{\alpha-1}\alpha + \left(1-\delta_k\right)\right] & (22) \\ & - U'(C_t)\frac{\partial \mathbf{K}}{\partial K_t} + \beta E_t V_K\left(K_{t+1}, N_t, A_{t+1}\right)\frac{\partial \mathbf{K}}{\partial K_t} & (23) \\ & + U'(C_t)\left[-\frac{\phi_n}{\varepsilon\chi}\left(\frac{N_t - \left(1-\delta_n\right) N_{t-1}}{\chi}\right)^{\frac{1}{\varepsilon}-1} + A_t\left(\frac{K_t}{N_t}\right)^{\alpha}\left(1-\alpha\right)\right]\frac{\partial \mathbf{N}}{\partial K_t} + \beta E_t V_N\left(K_{t+1}, N_t, A_{t+1}\right)\frac{\partial \mathbf{N}}{\partial K_t} & (24) \end{aligned}$$

But, now compare the terms in equation (23) to those in the first-order condition (K_{t+1}). The terms in (23) must sum to zero! The same is true for the term in (24). So that we have

$$V_K\left(K_t, N_{t-1}, A_t\right) = U'\left(C_t\right) \cdot \left[A_t \left(\frac{K_t}{N_t}\right)^{\alpha-1} \alpha + \left(1-\delta_k\right)\right]. \tag{25}$$

Back to the FOCs

So, using envelope theorem for both K and N, we find:

$$V_K\left(K_t, N_{t-1}, A_t\right) = U'\left(C_t\right) \cdot \left[A_t \left(\frac{K_t}{N_t}\right)^{\alpha-1} \alpha + \left(1-\delta_k\right)\right] \tag{26}$$

$$V_N\left(K_{t+1}, N_t, A_t\right) = U'\left(C_t\right) \cdot \left[\frac{\phi_n}{\varepsilon\chi}\left(\frac{N_t - \left(1-\delta_n\right)N_{t-1}}{\chi}\right)^{\frac{1}{\varepsilon}-1}\left(1-\delta_n\right)\right] \tag{27}$$

Combine the FOCs – (K_{t+1}), (N_t) – and Envelope conditions – (26) and (27), – to get:

$$U_t'(\cdot) = \beta E_t U_{t+1}'(\cdot)\left[A_{t+1}\left(\frac{K_{t+1}}{N_{t+1}}\right)^{\alpha-1}\alpha + \left(1-\delta_k\right)\right] \tag{28}$$

$$U_t'(\cdot)\left[\frac{\phi_n}{\varepsilon\chi}\left(\frac{N_t - \left(1-\delta_n\right)N_{t-1}}{\chi}\right)^{\frac{1}{\varepsilon}-1}\right] = U_t'(\cdot)\left[A_t\left(\frac{K_t}{N_t}\right)^{\alpha}\left(1-\alpha\right)\right] \tag{29}$$

$$+\beta E_t U_{t+1}'(\cdot)\left[\frac{\phi_n}{\varepsilon\chi}\left(\frac{N_{t+1} - \left(1-\delta_n\right)N_t}{\chi}\right)^{\frac{1}{\varepsilon}-1}\left(1-\delta_n\right)\right]. \tag{30}$$

Rearranging, and imposing our functional form for the matching functions, these can the identical "capital Euler equation" and "labor Euler equation" that we found for the planner problem.

2 The Decentralized Economy

Household Consider a household that chooses consumption and investment in shares of the representative firm. The household's objective is to maximize the following expected discounted sum of utilities:

$$\max_{\{C_t, S_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{\frac{C_t^{1-\sigma}}{1-\sigma} + \lambda_t \left[W_t N_t + S_{t-1}\left(D_t + P_t\right) - C_t - S_t P_t\right]\right\}$$

where C_t is the consumption at time t, σ is the coefficient of relative risk aversion, λ_t is the Lagrange multiplier on the budget constraint at time t, W_t is the wage rate at time t, N_t is the amount of labor supplied by the household at time t, S_t is the amount of shares of the representative firm held by the household at time t, D_t is the dividend paid by the representative firm at time t, P_t is the price of a share of the representative firm at time t, and β is the discount factor.

The household's first-order condition for consumption C_t is:

$$C_t^{-\sigma} = \lambda_t$$

The household's first-order condition for investment in shares S_t is:

$$\lambda_t P_t = \beta \mathbb{E}_t \left[\lambda_{t+1} \left(D_{t+1} + P_{t+1}\right)\right]$$
$$\rightarrow P_t = \beta \mathbb{E}_t \left[\underbrace{\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}}_{\text{Stochastic Discount Factor}} \left(P_{t+1} + D_{t+1}\right)\right]$$

And note that $S_t = 1$ for a representative agent model.

Firms The firm chooses the optimal levels of output, vacancies, investment, employment, and capital stock, and pays a wage w_t. The probability a vacancy gets filled, Q_t, is taken as exogenous.

The firm's objective function represents the discounted present value of profits, where profits are defined as revenue minus costs:

$$\begin{aligned} \max_{\{Y_t, I_t, V_t, K_{t+1}, N_{t+1}\}} E_0 \sum_{t=1}^{\infty} \beta^t \Bigg\{ & \frac{\lambda_t}{\lambda_0} \left[Y_t - w_t N_t - I_t - \phi_n V_t\right] \\ & + \frac{1}{\lambda_0} \Gamma_{1,t} \left[\left(1 - \delta_n\right) N_{t-1} + V_t Q_t - N_t\right] \\ & + \frac{1}{\lambda_0} \Gamma_{2,t} \left[\left(1 - \delta_k\right) K_t + I_t - K_{t+1}\right] \\ & + \frac{1}{\lambda_0} \Gamma_{3,t} \left[A_t K_t^{\alpha} N_t^{1-\alpha} - Y_t\right] \Bigg\} \end{aligned}$$

The firm's first order conditions are:

$$\frac{\lambda_t}{\lambda_0} = \frac{1}{\lambda_0} \Gamma_{3,t} \Rightarrow \lambda_t = \Gamma_{3,t} \tag{Y}$$
$$\frac{\lambda_t}{\lambda_0} = \frac{1}{\lambda_0} \Gamma_{2,t} \Rightarrow \lambda_t = \Gamma_{2,t} \tag{I}$$
$$\frac{\lambda_t}{\lambda_0} \phi_n = \frac{1}{\lambda_0} \Gamma_{1,t} Q_t \Rightarrow \Gamma_{1,t} = \lambda_t \frac{\phi_n}{Q_t} \tag{V}$$
$$\frac{\Gamma_{2,t}}{\lambda_0} = \beta E_t \left[\frac{\Gamma_{3,t+1}}{\lambda_0} A_{t+1} \alpha \left(\frac{K_{t+1}}{N_{t+1}}\right)^{\alpha - 1} + \frac{1}{\lambda_0} \Gamma_{2,t+1} \left(1 - \delta_k\right)\right] \tag{K}$$
$$0 = \frac{-\lambda_t}{\lambda_0} w_t - \frac{1}{\lambda_0} \Gamma_{1,t} + \frac{1}{\lambda_0} \Gamma_{3,t} A_t \left(\frac{K_t}{N_t}\right)^{\alpha} \left(\alpha - 1\right) + \frac{\beta}{\lambda_0} E_t \left[\Gamma_{1,t+1} \left(1 - \delta_n\right)\right] \tag{N}$$

We can reorganize these FOCs as:

$$\frac{\phi_n}{Q_t} = A_t \left(K_t / N_t\right)^{\alpha} \left(1 - \alpha\right) - w_t + \beta E_t \left[\frac{\lambda_{t+1}}{\lambda_t} \left(1 - \delta_n\right) \frac{\phi_n}{Q_{t+1}}\right] \tag{Decentralized}$$

Now recall the Social Planner's Problem gave us:

$$\frac{\phi_n}{M_v(\cdot)} = A_t\left(\frac{K_t}{N_t}\right)^{\alpha}(1-\alpha) + \beta E_t\left[\frac{\lambda_{t+1}}{\lambda_t}(1-\delta_n)\frac{\phi_n}{M_v(\cdot)}\right] \tag{Social Planner}$$

, where $M_v(\cdot) = \chi\varepsilon V_t^{\varepsilon-1}S_t^{1-\varepsilon}$.

Now let's examine the differences colored in the two equations above. Notice that the probability of a vacancy being matched is defined as

$$Q_t = \frac{\text{Matches}}{\text{Vacancies}} = \frac{\chi V_t^{\varepsilon}S_t^{1-\varepsilon}}{V_t} = \chi\left(\frac{S_t}{V_t}\right)^{1-\varepsilon}$$

, while the marginal increase in the number of jobs created by a marginal posting is

$$M_v(V_t, S_t) = \chi\varepsilon\left(\frac{S_t}{V_t}\right)^{1-\varepsilon}$$

.

If we plug in these expressions into our "labor Euler equations" in the decentralized and planner economy, we will arrive at:

$$\frac{\phi_n}{\chi}\left(\frac{V_t}{S_t}\right)^{1-\varepsilon} = MPL_t - w_t + \beta E_t\left[\frac{\lambda_{t+1}}{\lambda_t}(1-\delta_n)\frac{\phi_n}{\chi}\left(\frac{V_{t+1}}{S_{t+1}}\right)^{1-\varepsilon}\right] \tag{Decentralized}$$

$$\frac{\phi_n}{\chi\varepsilon}\left(\frac{V_t}{S_t}\right)^{1-\varepsilon} = MPL_t + \beta E_t\left[\frac{\lambda_{t+1}}{\lambda_t}(1-\delta_n)\frac{\phi_n}{\chi\varepsilon}\left(\frac{V_{t+1}}{S_{t+1}}\right)^{1-\varepsilon}\right], \tag{Social Planner}$$

where the two differences are marked in red and blue.

The first difference appears as in the decentralized economy, the firm could not internalize the fact that the probability of successful matching decreases (i.e. the labor market becomes more crowded) when an additional vacancy is posted. But the planner could internalize this and thus faces a higher perceived cost of hiring (as $\varepsilon < 1$).

The second difference appears as in the decentralized economy, the firm experiences lower benefits than the social benefits (MPL) since it only gets to keep the value of marginal product after compensating the marginal labor for its wage.

Now, let's think about how to make the two conditions coincide with each other such that the decentralized economy would be efficient. Such scenario would only happen if we have:

$$MPL_t - w_t = \varepsilon MPL_t \tag{31}$$

Nash Bargaining of Wages In a search model (like ours), there is a surplus generated by a match (namely MPL + the continuation value). For now, let us assume a worker gets zero earnings if they have no job. Then, loosely speaking, any wage between 0 and MPL plus continuation value is consistent with equilibrium, i.e. something that both workers and firms would prefer to walking away from the bargaining table.

Now we focus on a particular wage setting paradigm known as "Nash Bargaining", in which workers and firms bargain over the current wage, taking as given the wage that will be realized in future periods.

To analyze this bargaining problem, we need to compute the surplus that firms and workers earn if they stay in a given working arrangement. Denote $\bar{W}_t(w_t)$ to be the workers' surplus and $J_t(w_t)$ the firms' surplus. Given the relative bargaining power for both sides (η being the bargaining power of households and $1-\eta$ being the bargaining power of the firm), the "bargaining problem" maximizes the following Cobb-Douglas product:

$$\max_{w_t}\left[\bar{W}_t(w_t)\right]^{\eta}\left[J_t(w_t)^{1-\eta}\right],$$

which captures the idea that workers and firms share the surplus of a match according to their "bargaining powers".

Now let's try to write down the worker's surplus and the firm's surplus. If the worker and firm come to no agreement, and the bargaining collapses, then the firm will pay no wage and earn no profit: it's outside option is worth zero. On the other hand, if an agreement to pay wage w_t is made between both parties after bargaining, then the value of the match to the firm can be written as

$$J_t = \underbrace{MPL_t - w_t}_{\text{Today's Profits}} + \underbrace{\beta(1-\delta_n)E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\delta} J_{t+1}\right]}_{\text{Future Value if not separated}}. \tag{J}$$

We need to make some additional assumptions about what happens to the household if no agreement to work is reached. Our starting assumption is going to be that the household earns no wages (or no unemployment benefits). Under our slightly unusual assumptions $S_t =$ is fixed exogenously, so having an additional unemployed worker does not change the stock of potential workers the representative household can send out for employment tomorrow, and the continuation value of an unemployed worker is zero.

On the other hand, if the worker and firms agree on wage w_t, then the value of the match to worker is:

$$\bar{W}_t = \underbrace{w_t}_{\text{Today's Wage}} + \underbrace{\beta(1-\delta_n)E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\delta} \bar{W}_{t+1}\right]}_{\text{Future Value if not separated}} + \underbrace{\beta\delta_n E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\delta} \bar{U}_{t+1}\right]}_{\text{Future Value if separated=0}}. \tag{W}$$

With the expressions for $\bar{W}_t(w_t)$ and $J_t(w_t)$ in hand, we can finally attempt to solve the Nash Bargaining problem:

$$\max_{w_t}\left[\bar{W}_t(w_t)\right]^{\eta}\left[J_t(w_t)^{1-\eta}\right]$$

Taking the FOC w.r.t. w_t yields:

$$\eta\bar{W}_t(w_t)^{\eta-1}J_t(w_t)^{1-\eta} - \bar{W}_t(w_t)^{\eta}\cdot(1-\eta)J_t(w_t)^{-\eta} = 0$$

Rearranging this, we have:

$$\bar{W}_t(w_t) = \frac{\eta}{1-\eta}J_t(w_t) \tag{32}$$

We then substitute (32) into (W) to get

$$\frac{\eta}{1-\eta}J_t = w_t + \beta\left(1-\delta_n\right)E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\delta}\frac{\eta}{1-\eta}J_{t+1}\right].$$

Now, using (J) to write the left-hand-side differently:

$$\frac{\eta}{1-\eta}\left\{MPL_t - w_t + \beta\left(1-\delta_n\right)E\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\delta}J_{t+1}\right]\right\} = w_t + \beta\left(1-\delta_n\right)E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\delta}\frac{\eta}{1-\eta}J_{t+1}\right]$$

Simplifying this, we get:

$$\eta\left(MPL_t - w_t\right) = (1-\eta)w_t \tag{33}$$

$$\boxed{w_t = \eta\cdot MPL_t + (1-\eta)\cdot 0}, \tag{34}$$

which is the solution to the Nash Bargaining problem. Note that 0 is simply the value of the worker's outside option.

Efficiency Now recall that by comparing the labor Euler equation for the decentralized economy and the planner economy, we arrived at the conclusion that if (31) holds ($MPL_t - w_t = \varepsilon MPL_t$), then the decentralized economy would be efficient.

Under Nash Bargaining with $MPL_t - w_t = (1-\eta)$, this efficient condition would be achieved if

$$\eta = 1-\varepsilon. \tag{35}$$

This is known as the "Hosios efficiency condition".

To gain some intuition on why efficiency might still be achieved given the externalities in the decentralized model, we take the stance from the firm and consider the following two forces:

1. The firm pays the entire vacancy cost, but only receives a fraction of the surplus from the match, which suggests that firm's incentives to post vacancies are lower than optimum. This force is governed by η.

2. The firm does not rationalize the fact that the labor market becomes more congested as it posts its vacancies, which means that the firm's incentives to post vacancies are higher than optimum. This force is governed by ε, the curvature in the matching function.

Under the parameterization $\eta = 1-\varepsilon$, the two opposing externalities cancel out with each other and efficiency is achieved.

References

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