ECON 6170

Problem Set 6

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Exercise 1. Define $F : \mathbb{R} \rightrightarrows \mathbb{R}$ as follows:

$$F(x) = \begin{cases} [0,1] & x > 0 \\ \{1\} & x \leq 0 \end{cases}$$

Clearly, $F(0) = \{1\}$ is closed because finite sets are compact, and compact sets are closed and bounded. However, taking $\{x_n\}$ such that $x_n = \frac{1}{n}$, and $\{y_n\}$ such that $y_n = 0 \ \forall \ n$, we have that $x_n \to 1 \in f(0)$, $y_n \in F(x_n) \ \forall \ x_n$, but $y_n \to 0 \notin F(0)$, so F is not closed.

Exercise 2. False! Consider the example of $F : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by

$$F(x) = \begin{cases} \{0\} & x \neq 0 \\ \left(-\frac{1}{2}, \frac{1}{2}\right) & x = 0 \end{cases}$$

Using the topological definition of upper hemi-continuity, it's clear that F is upper hemi-continuous, taking $O \subseteq (-0.1, 0.1)$ at any $x \neq 0$ and $O = (-1/2, 1/2)$ at $x = 0$. However, $F(x)$ is clearly open at $x = 0$.

Exercise 3. Are the correspondences upper or lower hemi-continuous?

(a) This correspondence is upper hemi-continuous, but not lower hemi-continuous. We will use the topological definition for both. First, consider upper hemi-continuity. If $\exists \ O$ s.t. $F(x) \subseteq O$ for some x and open O, it must be the case that if $x < 3$, $\exists \ \delta > 0$ s.t. $B_\delta(2 - x), B_\delta(4 - x) \subseteq O$. Then we can choose $\varepsilon < \delta$, and have that (as long as x is sufficiently far from 3, where otherwise we could choose the distance between it and 3) $F(B_\varepsilon(x)) \subseteq O$. If $x = 3$, the same holds for the left, and since $3 - x \in [2 - x, 4 - x]$, choosing $\varepsilon < 1$ guarantees it for the right. Finally, if $x > 3$, $\exists \ \delta > 0$ s.t. $B_\delta(3 - x) \subseteq O$, meaning that for $\varepsilon < \delta$, $F(B_\varepsilon(x)) \subseteq O$.

This correspondence is not lower hemi-continuous because taking $x = 3$, the open ball $(1/2, 3/2)$ intersects $F(x)$ but does not intersect $F(x + \varepsilon)$ for any $0 < \varepsilon < 1/2$.

(b) This correspondence is neither upper nor lower hemi-continuous. We will use the topological definition again for both. First, take $O = (1/4, 13/4)$ at $x = 2$. Since $G(x) = [1, 3]$, $G(x) \subseteq O$. However, taking $\varepsilon = 1/8$, $G(x - \varepsilon) = \{1/8, 17/8\} \not\subseteq O$. Thus, G is not upper hemi-continuous. The same proof as part (a) works for demonstrating that G is not lower hemi-continuous, taking the open ball $O = (3/2, 5/2)$ at $x = 3$, which intersects $F(x)$ but not $F(x + \varepsilon)$ for $\varepsilon = 1/2$.

Exercise 4. Prove that the budget correspondence is continuous.

Proof. First, define the function $f_{m,p}(x) = m - p \cdot x$. This is an affine, and thus continuous, convex, and concave function for each m, p, and the budget correspondence is now equivalent to $\Gamma(p, m) = \{x \in \mathbb{R}^d_+ \mid p \cdot x \leq m\} = \{x \in \mathbb{R}^d_+ \mid f_{m,p}(x) \geq 0\}$. Since this is an upper contour set of a quasiconcave function, it is convex. Since it is a decreasing function bounded below by 0, it is bounded, and since $f_{m,p}(x)$ is affine and thus a continuous isomorphism, it is closed. Thus, Γ is compact-valued.

Take some open set $O \subseteq \mathbb{R}^d$ such that $\Gamma(p,m) \subseteq O$. Since Γ is compact valued and $f_{m,p}$ is continuous, and since $m \in \mathbb{R}_{++}$ meaning that $p \cdot 0 < m$, there exists $x \in \mathbb{R}^d_+$ such that $f_{m,p}(x) > 0$. Take $\delta > 0$ such that $f_{m,p}(y) > 0 \; \forall \; y \in B_\delta(x)$. Then, since the family of affine functions $f_{m',p'}$ is continuous in m', p', there exists $\varepsilon > 0$ such that if $(m',p') \in B_\varepsilon(m,p)$, $f_{m',p'}(y) \geq 0 \; \forall \; y \in B_\delta(x)$. Thus, $\Gamma(B_\varepsilon(m,p)) \subseteq O$, and Γ is upper hemi-continuous.

Take some open set $O \subseteq \mathbb{R}^d$ such that $\Gamma(p,m) \cap O \neq \varnothing$. Since $\Gamma(p,m)$ is compact and therefore closed, there exists x such that $B_\delta(x) \subseteq O$, and since $f_{m,p}$ is strictly decreasing, $p \cdot x < m$. Thus, since the family of affine functions $f_{m',p'}$ is continuous in m', p', there exists $\varepsilon > 0$ such that if $(m',p') \in B_\varepsilon(m,p)$, $f_{m',p'}(y) \geq 0 \; \forall \; y \in B_\delta(x)$. Thus, $\Gamma(B_\varepsilon(m,p)) \cap O \neq \varnothing$, and Γ is lower hemi-continuous. $\square$

Berge's Theorem of the Maximum tells us that if the agent's utility function is also continuous, the value function for the consumer's maximization problem is continuous and the set of maximizers for that function is an upper hemi-continuous correspondence.

Additional Exercises.

1. State whether each of the following are upper or lower hemi-continuous, and whether each has a closed graph.
   (a) This correspondence is upper hemi-continuous but not lower hemi-continuous, and it does have a closed graph. To see why it is upper, note that any open set that contains the interval will contain its endpoints, so will contain the singleton-valued portions entirely, and for any element x where $F(x)$ is not an interval, there exists ε such that $F(B_\varepsilon(x)) = F(x)$. To see why it is not lower, simply consider an open interval that intersects the interior of the interval but not the border. Since it is compact-valued, it being upper hemi-continuous suffices to show that it has a closed graph.

   (b) This correspondence is lower hemi-continuous but not upper hemi-continuous, and it does not have a closed graph. To see why it is lower hemi-continuous, consider that if $F(x) \cap O \neq \varnothing$ for any O, it must be the case that $F(x)$ is a singleton, and there exists $\varepsilon > 0$ such that $F(B_\varepsilon(x)) = F(x)$. It is not upper hemi-continuous, considering any open set that does not contain $F(x)$ when $F(x)$ is a singleton, which does contain $F(x)$ at the one point where $F(x) = \varnothing$. Similarly, take some $x_n \to x_0$, where $F(x_0) = \varnothing$. Any sequence y_n where $y_n \in F(x_n) \; \forall \; n$ will not limit to any point in $F(x_0)$, because $F(x_0)$ is empty.

   (c) This correspondence is upper hemi-continuous but not lower hemi-continuous, and it does have a closed graph. To see why it is upper hemi-continuous, observe that for any O such that $F(x) \subseteq O$, $\varnothing \subseteq O$. Thus, at any x, if $O \supseteq F(x)$, then $O \supseteq F(B_\varepsilon(x))$ for some $\varepsilon > 0$. However, at the endpoints of the line, no open set that intersects $F(x)$ will intersect the entire neighborhood, as on one side of x, $F(x)$ is empty. F has a closed graph because any sequence that limits to x where $F(x)$ is empty will have an infinite number of points in an empty neighborhood of x, since the endpoints of the two lines are closed.

   (d) This correspondence is not upper hemi-continuous or lower hemi-continuous, and it does not have a closed graph. To see why it is not upper hemi-continuous, consider the smallest open interval that contains the non-singleton interval. Since its upper bound is open, it will not contain the upper bound, so it will not contain $F(x)$ when shifted to the right by even a small amount. It is not lower hemi-continuous by the same argument as (a). It does not have a closed graph because limiting to the interval from the right, $x_n \to x$

   (e) This correspondence is neither upper hemi-continuous nor lower hemi-continuous, but it does have a closed graph. To see why it is neither upper nor lower hemi-continuous, use the example of the limiting point of the left side, call it x_0. There exists O such that $F(x_0) \subseteq O$, and thus $F(x_0) \cap O \neq \varnothing$, but for sufficiently small $\varepsilon > 0$, $F(x_0 + \varepsilon) \cap O = \varnothing$. It has a closed graph because any sequences attain their limit on the parts of the correspondence that are continuous functions, and the sequence $y_n \in F(x_n)$ (for $x_n \to x_0$ from above) diverges, so the conditions for the closed graph property hold.

2. Are the following upper or lower hemi-continuous at x_1 and x_2? Are they upper or lower hemi-continuous in general?

   (a) Γ_1 is lower hemi-continuous but not upper hemi-continuous at both x_1 and x_2. It is also lower hemi-continuous in general.

   (b) Γ_2 is not lower hemi-continuous but is upper hemi-continuous at both x_1 and x_2, and is upper hemi-continuous in general.

   (c) Γ_3 is upper hemi-continuous but now lower hemi-continuous at x_1, and is neither upper nor lower hemi-continuous at x_2. It is neither in general.