8. Fixed Point Theorems

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1 Recap

A fixed point of a self-mapping function $f : X \to X$ is a point $x \in X$ such that $f(x) = x$. For example, if $X = [0, 1]$, fixed points are all the points in the image of f that lie on the 45 degree line. We have already seen a fixed point theorem when you covered Euclidean topology.

Proposition. *Suppose $f : [0, 1] \to [0, 1]$ is continuous. Then, f has a fixed point.*

We present a number of fixed point theory that are commonly used in economics; e.g., proving existence of equilibrium in games. With a notable exception of the Contraction Mapping Theorem, most fixed point theory tells us that there is a solution, but not how to obtain that solution.

2 Lattice[1]

Theorem 1 (Tarski's fixed point thereom)**.** *Suppose $(X, \geq)$ is a compete lattice and a self-map $f : X \to X$ is an increasing function; i.e., $x > x' \Rightarrow f(x) \geq f(x')$. Then, f has a fixed point; i.e., there exists $x^* \in X$ such that $f(x^*) = x^*$. Moreover, the set of fixed point is a complete lattice so that it contains a smallest and largest fixed point.*

Proof. We will only show that: (i) there is a largest fixed point, and (ii) any nonempty subset of fixed points has an infimum in the set of all fixed points.

(i) Define $Z \coloneqq \{x \in X : f(x) \geq x\}$. Since X is complete, $\inf X \in X$ and because f is a self-map on X, $f(x) \geq \inf X$ for all $x \in X$. In particular, $f(\inf X) \geq \inf X$ so that $\inf X \in Z$; i.e., Z is nonempty. Since $Z \subseteq X$, by completeness of X, $\sup Z \in X$ and by definition, $\sup Z \geq z$ for all $z \in Z$. Since f is increasing and by definition of Z, we must have

$$f(\sup Z) \geq f(z) \geq z \;\forall z \in Z.$$

Therefore, $f(\sup Z)$ is an upper bound of Z. By definition, $\sup Z$ is the least upper bound of Z and so $f(\sup Z) \geq \sup Z$. Since f is increasing, we also have $f(f(\sup Z)) \geq f(\sup Z)$; i.e., $f(\sup Z) \in Z$. By definition $\sup Z$, it follows that $\sup Z \geq f(\sup Z)$. Hence, $\sup Z$ is a fixed point. This must also be the greatest fixed point because any fixed point must be contained in Z.

[1] The section is based on lecture notes by Federico Echenique and John Quah.

(ii) Let $\mathcal{E} \subseteq X$ be the set of fixed points of f, which is nonempty by part (i), and fix any nonempty subset $E \subseteq \mathcal{E}$. Define $Y \coloneqq \{x \in X : \inf E \geq x\}$ (set of lower bounds of E). We proceed as follows: (1) show that Y is a complete lattice; (2) f restricted to Y, denoted $f|_Y$, is a self-map on Y; (3) conclude from part (i) that $f|_Y$ has a greatest fixed point $\overline{e} \in \mathcal{E}$ that that equals $\inf E$ so that $\inf E \in \mathcal{E}$.

(1) We wish to show that for any nonempty subset $S \subseteq Y$, $\sup S \in Y$ and $\inf S \in Y$. Fix a nonempty $S \subseteq Y$. Since $S \subseteq X$ and X is a complete lattice, $\sup S \in X$ and $\inf S \in X$. By definition of Y, $\inf E \geq y$ for all $y \in Y$ so that $\inf E$ is an upper bound of Y. Because $\sup Y$ is the least upper bound, we must have $\inf E \geq \sup Y$ and so $\sup Y \in Y$. Because $S \subseteq Y$, we must have $\sup Y \geq \sup S$ so that $\inf E \geq \sup S$; i.e., $\sup S \in Y$. Since $\sup S \geq \inf S$, we must also have $\inf S \in Y$.

(2) For any $e \in E$, we have $e \geq \inf E$ so that $e = f(e) \geq f(\inf E)$; i.e., $f(\inf E)$ is a lower bound of E. Since $\inf E$ is the greatest lower bound of E, we must have $\inf E \geq f(\inf E)$ so that $f(\inf E) \in Y$. Moreover, for all $y \in Y$, $\inf E \geq y$ so that $\inf E \geq f(\inf E) \geq f(y)$. Hence, $f|_Y : Y \to Y$; i.e., $f|_Y$ is a self-map on Y.

(3) Since $f|_Y$ is an increasing self-map on a complete lattice Y, by (i), it has a greatest fixed point $\overline{e} \in Y$. Since $\overline{e}$ must be fixed point of f, we have $\overline{e} \in \mathcal{E}$. Moreover, if $e \in \mathcal{E}$ is a lower bound on E, $\inf E \geq e$ so that $e \in Y$. Then, e is a fixed point of $f|_Y$ and we must have $\overline{e} \geq e$. Hence, $\overline{e}$ is the greatest lower bound of E in $\mathcal{E}$; i.e., $\overline{e} = \inf E \in \mathcal{E}$. ■

Remark 1. The proof of the theorem tells us that the largest fixed point is given by $\sup\{x \in X : f(x) \geq x\}$ and the smallest fixed point is given by $\inf\{x \in X : x \geq f(x)\}$.

Exercise 1. Complete the proof of Theorem 1; i.e., show that there is a smallest fixed point and any nonempty subset of fixed points has a supremum in the set of all fixed points.

Since $(X = [0,1], \geq)$ is a complete lattice, the following is immediate.

Corollary 1. *Every increasing self-map on* $[0,1]$ *has a fixed point.*

The requirement that f is increasing is crucial. Try drawing functions $f : [0,1] \to [0,1]$ that is decreasing and increasing and see how they can/cannot cross the 45 degree line (fixed point are the points on the 45 degree line).

Proposition 1. *Suppose* $(X, \geq)$ *is a complete lattice,* $(\Theta, \geq)$ *is a partially ordered, and* $f : X \times \Theta \to X$ *is such that* $f(x,\theta)$ *increasing in* x *for any given* $\theta \in \Theta$ *and increasing in* θ *for any given* $x \in X$*. Then, the largest and the smallest fixed points of* $f(\cdot,\theta)$ *exist and they are increasing in* θ*.*

Proof. We only show that the largest fixed point is increasing in θ. Fix $\theta'' > \theta'$. Since $f(x,\theta)$ is increasing in θ for any $x \in X$, $f(x,\theta'') \geq f(x,\theta')$, which, in turn, implies that

$$Z' \coloneqq \{x \in X : f(x,\theta') \geq x\} \subseteq \{x \in X : f(x,\theta'') \geq x\} \eqqcolon Z''.$$

By Tarski's fixed point theorem, the largest fixed points in Z' and Z'' exist and, in fact, are given by $\overline{x}(\theta') \coloneqq \sup Z'$ and $\overline{x}(\theta'') \coloneqq \sup Z''$. Since $Z' \subseteq Z''$, we must have $\overline{x}(\theta'') \geq \overline{x}(\theta')$. ■

Exercise 2. Show that the smallest fixed point is also increasing in θ in the proposition above.

2.1 Example: Pure-strategy Nash equilibria in quasi-supermodular games

Definition 1. A (complete information) *game* is a tuple $(N, (S_i, u_i)_{i \in N})$, where $N \coloneqq \{1, \dots, n\}$ is the set of players, S_i is the set of player $i \in N$'s strategies, and $u_i : S \to \mathbb{R}$ is player i's utility when players choose strategy profile $s \in S \coloneqq \times_{i \in N} S_i$. A game is *quasi-supermodular* if:

- ▷ each S_i is a subcomplete sublattice of $\mathbb{R}^{d_i}$;
- ▷ $u_i(s_i, s_{-i})$ is quasi-supermodular in $s_i \in S_i$ for any $s_{-i} \in S_{-i}$;[2]
- ▷ $u_i(s_i, s_{-i})$ has single-crossing differences in (s_i, s_{-i}).

A strategy profile $s^* \in S$ is a *pure-strategy Nash equilibrium* of a game $(N, (S_i, u_i)_{i \in N})$ if, for all $i \in N$,

$$u_i\left(s_i^*, s_{-i}^*\right) \geq u_i\left(s_i, s_{-i}^*\right) \ \forall s_i \in S_i.$$

Theorem 2. *Fix a quasi-supermodular game $(N, (S_i, u_i)_{i \in N})$ and assume that, for all $i \in N$, $u_i(\cdot, s_{-i})$ is continuous for all $s_{-i} \in S_{-i}$. Quasi-supermodular games have pure-strategy Nash equilibria.*

Proof. Denote player $i \in N$'s best response to $s_{-i} \in S_{-i}$ as

$$B_i\left(s_{-i}\right) \coloneqq \underset{s_i \in S_i}{\arg\max}\ u_i\left(s_i, s_{-i}\right).$$

Since S_i is compact (why?) and $u_i(\cdot, s_{-i})$ is continuous, $B_i(s_{-i})$ is nonempty by extreme value theorem. The theorem of the maximum also gives that $B_i(s_{-i})$ is compact. Moreover, the fact that $u_i(, s_{-i})$ is quasi-supermodular means that $B_i(s_{-i})$ is a sublattice (why?). Together, these imply that $B_i(s_{-i})$ is a subcomplete sublattice and thus contains its supremum and infimum. The monotone comparative statics theorem implies that that if $s'_{-i} \geq s_{-i}$, then

$$\sup B_i\left(s'_{-i}\right) \geq \sup B_i\left(s_{-i}\right) \text{ and } \inf B_i\left(s'_{-i}\right) \geq \inf B_i\left(s_{-i}\right).$$

Thus, $\sup B_i$ and $\inf B_i$ are both increasing. Define $\overline{B}, \underline{B} : S \mapsto S$ by $\overline{B}(s) \coloneqq (\sup B_i(s_{-i}))_{i \in N}$ and $\underline{B}(s) \coloneqq (\inf B_i(s_{-i}))_{i \in N}$. Since the product set of complete lattices is a complete lattice, that S_i is a complete lattice (by assumption), means that $S \coloneqq \prod_{i \in N} S_i$ is also a complete lattice. Since $\overline{B}(s)$ and $\underline{B}(s)$ are both increasing functions, by Tarski's fixed point theorem, there exist $\overline{s}^*, \underline{s}^* \in S$ such that $\overline{s}^* = \overline{B}(\overline{s}^*)$ and $\underline{s}^* = \underline{B}(\underline{s}^*)$. Thus, not only there exists a Nash equilibrium, there also exists the largest and the smallest Nash equilibria. ■

Remark 2. In fact, we know that $\overline{s}^* = \sup\{s \in S : \overline{B}(s) \geq s\}$ and $\underline{s}^* = \inf\{s \in S : \underline{B}(s) \leq s\}$.

Proposition 2. *Fix a quasi-supermodular game $(N, (S_i, u_i)_{i \in N})$ and assume that, for all $i \in N$, $u_i(\cdot, s_{-i}, , \theta_i)$ is continuous for all $s_{-i} \in S_{-i}$ and all $\theta_i \in \Theta_i$ where $(\Theta_i, \geq)$ is a partially ordered set. Suppose $u_i(s_i, s_{-i}, \theta_i)$ obeys single-crossing differences in $(s_i, (s_{-i}, \theta_i))$ for all $i \in N$. Then, the largest and the smallest Nash equilibria of the game are both increasing in $(\theta_i)_{i \in N}$.*

[2]Recall that s_{-i} refers to the strategy of all players other than i and so $s_{-i} \in S_{-i} \coloneqq \times_{j \in N \setminus \{i\}} S_j$.

Proof. Since $u_i(s_i, s_{-i}, \theta_i)$ obeys single-crossing differences in $(s_i, (s_{-i}, \theta_i))$, and u_i is quasi-supermodular in s_i, then, by the monotone comparative static theorem, we have that

$$B_i\left(s''_{-i}, \theta''_i\right) \geq_S B_i\left(s'_{-i}, \theta'_i\right) \ \forall \left(s''_{-i}, \theta''_i\right) \geq \left(s'_{-i}, \theta'_i\right).$$

Hence, this implies that $\sup B_i(s_{-i}, \theta_i)$ and $\inf B_i(s_{-i}, \theta_i)$ are both increasing in θ_i. By the previous result, the largest Nash equilibrium of the game parameterised by θ is $\sup\{s \in S : \overline{B}(s, \theta) \geq s\}$, where $\overline{B}(s, \theta) \equiv (\sup B_i(s_{-i}, \theta_i))_{i \in N}$. Since $\sup B_i$ is increasing in θ_i,

$$\overline{B}\left(s, \theta''\right) \geq \overline{B}\left(s, \theta'\right) \ \forall \theta'' > \theta',$$

Hence,

$$\left\{s \in S : \overline{B}\left(s, \theta'\right) \geq s\right\} \subseteq \left\{s \in S : \overline{B}\left(s, \theta''\right) \geq s\right\} \ \forall \theta'' > \theta',$$

which implies that

$$\sup\left\{s \in S : \overline{B}\left(s, \theta''\right) \geq s\right\} \geq \sup\left\{s \in S : \overline{B}\left(s, \theta'\right) \geq s\right\} \ \forall \theta'' > \theta'.$$

Similarly for the smallest Nash equilibrium. ■

2.2 Example: Stable matching

Definition 2. A *one-to-one matching market* is a tuple $(M, W, (\succsim_m)_{m \in M}, (\succsim_w)_{w \in W})$ where

(i) M and W are disjoint finite sets;

(ii) for each $m \in M$ $\succsim_m$ is a total order over $W \cup \{\varnothing\}$ such that for any distinct $w, w' \in W \cup \{\varnothing\}$, $w \succsim_m w'$ or $w' \succsim_m w$ but not both;

(iii) for each $w \in W$, $\succsim_w$ is a total order over $M \cup \{\varnothing\}$ such that for any distinct $m, m' \in M \cup \{\varnothing\}$, $m \succsim_m m'$ or $m' \succsim_m m$ but not both.

For any $m \in M$, $w \in W$ is *acceptable* to m if $w \succsim_m \varnothing$. Analogously, for any $w \in W$, $m \in M$ is *acceptable* to w if $m \succsim_m \varnothing$. A *fantasy* is a function $\nu : M \cup W \to M \cup W$ such that

$$\nu(m) \in W \cup \{\varnothing\} \ \forall m \in M \text{ and } \nu(w) \in M \cup \{\varnothing\} \ \forall w \in W.$$

Let V denote the set of all fantasies. For any two fantasies $\nu, \nu' \in V$, we say that ν is *less than* ν', denoted $\nu \leq \nu'$ if

$$\nu'(m) \succsim_m \nu(m) \ \forall m \in M \text{ and } \nu(w) \succsim_w \nu'(w) \ \forall w \in W.$$

Lemma 1. *$(V, \leq)$ is a lattice.*

Proof. Take any $\nu, \nu' \in V$. We want to show that $\nu \vee \nu' \in V$ and $\nu \wedge \nu' \in V$. We only show the first. For any $m \in M$, since $\succsim_m$ is a total order, there exists $\overline{w}_m \in W \cup \{\varnothing\}$ such that $\overline{\nu}_m = \sup\{\nu(m), \nu'(m)\}$. Similarly, for any $w \in W$, there exists $\overline{m}_m \in M \cup \{\varnothing\}$ such that $\overline{m}_w = \sup\{\nu(w), \nu'(w)\}$. Then, $(\nu \vee \nu')(m) = \overline{w}_m$ and $(\nu \vee \nu')(w) = \overline{m}_w$ for all $m \in M$ and all $w \in W$ and clearly $\nu \wedge \nu' \in V$. ■

Definition 3. A *matching* is a fantasy μ with the property that

$$w = \mu(m) \Leftrightarrow \mu(w) = m \ \forall m \in M \ \forall w \in W.$$

A matching μ is *individually rational* if $\mu(a) \succsim_a \varnothing$ for all $a \in M \cup W$. A pair $(m, w) \in M \times W$ is a *blocking pair* for μ if $m \succ_w \mu(w)$ and $w \succ_m \mu(m)$. A matching is *stable* if it is individually rational and there are no blocking pairs.

Remark 3. The canonical example is the marriage market (M for men and W for women).[3] We will follow this example purely for historical reasons.

For any fantasy ν and any $m \in M$ and $w \in W$,

$$\begin{aligned} A(m,\nu) &:= \{w' \in W : m \succsim_{w'} \nu(w')\}, \\ A(w,\nu) &:= \{m' \in M : w \succsim_{m'} \nu(m')\}. \end{aligned}$$

For example, $A(m, \nu)$ is the set of women who would prefer to match with m over the man specified by the fantasy ν. Define a function $T : V \to V$ by letting $(T\nu)(m)$ be the optimal choice according to $\succsim_m$ in $A(m, \nu) \cup \{\varnothing\}$ for any $m \in M$; i.e., for all $m \in M$,

$$(T\nu)(m) \succsim_m w \ \forall w \in A(m,\nu).$$

Similarly, let $(T\nu)(w)$ being the optimal choice according to $\succsim_w$ in $A(w, \nu) \cup \{\varnothing\}$ for any $w \in W$; i.e., for all $w \in W$,

$$(T\nu)(w) \succsim_w m \ \forall m \in A(w,\nu).$$

By construction, T is a self-map and so to apply Tarski's fixed point theorem, we need to show that T is increasing.

Lemma 2. *T is monotone increasing; i.e., for any $\nu \leq \nu'$, we have $T\nu \leq T\nu'$.*

Proof. Fix $\nu \leq \nu'$ and some $m \in M$, and take any $w \in A(m, \nu)$. Then, $m \succsim_w \nu(w) \succsim_w \nu'(w)$ so that $w \in A(m, \nu')$. Hence, $A(m, \nu) \subseteq A(m, \nu')$. Analogous argument shows that $A(w, \nu') \subseteq A(w, \nu)$. Since the best element from a larger set cannot be worse than from a smaller set, $(T\nu')(m) \succsim_m (T\nu)(m)$ and $(T\nu)(w) \succsim_w (T\nu')(w)$. Hence, $T\nu \leq T\nu'$. ∎

Lemma 3. *Any fixed point of T is a stable matching. If μ is a stable matching, then it is a fixed point of T.*

Proof. Suppose that $\nu \in V$ is a fixed point of T; i.e., $\nu = T\nu$. We first show that ν is a matching. Toward a contradiction, suppose there exists $(m, w) \in M \times W$ with $w = \nu(m)$ and $m \neq \nu(w)$ (the proof for the other case is analogous). By definition of $A(m, \nu)$, that $w = \nu(m)$ implies $m \in A(w, \nu)$. So, $m \neq \nu(w) = T\nu(w)$ means that $\nu(w) \succ_w m$. But then $m \notin A(w, \nu)$ which contradicts that $w = \nu(m) = (T\nu)(m)$. We now argue that ν is stable. By construction, ν is individually rational as $\nu(a) = (T\nu)(a) \succsim_a \varnothing$ for any $a \in M \cup W$. Fix some $(m, w) \in M \times W$. If $m \succ_w \nu(w)$, then $w \in A(m, \nu)$. By definition of T, $(T\nu)(m) = \nu(m) \succsim_m w$ and so $w \not\succ_m \nu(m)$; i.e., (m, w) cannot be a blocking pair.

[3] The seminar paper is Gale and Shapley (1962) which perhaps explains the labelling.

Conversely, let μ be a stable matching. We wish to show that $\mu = T\mu$. By way of contradiction, suppose there exists an element of $A(m,\mu) \cup \{\varnothing\}$ that is strictly better than $\mu(m)$ for $\succsim_w$. By individual rationality, $\mu(a) \succsim_a \varnothing$ for all $a \in M \cup W$ so this element cannot be $\varnothing$. So let $w \in A(m,\mu)$ be such that $w \succ_m \mu(m)$. We obtain a contradiction if we can show that (m,w) forms a blocking pair. To that end, note that, $w \neq \mu(m)$ implies that $m \neq \mu(w)$ as μ is a matching. Then, $w \in A(m,\mu)$ implies that $m \succ_w \mu(w)$ because $m \succsim_w \mu(w)$ and $\succsim_w$ is a strict preference. Then, (m,w) would form a blocking pair. ∎

Theorem 3. *The set $S \subseteq V$ of stable matching is nonempty and $(S, \leq)$ is a lattice. Moreover, there exists two stable matchings μ_M and μ_W such that, for any stable matching μ,*

$$\mu_M(m) \succsim_m \mu(m) \succsim_m \mu_W(m) \text{ and } \mu_W(w) \succsim_w \mu(w) \succsim_w \mu_M(w)$$

for all $m \in M$ and all $w \in W$.

Proof. By Tarski's fixed point theorem, we know that T, as an increasing self-map on V, has a fixed point. By the previous lemma, this fixed point is a stable matching. Hence, S is nonempty. That $(S, \leq)$ is a lattice also follows from the fact that set of fixed points of T forms a complete lattice. It also follows that there is a largest and smallest fixed points, μ_M and μ_W with the properties above. ∎

Remark 4. Above establishes that there are *men-preferred* and *women-preferred* stable matchings.

Exercise 3. Prove that the set of stable matching is a sublattice of $(V, \leq)$ and that, for any two stable matchings μ and μ': (i) $(\mu \vee \mu')(m)$ is preferred with respect to $\succsim_m$ over $\mu(m)$ and $\mu'(m)$; (ii) $(\mu \wedge \mu')(m)$ is the worse with respect to $\succsim_m$ than $\mu(m)$ and $\mu(m')$.

Remark 5. The "usual" proof of the theorem is a constructive one—by running a *deferred acceptance algorithm* (DAA).

3 Continuity

Here are the two fixed point theorem that is commonly used in economics that exploits continuity in functions.

Theorem 4 (Brouwer's fixed point theorem)**.** *Let $X \subseteq \mathbb{R}^d$ be nonempty, compact and convex. Suppose that $f : X \to X$ is continuous. Then, f has a fixed point; i.e., there exists $x^* \in X$ such that $x^* = f(x^*)$.*

While Brouwer's fixed point theorem deals with functions, Kakutani's fixed point theorem deals with correspondences. We state the theorem without proof.

Theorem 5 (Kakutani's fixed point theorem)**.** *Let $X \subseteq \mathbb{R}^d$ be nonempty, compact and convex. Suppose that $f : X \rightrightarrows X$ is nonempty, closed-valued, convex-valued and upper hemicontinuous. Then, F has a fixed point; i.e., there exists $x^* \in S$ such that $x^* \in F(x^*)$.*

3.1 Example: Mixed-strategy Nash equilibrium

We prove the existence of a mixed-strategy Nash equilibrium using Brouwer's theorem and also using Kakutani's theorem.

Definition 4. A game $G = (N, (S_i, u_i)_{i \in N})$ is *finite* if S_i is finite for all $i \in N$. A *mixed extension* of G, denoted $\overline{G}$, is a tuple $(N, (\Delta S_i, \bar{u}_i)_{i=1}^N)$, where

$$\Delta S_i := \left\{ \sigma_i \in \mathbb{R}_+^{S_i} : \sum_{s_i \in S_i} \sigma_i(s_i) = 1 \right\}$$

and $\overline{u}_i : \times_{i=1}^N \Delta S_i \to \mathbb{R}$ is defined

$$\overline{u}_i(\sigma_i, \sigma_{-i}) := \sum_{(s_i, s_{-i}) \in S} \left(\prod_{j=1}^N \sigma_j(s_j) \right) \sigma_i(s_i) \, u_i(s_i, s_{-i}).$$

A mixed-strategy Nash equilibrium of G is a Nash equilibrium of the mixed extension of G; i.e., $\sigma^* \in \Delta S$ such that

$$\overline{u}_i\left(\sigma_i^*, \sigma_{-i}^*\right) \geq \overline{u}_i\left(\sigma_i, \sigma_i^*\right) \ \forall \sigma_i \in \Delta S_i.$$

Remark 6. In the mixed extension of the game, players are allowed to randomise over pure strategies and $\sigma_i(s_i)$ specifies the probability with each player i plays $s_i \in S_i$. Note that $\sigma_i = (\sigma_i(s_i))_{s_i \in S_i}$; i.e., each player's mixed strategy is a vector of probabilities.

Theorem 6 (Existence of Nash equilibrium)**.** *Let G be a finite game, then there exists a mixed-strategy Nash equilibrium.*

We give two proofs of this result. The first relies on the Brouwer's fixed point theorem. The second relies on Kakutani's fixed point theorem. Let us establish some intermediate results that we need in both cases.

Lemma 4. *For each $i \in N$, ΔS_i is convex. Therefore, ΔS is convex.*

Proof. Take any $\sigma_i, \sigma_i' \in \Delta S_i$ any $\lambda \in (0,1)$. Then,

$$\lambda \sigma_i + (1-\lambda)\sigma_i' = \left(\lambda \sigma_i(s_i) + (1-\lambda)\sigma_i(s_i)\right)_{s_i \in S_i},$$

where

$$\lambda \sigma_i(s_i) + (1-\lambda)\sigma_i(s_i) \in [0,1]$$

and

$$\sum_{s_i \in S_i} \lambda \sigma_i(s_i) + (1-\lambda)\sigma_i(s_i) = \lambda \sum_{s_i \in S_i} \sigma_i(s_i) + (1-\lambda) \sum_{s_i \in S_i} \sigma_i(s_i) = 1.$$

Hence, $\lambda\sigma_i + (1-\lambda)\sigma_i' \in \Delta(S_i)$; i.e., ΔS_i is convex. Since ΔS is a product set of convex set, it is also convex. ■

Lemma 5. *For each $i \in N$, ΔS_i is compact. Therefore, ΔS is compact.*

Proof. By Balzano-Wierstrass Theorem, it suffices to show that ΔS_i is closed and bounded. ΔS_i is clearly bounded (by a vector of zeros and ones). To show that ΔS_i is closed, take any sequence

$(\sigma_{i,k})_k$ in ΔS_i such that $\sigma_{i,k} \to \sigma_i$. Since $0 \leq \sigma_{i,k}(s_i) \leq 1$ for all $s_i \in S_i$ and $k \in \mathbb{N}$, we must have $0 \leq \sigma_i(s_i) \leq 1$. Moreover, since $\sum_{s_i \in S_i} \sigma_{i,k}(s_i) = 1$ for all $k \in \mathbb{N}$, we must also have $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$. That is, $\sigma_i \in \Delta S_i$ so that ΔS_i is closed. ∎

Proof using Brouwer's theorem. We first prove the result using Brouwer's theorem.[4] Define $\Delta S \coloneqq \times_{i \in N} S_i$ and $f : \Delta S \to \Delta S$ as follows: for any $\sigma \in \Delta S$, for each player $i \in N$ and each $s_i \in S_i$,

$$f_i(\sigma, s_i) \coloneqq \frac{\sigma_i(s_{ij}) + \max\{0, \overline{u}_i(\delta_{s_i}, \sigma_{-i}) - \overline{u}_i(\sigma)\}}{1 + \sum_{s_i' \in S_i} \max\left\{0, \overline{u}_i\left(\delta_{s_i'}, \sigma_{-i}\right) - \overline{u}_i(\sigma)\right\}},$$

where $\delta_{s_i} \in \Delta S_i$ is such that $\delta(s_i') = 1$ if and only if $s_i' = s_i$. For each $i \in N$, define $f_i(\sigma) \coloneqq (f_i(\sigma, s_i))_{s_i \in S_i}$ and $f(\sigma) \coloneqq (f_i(\sigma))_{i \in N}$. Observe that, for any $i \in N$,

$$\sum_{s_i \in S_i} f_i(\sigma, s_i) = 1 \text{ and } f_i(\sigma, s_i) \geq 0 \ \forall s_i \in S_i.$$

Hence, $f_i(\sigma) \in \Delta S_i$ for all $i \in N$ and $f(\sigma) \in \Delta S$. That is, f is a self-map on ΔS. Moreover, observe that $f_i(\cdot, s_i)$ is continuous for each $s_i \in S_i$ (numerator and the denominators are both continuous and the denominator is bounded away from zero). Thus, f is a continuous self-map on ΔS that is nonempty, compact and convex. The Brouwer's fixed point theorem gives a point σ^* that is a fixed point of f; i.e., $f(\sigma^*) = \sigma^*$ and $f_i(\sigma^*, s_i) = \sigma_i^*(s_i)$ for all $i \in N$ and all $s_i \in S_i$. In other words,

$$\sigma_i^*(s_i) \sum_{s_i' \in S_i} \max\left\{0, \overline{u}_i\left(\delta_{s_i'}, \sigma_{-i}^*\right) - \overline{u}_i(\sigma^*)\right\} = \max\left\{0, \overline{u}_i\left(\delta_{s_i}, \sigma_{-i}^*\right) - \overline{u}_i(\sigma^*)\right\}.$$

Multiplying both sides by $u_i(\delta_{s_i}, \sigma_{-i}^*) - u_i(\sigma^*)$ and summing over s_i gives

$$\begin{aligned}
&\sum_{s_i \in S_i} \sigma_i^*(s_i) \left[\overline{u}_i\left(\delta_{s_i}, \sigma_{-i}^*\right) - \overline{u}_i(\sigma^*)\right] \sum_{s_i' \in S_i} \max\left\{0, \overline{u}_i\left(\delta_{s_i'}, \sigma_{-i}^*\right) - \overline{u}_i(\sigma^*)\right\} \\
&\quad = \sum_{s_i \in S_i} \left[\overline{u}_i\left(\delta_{s_i}, \sigma_{-i}^*\right) - \overline{u}_i(\sigma^*)\right] \max\left\{0, u_i\left(\delta_{s_i}, \sigma_{-i}^*\right) - \overline{u}_i(\sigma^*)\right\}.
\end{aligned}$$

Observe that the left-hand side is zero because $\sum_{s_i \in S_i} \sigma_i^*(s_i) = 1$ and $\sum_{s_i \in S_i} \sigma_i^*(s_i)\overline{u}_i(\delta_{s_i}, \sigma_{-i}^*) = \overline{u}_i(\sigma^*)$. Hence,

$$0 = \sum_{s_i \in S_i} \left[\overline{u}_i\left(\delta_{s_i}, \sigma_{-i}^*\right) - \overline{u}_i(\sigma^*)\right] \max\left\{0, u_i\left(\delta_{s_i}, \sigma_{-i}^*\right) - \overline{u}_i(\sigma^*)\right\}.$$

But the summation on the right-hand side can be zero only if $\overline{u}_i(\delta_{s_i}, \sigma_{-i}^*) \leq \overline{u}_i(\sigma^*)$ for all $s_i \in S_i$ and all $i \in N$. But any $\sigma_i \in \Delta S_i$, $\overline{u}_i(\sigma_i, \sigma_{-i}^*)$ is a convex combination of $\{\overline{u}_i(\delta_{s_i}, \sigma_{-i}^*)\}_{s_i \in S_i}$ so we must have $\overline{u}_i(\sigma^*) \geq u_i(\sigma_i, \sigma_{-i}^*)$ for all $\sigma_i \in \Delta S_i$. Thus, σ^* is a mixed-strategy Nash equilibrium. ∎

Proof using Kakutani's theorem. For any $i \in N$, define the best response correspondence by

$$B_i(\sigma_{-i}) \coloneqq \underset{\sigma_i' \in \Delta S_i}{\arg\max}\, \overline{u}_i(\sigma_i', \sigma_{-i}).$$

Notice that B_i implies the equilibrium condition for player i. Since $\overline{u}_i$ is linear in σ_i by construction,

[4] The proof is from Jehle and Reny.

it is continuous. Moreover, ΔS_i is compact so that by the theorem of maximum, B_i is nonempty, compact-valued and upper hemicontinuous on $\Delta S_{-i} := \times_{j \neq i} \Delta S_j$. Moreover, since $\overline{u}_i$ is linear in σ, B_i is convex-valued as well (i.e. $B_i(\sigma_i, \sigma_{-i})$ is convex for all $\sigma_i \in \Delta S_i$). Define $B : \Delta S \rightrightarrows \Delta S$ by

$$B(\sigma) := (B_i(\sigma_{-i}))_{i \in N}.$$

Then B inherits the properties of B_i's so that B is a nonempty-, compact-, convex-valued, upper-hemicontinuous correspondence. Since ΔS is compact and convex, we can then appeal to Kakutani's fixed point theorem to conclude that there exists $\sigma^* \in \Delta S$ such that $\sigma^* \in B(\sigma^*)$ and σ^* is a mixed-strategy Nash equilibrium. ■

4 Contraction

Definition 5. A *metric* on an nonempty set X is a function $\rho : X^2 \to \mathbb{R}$ that satisfies the following conditions:

- ▷ (nonnegativity) $\rho(x, y) \geq 0 \ \forall x, y \in X$;
- ▷ (identity of indiscernibles) $\rho(x, y) = 0 \Leftrightarrow x = y$;
- ▷ (symmetry) $\rho(x, y) = \rho(y, x) \ \forall x, y \in X$;
- ▷ (triangle inequality) $\rho(x, y) \leq \rho(x, z) + \rho(y, z) \ \forall x, y, z \in X$.

A *metric space* is a pair (X, ρ) where X is a nonempty set and ρ is on X.

Example 1 (Examples of metric spaces)**.**

(i) Euclidean metric space. $(\mathbb{R}^d, \|\cdot\|_d)$, where $\|\cdot\|_d$ is the *Euclidean distance* given by

$$\|\mathbf{x} - \mathbf{y}\|_d = \sqrt{\sum_{i=1}^{d} (x_i - y_i)^2}.$$

(ii) Discrete metric space. (X, d_{discrete}), where $X \neq \varnothing$ and

$$d_{\text{discrete}}(x, y) = \mathbf{1}_{\{x \neq y\}}.$$

(iii) Product metric spaces. If (X, ρ_X) and (Y, ρ_Y) are metric spaces, then $(X \times Y, \rho)$ is a metric space where

$$\rho((x_1, y_1), (x_2, y_2)) := [(\rho_X(x_1, x_2))^p + (\rho_Y(y_1, y_2))^p]^{1/p} \tag{1}$$

for any $p \geq 1$.

(iv) Space of continuous functions. Given a nonempty set X, let $\mathbf{C}(X)$ denote the set of continuous real-valued functions on X. Then, $(\mathbf{C}(X), \ \|\cdot\|_\infty)$ is a metric space, where $\|\cdot\|_\infty$ is the *sup-norm* given by

$$\|f\|_\infty := \sup\{|f(x)| : x \in X\}.$$

Definition 6. Given a metric space (X, ρ), a sequence $(x_n)_n$ in X is *Cauchy* if, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\rho(x_n, x_m) < \epsilon$ for all $m, n > N$. A metric space (X, ρ) is *complete* if every Cauchy sequence is convergent; i.e., if X contains limit points of all Cauchy sequences in X.

Remark 7. Just as when X was a Euclidean space, one can show that (i) every convergent sequence is Cauchy but not every Cauchy sequence is convergent; (ii) A sequence in product metric space is Cauchy if and only if each component-wise sequence is Cauchy; (iii) Cauchy sequences are bounded and has at most one limit point. However, not all metric spaces are complete (e.g., $(\mathbb{Q}, |\cdot|)$ is a metric space that is not complete).

Fact 1. $(\mathbf{C}(X), \|\cdot\|_\infty)$ *is a complete metric space.*

Let (X, ρ) be a metric space. A self-map $f : X \to X$ is a *contraction* (with modulus r) if there exists $r \in [0, 1)$ such that

$$\rho(f(x), f(y)) \leq r\rho(x, y) \;\; \forall x, y \in X.$$

Intuitively, the distance between x and y contracts—i.e., shrinks—when we apply the function f to both points. Note that r is a Lipchitz constant so that a if f if is contraction then f must be continuous. The following establishes that this contraction property of a self-map is sufficient to guarantee the existence of a *unique* fixed point. The result is called the contraction napping theorem or the Banach contraction principle.

Theorem 7 (Contraction Mapping Theorem)**.** *Let (X, ρ) be a complete metric space and $f : X \to X$ be a contraction. Then, f has a unique fixed point; i.e., there exists a unique $x^* \in X$ such that $x^* = f(x^*)$.*

Proof. Fix $x_0 \in X$. Define recursively a sequence $(x_n)_n$ in X by $x_{n+1} := f(x_n)$. Since f is a contraction, for some $r \in [0, 1)$, for each $n \in \mathbb{N}$,

$$\begin{aligned}\rho(x_{n+1}, x_n) &= \rho(f(x_n), x_n) \\ &= \rho(f(f(x_{n-1})), f(x_{n-1})) \\ &\leq r\rho(f(x_{n-1}), x_{n-1})\end{aligned}$$

Note also that

$$\begin{aligned}\rho(f(x_{n-1}), x_{n-1}) &= \rho(f(f(x_{n-2})), f(x_{n-2})) \\ &= r\rho(f(x_{n-2}), x_{n-2}) \\ \Rightarrow \rho(x_{n+1}, x_n) &\leq r^2\rho(f(x_{n-2}), x_{n-2}).\end{aligned}$$

By induction, we may conclude that

$$\rho(x_{n+1}, x_n) \leq r^n\rho(f(x_0), x_0). \tag{2}$$

By Triangular Inequality, we have that

$$\rho(x_n, x_m) \leq \rho(x_n, z) + \rho(z, x_m) \;\; \forall z \in X.$$

Let $m = n + 2$, then we can let $z = x_{n+1}$ to obtain that:

$$\rho\left(x_n, x_{n+2}\right) \leq \rho\left(x_n, x_{n+1}\right) + \rho\left(x_{n+1}, x_{n+2}\right)$$

Now suppose $m = n + 3$, letting $z = x_{n+1}$ again, we obtain

$$\rho\left(x_n, x_{n+3}\right) \leq \rho\left(x_n, x_{n+1}\right) + \rho\left(x_{n+1}, x_{n+3}\right).$$

Consider $\rho(x_{n+1}, x_{n+3})$, using the Triangular Inequality again with $z = x_{n+2}$, we have

$$\rho\left(x_{n+1}, x_{n+3}\right) \leq \rho\left(x_{n+1}, x_{n+2}\right) + \rho\left(x_{n+2}, x_{n+3}\right).$$

Combining the two,

$$\rho\left(x_n, x_{n+3}\right) \leq \rho\left(x_n, x_{n+1}\right) + \rho\left(x_{n+1}, x_{n+2}\right) + \rho\left(x_{n+2}, x_{n+3}\right).$$

Thus, for any $n, m \in \mathbb{N}$ with $m > n$,

$$\rho\left(x_n, x_m\right) \leq \sum_{j=n}^{m-1} \rho\left(x_n, x_{n+1}\right).$$

We can now use 2 to obtain that

$$\rho\left(x_n, x_m\right) \leq \sum_{j=n}^{m-1} r^j \rho\left(f\left(x_0\right), x_0\right).$$

Note that

$$\begin{aligned}
\sum_{j=n}^{m-1} r^j &\leq \sum_{j=n}^{m-1} r^j + \sum_{j=m}^{\infty} r^j = \sum_{j=n}^{\infty} r^j \\
&= r^n + r^{n+1} + \cdots = r^n\left(1 + r + r^2 + \cdots\right) = r^n \sum_{i=0}^{\infty} r^i \\
&= \frac{r^n}{1-r}.
\end{aligned}$$

Hence, we can write

$$\rho\left(x_n, x_m\right) \leq \frac{r^n}{1-r} \rho\left(f\left(x_0\right), x_0\right).$$

For any $\epsilon > 0$, let $N \in \mathbb{N}$ be such that

$$\frac{r^N}{1-r} r\left(f\left(x_0\right), x_0\right) = \epsilon.$$

Then for any $m, n > N$, $r^n, r^m < r^N$ (assuming, without loss of generality, $m > n$) so that

$$\rho\left(x_n, x_m\right) \leq \frac{r^n}{1-r} \rho\left(f\left(x_0\right), x_0\right) < \frac{r^N}{1-r} \rho\left(f\left(x_0\right), x_0\right).$$

Hence, $(x_n)_n$ is a Cauchy sequence. Since $(x_n)_n$ is a Cauchy sequence and (X, ρ) is complete, then

$x_n \to x^* \in X$ so that:

$$\lim_{n\to\infty} \rho\left(f\left(x_n\right), x_n\right) = \rho\left(f\left(x^*\right), x^*\right).$$

Now, because

$$0 \leq \rho\left(\underbrace{f\left(x_n\right)}_{=x_{n+1}}, x_n\right) \leq \frac{r^n}{1-r}\rho\left(f\left(x_0\right), x_0\right),$$

as $n \to \infty$, $\rho(f(x_n), x_n) \to 0$. Therefore,

$$\lim_{n\to\infty} \rho\left(f\left(x_n\right), x_n\right) = \rho\left(f\left(x^*\right), x^*\right) = 0;$$

so that $x^* = f(x^*)$.

For uniqueness, suppose that $x^* = f(x^*)$ and $y^* = f(y^*)$ for some $x^*, y^* \in X$. Then,

$$\rho\left(x^*, y^*\right) = \rho\left(f\left(x^*\right), f\left(y^*\right)\right) \leq r\rho\left(x^*, f^*\right)$$

which implies that $\rho\left(x^*, y^*\right) = 0$ (since $r \neq 0$) and hence $x^* = y^*$. ■

Remark 8. Observe that the proof tells us that we can start from any point in X and get to a fixed point by iteration. Thus, unlike most other fixed point theorems, Contraction Mapping Theorem tells us how to find the unique fixed point as well as proving its existence. You will use this property a lot in dynamic programming (e.g., in macroeconomics although these can crop up in IO as well as in dynamic games).

Let us give some sufficient conditions for a function to be a contraction.

Proposition 3 (Blackwell's Test)**.** *Let X be a compact metric space. Suppose that $W : \mathbf{C}(X) \to \mathbf{C}(X)$ is a self-map on $\mathbf{C}(X)$. Then, W is a contraction with modulus β if*

(i) W is increasing, i.e., $W(f) \geq W(g)$ for all $f, g \in \mathbf{C}(X)$ such that $f \geq g$;

(ii) for all $(f, \alpha) \in \mathbf{C}(X) \times \mathbb{R}_+$, there exists $\beta \in (0,1)$ such that $W(f+\alpha) \leq W(f) + \beta\alpha$.

Proof. Suppose that $f \geq g$, which implies that $f + \|f-g\|_\infty \geq g$. Since W is increasing, this implies that

$$W\left(f + \|f-g\|_\infty\right) \geq W\left(g\right).$$

Using property (ii),

$$W\left(f\right) + \beta\left\|f-g\right\|_\infty \geq W\left(f + \|f-g\|\right) \geq W\left(g\right),$$

which rearranges to

$$\beta\left\|f-g\right\|_\infty \geq W\left(g\right) - W\left(f\right).$$

Now suppose that $g \geq f$, then we can obtain that

$$\beta\left\|f-g\right\|_\infty \geq W\left(f\right) - W\left(g\right).$$

That is,

$$\beta\left\|f-g\right\|_\infty \geq \left|W\left(f\right) - W\left(g\right)\right|.$$

This inequality holds for all $x \in X$, and in particular, it holds for $x^* = \arg\max_{x \in X} |W(f) - W(g)|$. That is, there exists $\beta \in (0,1)$ such that

$$\rho(W(f), W(g)) \leq \beta \rho(f, g)$$

so that W is a contraction. ■

4.1 Example: Bellman equation

In many economic problems, especially when dynamics are considered, the following type of problem are of interest: given $x_0 \in X$,

$$V^*(x_0) = \max_{(x_t)_{t=0}^{\infty} \subset X} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \text{ s.t. } x_t \in \Gamma(x_{t-1}) \ \forall t \in \mathbb{N}, \tag{3}$$

where $\beta \in (0,1)$, $X \subseteq \mathbb{R}$ is a compact metric space, F is bounded and continuous and $\Gamma : X \rightrightarrows X$ is a compact-valued and continuous correspondence. We can interpret equation (3) as follows: A forward-looking economic agent is optimising the life-time objective, with the same objective function (say utility) in each period, and β being the discount factor. The problem is complicated because, in each period, current choice x_t may enter into current period flow payoff and the next period flow payoff (since F depends on both x_t and x_{t+1}). Moreover, current choice x_t may affect the feasible choices in the future (since $x_t \in \Gamma(x_{t-1})$). It is well-known that this problem, under the conditions provided above, has a solution.

We take as given the Principle of Optimality which tells us that we may consider the first-period decision separately, setting aside all future decisions.[5] So consider the choice of x_1 by rearranging equation (3) as below:

$$V^*(x_0) = \max_{x_1 \in \Gamma(x_0)} \left[F(x_0, x_1) + \beta \left[\max_{(x_t)_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} F(x_t, x_{t+1}) \text{ s.t. } x_t \in \Gamma(x_{t-1}) \ \forall t \geq 1 \right] \right].$$

The inner maximisation problem is, in fact, the value of the $t = 1$ decision problem, given x_1. Thus, we can write the expression above as a recursive definition of the value function V^*:

$$V^*(x_0) = \max_{x_1 \in \Gamma(x_0)} \left[F(x_0, x_1) + \beta V^*(x_1) \right].$$

The Bellman operator $W : \mathbf{C}(X) \to \mathbf{C}(X)$ is defined as

$$W(V)(x) \coloneqq \max_{y \in \Gamma(x)} F(x, y) + \beta V(y) \ \forall x \in X.$$

By the Contraction Mapping Theorem (Theorem 7), if W is a contraction, since $\mathbf{C}(X)$ is a complete metric space, there exists a unique fixed point. That is, there exists a unique $V : X \to X$ such that

$$V(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta V(y), \tag{4}$$

[5] We need Γ to be non-empty valued, $\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$ exists for all $x_0 \in X$ and $(x_0, x_1, \ldots) \in \Gamma(x_0)$ (i.e., all feasible plans can be evaluated) and the solution to Bellman equation satisfies the transversality condition $\lim_{n\to\infty} \beta^n V(x_n) = 0$ holds for all $(x_0, x_1, \ldots) \in \Gamma(x_0)$ and all $x_0 \in X$.

which then implies that $V = V^*$. This gives a useful characterisation of the solution of the problem equation (3), which we call equation (4) the *Bellman equation*.

To show that the Bellman operator is a contraction, we can use the Blackwell's test.

Claim 1. The Bellman operator

$$W(V)(x) := \max_{y \in \Gamma(x)} F(x, y) + \beta V(y) \;\; \forall x \in X$$

is a contraction.

Proof. First, if $\beta > 0$, then, $V(x) \geq V'(x)$ for all $x \in X$ implies that

$$W(V)(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta V(y) \geq \max_{y \in \Gamma(x)} F(x, y) + \beta V'(y) = W(V')(x)$$

so that W is increasing. For $\alpha \in \mathbb{R}_+$,

$$\begin{aligned} W(V + \alpha)(x) &= \max_{y \in \Gamma(x)} F(x, y) + \beta [V(y) + \alpha] \\ &= \left[\max_{y \in \Gamma(x)} F(x, y) + V(y)\right] + \beta\alpha \\ &= W(V)(x) + \beta\alpha. \end{aligned}$$

Since both conditions of the Blackwell's test are satisfied, we conclude that W is a contraction. ∎

Remark 9. Define $g(x)$ as the policy function that solves the right-hand side of (4). We do not (at the moment) know whether V is differentiable. However, by our monotone comparative statics results, if F has single-crossing differences in (y, x) and Γ is increasing (in the set-inclusion order), then the objective function of the problem has single-crossing differences. If we also know that the objective function is quasi-supermodular in y for all x, then it follows that $g(x)$ is increasing in x.