# Econ 6190: Econometrics I

# Confidence Interval

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2024 Fall

# Contents


- Finding Confidence Interval by Pivotal Quantities
- Finding Confidence Interval by Test Inversion
- Evaluation of Confidence Interval

# Reference

- Hansen Ch. 14

# 1. Motivation

# Interval estimation

- We've seen point estimation of a parameter $\theta$: report a single value as a guess of $\theta$
- In this note, we consider **interval estimation** as a tool to report **estimation uncertainty**
- **Definition**: Given sample $\mathbf{X} = \{X_1, X_2 \ldots X_n\}$, an interval estimator of a real-valued parameter $\theta$ is an interval $C = C(\mathbf{X}) = [L(\mathbf{X}), U(\mathbf{X})]$
  - $L(\mathbf{X})$ and $U(\mathbf{X})$ are functions of $\mathbf{X}$ so they are random
  - For $\mathbf{X} = \mathbf{x}$, $[L(\mathbf{x}), U(\mathbf{x})]$ are realized values of the interval
  - If $L(\mathbf{X}) = -\infty$, we have one-sided interval $(-\infty, U(\mathbf{X})]$
  - If $L(\mathbf{X}) = \infty$, we have one-sided interval $[L(\mathbf{X}), +\infty)$

# Example: interval estimator for normal mean

- Consider a random sample $\{X_1, X_2, X_3, X_4\}$ from $\mathrm{N}(\mu, 1)$
- An interval estimator for $\mu$ could be $[\bar{X} - 1, \bar{X} + 1]$: we assert that $\mu$ is in this interval
- Reporting $[\bar{X} - 1, \bar{X} + 1]$ is less precise than report $\bar{X}$
- Why do we want to report $[\bar{X} - 1, \bar{X} + 1]$ instead of $\bar{X}$?
    - By giving up some precision, we gain some confidence that our assertion is true

- Note $P\{\bar{X} = \mu\} = 0$
- However

$$
\begin{aligned}
&P\{\bar{X} - 1 \leq \mu \leq \bar{X} + 1\} \\
=&P\{-1 \leq \bar{X} - \mu \leq 1\} \\
=&P\left\{-2 \leq \frac{\bar{X} - \mu}{\sqrt{\frac{1}{4}}} \leq -2\right\} \\
=&P\{-2 \leq Z \leq -2\} \qquad \left(\frac{\bar{X} - \mu}{\sqrt{\frac{1}{4}}} \text{ is standard normal}\right) \\
=&0.9544
\end{aligned}
$$

- We have 95% chance of **covering** $\mu$

# Coverage probability and confidence interval

- **Definition**: For an interval estimator $[L(\mathbf{X}), U(\mathbf{X})]$ of parameter $\theta$, the **coverage probability** of $[L(\mathbf{X}), U(\mathbf{X})]$ is the probability that the random interval contains the true $\theta$, denoted by

$$P\{L(\mathbf{X}) \leq \theta \leq U(\mathbf{X})\}, \text{ or } P\{\theta \in [L(\mathbf{X}), U(\mathbf{X})]\}$$

  - The probability statements refers to $\mathbf{X}$ and depends on its distribution $F$
  - Equivalent to $P\{L(\mathbf{X}) \leq \theta, U(\mathbf{X}) \geq \theta\}$

- **Definition**: A $1-\alpha$ confidence interval for $\theta$ is an interval estimator $[L(\mathbf{X}), U(\mathbf{X})]$ which has coverage probability $1-\alpha$

# Asymptotic confidence interval

- When the finite sample distribution is unknown we can approximate the coverage probability by its asymptotic limit
- The **asymptotic** coverage probability of interval estimator $[L(\mathbf{X}), U(\mathbf{X})]$ is

$$\liminf_{n\to\infty} P\{\theta \in [L(\mathbf{X}), U(\mathbf{X})]\}$$

- An $1-\alpha$ **asymptotic** confidence interval for $\theta$ is an interval estimator $[L(\mathbf{X}), U(\mathbf{X})]$ with asymptotic coverage probability $1-\alpha$

# 2. Finding Confidence Interval by Pivotal Quantities

# Pivotal quantity

- **Definition**: A random variable $Q(\mathbf{X}, \theta) = Q(X_1, X_2, \dots X_n, \theta)$ is a pivotal quantity (or pivot) if the distribution of $Q(\mathbf{X}, \theta)$ is independent of parameters $\theta$. That is, if $\mathbf{X} \sim F(\mathbf{x}, \theta)$, then $Q(\mathbf{X}, \theta)$ has the same distribution for all values of $\theta$
- Example
  - Let $\{X_1, X_2, \dots X_n\}$ be a random sample from $\mathrm{N}(\mu, \sigma^2)$
  - Then the t statistic
$$\frac{\bar{X} - \mu}{\frac{s}{\sqrt{n}}}$$
    is a pivot since it follows a $t_{n-1}$ distribution and does not depend on $\mu$ or $\sigma^2$
- Once we have a pivot, finding confidence interval is easy

# Example: confidence interval for normal mean

- Again let $\{X_1, X_2, \dots X_n\}$ be a random sample from $\mathrm{N}(\mu, \sigma^2)$. Then

$$\frac{\bar{X} - \mu}{\frac{s}{\sqrt{n}}} \sim t_{n-1}$$

- Specify a coverage probability $1 - \alpha$
- Let $q_{1-\alpha/2}$ be the $(1 - \alpha/2)$-th quantile of $t_{n-1}$. Then

$$\begin{aligned} P\left\{-q_{1-\alpha/2} \leq \frac{\bar{X} - \mu}{\frac{s}{\sqrt{n}}} \leq q_{1-\alpha/2}\right\} &= 1 - \alpha \\ \Rightarrow P\left\{\bar{X} - q_{1-\alpha/2}\frac{s}{\sqrt{n}} \leq \mu \leq q_{1-\alpha/2}\frac{s}{\sqrt{n}} + \bar{X}\right\} &= 1 - \alpha \end{aligned}$$

- Thus a $1 - \alpha$ confidence interval for $\mu$ is

$$[\bar{X} - q_{1-\alpha/2}\frac{s}{\sqrt{n}}, \bar{X} + q_{1-\alpha/2}\frac{s}{\sqrt{n}}]$$

# Example: confidence interval for normal variance

- Still let $\{X_1, X_2, \dots X_n\}$ be a random sample from $\mathrm{N}(\mu, \sigma^2)$. Then
$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$$
- Specify a coverage probability $1-\alpha$
- Let $c_{\alpha/2}, c_{1-\alpha/2}$ be the $\alpha/2$-th and $(1-\alpha/2)$-th quantile of $\chi^2_{n-1}$. Then
$$\begin{aligned} P\left\{c_{\alpha/2} \le \frac{(n-1)s^2}{\sigma^2} \le c_{1-\alpha/2}\right\} &= 1-\alpha \\ \Rightarrow P\left\{\frac{(n-1)s^2}{c_{1-\alpha/2}} \le \sigma^2 \le \frac{(n-1)s^2}{c_{\alpha/2}}\right\} &= 1-\alpha \end{aligned}$$
- Thus a $1-\alpha$ confidence interval for $\sigma^2$ is $\left[\frac{(n-1)s^2}{c_{1-\alpha/2}}, \frac{(n-1)s^2}{c_{\alpha/2}}\right]$

# Example: asymptotic confidence intervals for non-normal mean

- Let $\{X_1, X_2, \dots X_n\}$ be a random sample from $F$ with mean $\mu$ and variance $\sigma^2$
- By central limit theorem, as $n \to \infty$

$$\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{d} \mathrm{N}(0, 1)$$

- By weak law of large numbers, $s$ is a consistent estimator of $\sigma$
- Hence by continuous mapping theorem

$$\frac{\bar{X} - \mu}{\frac{s}{\sqrt{n}}} \xrightarrow{d} \mathrm{N}(0, 1)$$

  and is an **asymptotic** pivot

- Specify a coverage probability $1 - \alpha$
- Let $z_{1-\alpha/2}$ be the $(1 - \alpha/2)$-th quantile of $\mathrm{N}(0, 1)$. Then as $n \to \infty$

$$
\begin{aligned}
&P\left\{-z_{1-\alpha/2} \le \frac{\bar{X} - \mu}{\frac{s}{\sqrt{n}}} \le z_{1-\alpha/2}\right\} && \to 1 - \alpha \\
\Rightarrow &P\left\{\bar{X} - z_{1-\alpha/2}\frac{s}{\sqrt{n}} \le \mu \le \bar{X} + z_{1-\alpha/2}\frac{s}{\sqrt{n}}\right\} && \to 1 - \alpha
\end{aligned}
$$

- Thus an **asymptotic** $1 - \alpha$ confidence interval for $\mu$ is

$$
[\bar{X} - z_{1-\alpha/2}\frac{s}{\sqrt{n}}, \bar{X} + z_{1-\alpha/2}\frac{s}{\sqrt{n}}]
$$

# Example: asymptotic confidence intervals for estimated parameters

- Let $\hat{\theta}$ be an estimator of scalar valued parameter $\theta$ satisfying

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} \mathrm{N}(0, V)$$

as $n \to \infty$ and $\hat{V}$ is a consistent estimator of $V$

- Standard error for $\hat{\theta}$ is given by $s(\hat{\theta}) = \sqrt{\frac{\hat{V}}{n}}$
- By continuous mapping theorem

$$\frac{\hat{\theta} - \theta}{s(\hat{\theta})} \xrightarrow{d} \mathrm{N}(0, 1)$$

implying $[\hat{\theta} - z_{1-\alpha/2} s(\hat{\theta}), \hat{\theta} + z_{1-\alpha/2} s(\hat{\theta})]$ is an asymptotic $1 - \alpha$ confidence interval

# 3. Finding Confidence Interval by Test Inversion

# Test inversion

- A general way of getting confidence interval is by **test inversion**
- For a parameter $\theta \in \Theta$, consider testing

$$\mathbb{H}_0 : \theta = \theta_0, \ \mathbb{H}_1 : \theta \neq \theta_0$$

- Suppose we have a test statistic $T(\theta_0)$ and a critical value $c$ such that the decision rule

$$\begin{array}{rl} \text{accept } \mathbb{H}_0 & \text{if } T(\theta_0) \leq c \\ \text{reject } \mathbb{H}_0 & \text{if } T(\theta_0) > c \end{array}$$

  has size $\alpha$
- Define the set

$$C = \{\theta \in \Theta : T(\theta) \leq c\}$$

  as **the set of $\theta$ not rejected by the test**
- This test inversion set $C$ is a valid choice of confidence interval

# Theorem

- If $T(\theta_0)$ has exact size $\alpha$ for all $\theta_0 \in \Theta$, then

$$C = \{\theta \in \Theta : T(\theta) \leq c\}$$

is a $1 - \alpha$ confidence interval for $\theta$

- If $T(\theta_0)$ has asymptotic size $\alpha$ for all $\theta_0 \in \Theta$, then $C$ is a asymptotic $1 - \alpha$ confidence interval for $\theta$

- **Proof**: Let the true value be $\theta_0$. Then

$$\begin{aligned} P\{\theta_0 \in C\} &= P\{T(\theta_0) \leq c\} \\ &= 1 - P\{T(\theta_0) > c\} \\ &= 1 - \alpha \end{aligned}$$

where the last equality holds since $T(\theta_0)$ has exact size $\alpha$

If $T(\theta)$ has asymptotic size $\alpha$, then applying limit to the second line yields the conclusion

# Example: asymptotic confidence intervals for estimated parameters

- Again if $\frac{\hat{\theta}-\theta}{s(\hat{\theta})} \xrightarrow{d} \mathrm{N}(0,1)$, then an asymptotic size $\alpha$ test for

$$\mathbb{H}_0 : \theta = \theta_0,\ \mathbb{H}_1 : \theta \neq \theta_0$$

is

$$\begin{array}{ll} \text{accept } \mathbb{H}_0 & \text{if } |T(\theta_0)| \leq z_{1-\alpha/2} \\ \text{reject } \mathbb{H}_0 & \text{if } |T(\theta_0)| > z_{1-\alpha/2} \end{array}$$

where

$$T(\theta_0) = \frac{\hat{\theta} - \theta_0}{s(\hat{\theta})}$$

and $z_{1-\alpha/2}$ is the $(1 - \alpha/2)$-th quantile of $\mathrm{N}(0,1)$

- The test inversion confidence interval is

$$
\begin{aligned}
C &= \{\theta \in \Theta : |T(\theta)| \leq z_{1-\alpha/2}\} \\
&= \{\theta \in \Theta : -z_{1-\alpha/2} \leq \frac{\hat{\theta} - \theta}{s(\hat{\theta})} \leq z_{1-\alpha/2}\} \\
&= \{\theta \in \Theta : \hat{\theta} - z_{1-\alpha/2} s(\hat{\theta}) \leq \theta \leq \hat{\theta} + z_{1-\alpha/2} s(\hat{\theta})\}
\end{aligned}
$$

  which is the same as what we derived in the previous section

- In fact, all the confidence intervals derived by using pivotal quantities rely on test inversion

# Example: inverting Likelihood Ratio Test

- Consider a parametric model $f(x|\theta)$ with log likelihood $\ell_n(\theta) = \sum_{i=1}^{n} \log f(X_i|\theta)$
- The likelihood ratio statistic for testing $\mathbb{H}_0 : \theta = \theta_0, \ \mathbb{H}_1 : \theta \neq \theta_0$ is

$$LR_n(\theta_0) = 2(\max_{\theta \in \Theta} \ell_n(\theta) - \ell_n(\theta_0))$$

- Since $LR_n(\theta_0) \xrightarrow{d} \chi^2_{\dim(\theta)}$, an asymptotic size $\alpha$ test is

$$\begin{array}{ll} \text{accept } \mathbb{H}_0 & \text{if } LR_n(\theta_0) \leq q_{1-\alpha} \\ \text{reject } \mathbb{H}_0 & \text{if } LR_n(\theta_0) > q_{1-\alpha} \end{array}$$

  where $q_{1-\alpha}$ is the $1-\alpha$-th quantile of $\chi^2_{\dim(\theta)}$
- Hence a test inversion $1-\alpha$ confidence interval is

$$\{\theta \in \Theta : LR_n(\theta) \leq q_{1-\alpha}\}$$

# 3. Evaluation of confidence interval

# Length and coverage trade off

- For the same problem, we can find many different confidence intervals
- Naturally we want small length and large coverage probability
- We can always have large coverage probability by increasing the length of the interval
  - $(-\infty, +\infty)$ has coverage probability 1 but not useful
- One method is to minimize length subject to a specified coverage probability

# Example: optimizing interval length for normal mean

- Let $\{X_1, X_2, \ldots X_n\}$ be a random sample from $\mathrm{N}(\mu, \sigma^2)$ with known $\sigma^2$. Then

$$Z = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim \mathrm{N}(0, 1)$$

  is a pivot

- Any number $a$ and $b$ such that

$$P\left\{a \leq \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \leq b\right\} = 1 - \alpha$$

  gives $1 - \alpha$ confidence interval

$$\left\{\mu : \bar{X} - b\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} - a\frac{\sigma}{\sqrt{n}}\right\}$$

- The length of the confidence interval is $(b-a)\frac{\sigma}{\sqrt{n}}$
- The constrained optimization problem we can consider is

$$\min(b-a)\ s.t.\ P\{a \le Z \le b\} = 1-\alpha$$

*Three* 90% *normal confidence intervals*

| $a$ | $b$ | Probability | $b-a$ |
|---|---|---|---|
| $-1.34$ | $2.33$ | $P\{Z < a\} = .09,\ P\{Z > b\} = .01$ | $3.67$ |
| $-1.44$ | $1.96$ | $P\{Z < a\} = .075,\ P\{Z > b\} = .025$ | $3.40$ |
| $-1.65$ | $1.65$ | $P\{Z < a\} = .05,\ P\{Z > b\} = .05$ | $3.30$ |

- In this case, splitting $\alpha$ equally in the two tails results the shortest interval

# Theorem

- Definition: A pdf $f(x)$ is *unimodal* if there exists $x^*$ such that $f(x)$ is nondecreasing for $x \leq x^*$ and $f(x)$ is non increasing for $x \geq x^*$

- Let $f(x)$ be a unimodal pdf. If $[a, b]$ satisfies

  1. $\int_a^b f(x)dx = 1 - \alpha$
  2. $f(a) = f(b) > 0$
  3. $a \leq x^* \leq b$ where $x^*$ is a mode of $f(x)$

  Then $[a, b]$ is the shortest among all intervals satisfy 1