ECON 6170 Section 11

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1 Partial Orders and Lattices

Definition 1. The natural order on $\mathbb{R}^d$ is given by

$$y \geq x \text{ if } (y_i \geq x_i \text{ for } i = 1, 2, \ldots, n)$$

We write $y > x$ if $y \geq x$ but not $x \geq y$.

Definition 2. A **partial order**, $\geq$, on a set X, is a binary relation satisfying for all $x, y, z \in X$

(i) $x \geq x$. [reflexivity]

(ii) If $x \geq y$ and $y \geq z$, then $x \geq z$. [transitivity]

(iii) If $x \geq y$ and $y \geq x$, then $x = y$. [antisymmetry]

We call $(X, \geq)$ a partially ordered set.

Definition 3. If $\geq$ also satisfies completeness, (iv) $x \geq y$ or $y \geq x$ for all $x, y \in X$, then we say that $\geq$ is a total order.

Section Exercise 1. Determine whether each of the following is a partial order. Of those that are partial orders, determine if they are also total orders.

(i) The natural order on $\mathbb{R}^d$.
Yes. If $x_i \geq y_i$ and $y_i \geq z_i$ then $x_i \geq z_i$ for all i. Of course, $x_i \geq x_i$ for all i. And if $x_i \geq y_i$ and $y_i \geq x_i$ then $x_i = y_i$ for all i. It is a total order iff $n = 1$. For example, $(1, 2) \ngeq (2, 1)$ and $(2, 1) \ngeq (1, 2)$.

(ii) The strict natural order $>$ on $\mathbb{R}^d$.
No. It cannot be the case that $x > x$, as this would require it not to be the case that $x \geq x$. So we don't have reflexivity, but you can show that we do have transitivity and antisymmetry.

(iii) The equality relation $=$ on $\mathbb{R}^d$.
Yes. Immediately, we an see that we have transitivity, reflexivity, and antisymmetry. It is not a total order, as $x \neq y$ for almost all pairs (x, y).

(iv) The lexicographic order on $\mathbb{R}^2$, which is given by $x \geq y$ if (1) $x_1 > y_1$ or (2) $x_1 = y_1$ and $x_2 \geq y_2$.

Yes. Reflexivity will hold using (2). For antisymmetry, $x \geq y$ and $y \geq x$ can only hold via (2), so $y = x$.

For transitivity, consider three cases. In the first $x = y$ and $y \geq z$, so $x \geq z$. In the second $x \geq y$ and $y = z$, so $x \geq z$. In the final case, $x > y$ and $y > z$. If $x_1 > y_1$ or $y_1 > z_1$ then $x_1 > z_1$; if $x_1 = y_1 = z_1$, then $x_2 \geq y_2 \geq z_2$. It is also a total order.

(v) The inclusion order $\subseteq$ on 2^X.
Yes. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. $A \subseteq A$. And if $A \subseteq B$ and $B \subseteq A$, then $A = B$. It is a total order iff $\#X \leq 1$. For example, $x \neq y$ implies $\{x\} \nsubseteq \{y\}$ and $\{y\} \nsubseteq \{x\}$.

(vi) The linear preference relation $\succsim$ on $\mathbb{R}^d$, which is given by $y \succsim x$ if $y_1 + \cdots + y_n \geq x_1 + \cdots + x_d$.
No. We do not have antisymmetry—for example, $(1,1) \succsim (2,0)$ and $(2,0) \succsim (1,1)$, but $(1,1) \neq (2,0)$. You can show that transitivity and reflexivity do hold, however.

Definition 4. Let $(X, \geq)$ be a partially ordered set (poset). Given any $x, y \in X$, the *join* of x and y is

$$x \vee y := \sup \{x, y\}$$

That is, (i) $x \vee y \geq x$ and $x \vee y \geq y$ and (ii) if $z \in X$ satisfies $z \geq x$ and $z \geq y$, then $z \geq x \vee y$.

Similarly, the *meet* of x and y is

$$x \wedge y := \inf \{x, y\}.$$

Definition 5. A poset $(X, \geq)$ is a *lattice* if $x \vee y \in X$ and $x \wedge y \in X$ for all $x, y \in X$.

Definition 6. A subset $S \subseteq X$ is a *sublattice of* $X \subseteq \mathbb{R}^d$ if $x \vee y \in S$ and $x \wedge y \in S$ for all $x, y \in S$.

Note that all sublattices of a lattice $(X, \geq)$ are also lattices in their own right, but not all subsets of X that are lattices with respect to $\geq$ are sublattices of $(X, \geq)$.[1] This is illustrated in the following diagram and example.

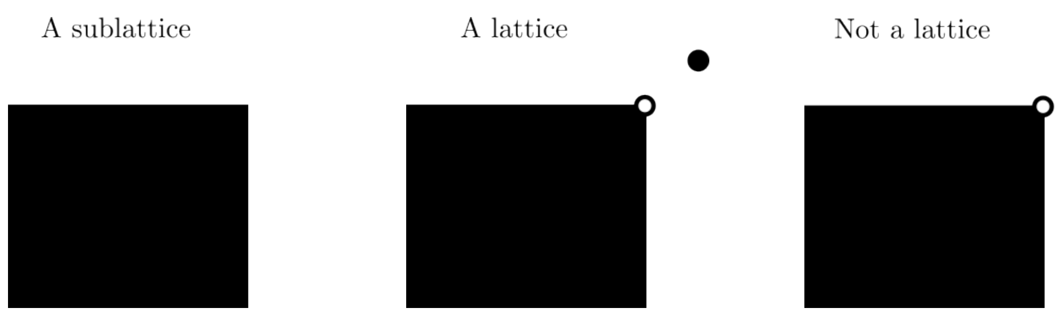

Figure 1: The third set is not a lattice with respect to the natural order because if x is the north-west corner point and y is the south-east corner point, then $\{x, y\}$ has no upper bound, and $x \vee y$ does not exist. The second set is a lattice because the isolated point serves as the join for pairs of points from the north and east boundaries. This isolated point is not the join that would be given in $\mathbb{R}^2$, so this is not a sublattice of $\mathbb{R}^2$.

[1] Basically, sublattice $\Longrightarrow$ (subset and lattice); but (subset and lattice) $\not\Longrightarrow$ sublattice.

Section Exercise 2 (Extension of Example 5 from Lecture Notes)**.** Identify (1) which of the following subsets of $\mathbb{R}^2$ are lattices with respect to the natural order $\geq$, and (2) which are also sublattices of $(\mathbb{R}^2, \geq)$:

(i) $S_1 := \{(1,1),(1,2),(2,1),(2,2)\}$.
S_1 is a sublattice of $\mathbb{R}^2$ because $x \vee y$ and $x \wedge y$ are in S_1 for all $x, y \in S_1$. It is therefore also a lattice in its own right.

(ii) $S_2 := \{(1,1),(1,2),(2,1),(3,3)\}$.
S_2 is not a sublattice of $\mathbb{R}^2$ because $(2,1) \vee (1,2) = (2,2) \notin S_2$. However, it is a lattice because (a) $(3,3) \geq (1,2)$, (b) $(3,3) \geq (2,1)$ and (c) $(3,3)$ is the only (and therefore least) element of S_2 satisfying both (a) and (b). We can therefore write $(2,1) \vee_2 (1,2) = (3,3)$, where $\vee_2$ is the meet operation implied by the lattice $(S_2, \geq)$.

(iii) $S_3 := \{(1,1),(1,2),(2,1)\}$.
S_3 is not a lattice because it does not contain any element satisfying $x \geq (2,1)$ and $x \geq (1,2)$. Therefore it cannot contain a least such element.

(iv) $S_4 := \{(1,1),(1,2),(2,1),(2,3),(3,2),(4,4)\}$.
S_4 is not a lattice because all $x \in \{(3,2),(2,3)\}$ satisfy $x \geq (2,1)$ and $x \geq (1,2)$, but neither $(3,2) \leq (2,3)$ nor $(2,3) \leq (3,2)$. In other words, $\{(2,1),(1,2)\}$ has upper bounds but no *least* upper bound.

Section Exercise 3. Show that S and T are sublattices of lattices $(\mathbb{R}^d, \geq)$ and $(\mathbb{R}^m, \geq)$, respectively if and only if $S \times T$ is a sublattice of $(\mathbb{R}^d \times \mathbb{R}^m, \geq)$.

Let $z := (x,y), z' := (x',y') \in S \times T$. Then $x \vee x', x \wedge x' \in S$ and $y \vee y', y \wedge y' \in T$ if and only if $z \vee z' = (x \vee x', y \vee y'), z \wedge z' = (x \wedge x', y \wedge y') \in S \times T$.

2 Supermodularity

Definition 7. Let Z be a sublattice of $(\mathbb{R}^d, \geq)$. A function $f : Z \to \mathbb{R}$ is supermodular if

$$f(z) + f(z') \leq f(z \vee z') + f(z \wedge z')$$

for all $z, z' \in Z$.

Section Exercise 4. Determine whether the following functions $f : \mathbb{R}^2 \to \mathbb{R}$ are supermodular:

(i) $\max\{x, y\}$
No. Consider $(0,1)$ and $(1,0)$. $(0,1) \vee (1,0) = (1,1)$ and $(0,1) \wedge (1,0) = (0,0)$. Therefore, $f(z \vee z') + f(z \wedge z') = 1 + 0 < f(z) + f(z')$.

(ii) $\min\{x, y\}$
Yes. Let $z := (x,y)$ and $z' := (x',y')$. Suppose, WLOG, that $\min\{x,y\} \leq \min\{x',y'\}$. Then $\min\{x \wedge x', y \wedge y'\} = \min\{x, x', y, y'\} = \min\{x, y\}$. And $\min\{x \vee x', y \vee y'\} \geq \min\{x', y'\}$.

(iii) xy

Yes. We can use the criterion of Proposition 3:

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = 1 \geq 0$$

(iv) x/y with $y \neq 0$
No.

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = -\frac{1}{y^2} < 0$$

Definition 8. Let Z be a sublattice of $(\mathbb{R}^d, \geq)$. A function $f : Z \to \mathbb{R}$ is quasi-supermodular if

$$f(z) \geq f(z \wedge z') \implies f(z \vee z') \geq f(z')$$

and

$$f(z) > f(z \wedge z') \implies f(z \vee z') > f(z')$$

for all $z, z' \in Z$.

Section Exercise 5. Is $\max\{x, y\}$ quasi-supermodular?

The first condition does hold for $\max\{x, y\}$:

$$\max\{x \vee x', y \vee y'\} = \max\{x, x', y, y'\} \geq \max\{x', y'\}$$

is always true. But our counterexample in the previous exercise contradicts the second condition:

$$\max\{0, 1\} = 1 > 0 = \max\{0, 0\} = \max\{0 \wedge 1, 1 \wedge 0\}$$

but

$$\max\{0 \vee 1, 1 \vee 0\} = \max\{1, 1\} = 1 = \max\{1, 0\}$$