ECON 6170 Section 6

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The Relative Topology

The Correspondence notes use the concepts of relative closedness and relative openness. Let $S \subseteq \mathbb{R}^d$.

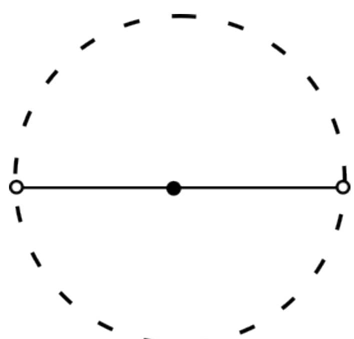
Definition 1. A set $U \subseteq S$ is open relative to S if every point $x \in U$ is contained in a relatively open ball $B_\varepsilon(x) \cap S \subseteq U$.

Example 1.

(i) Consider $\mathbb{R}_+ \subseteq \mathbb{R}$. $[0,1)$ is open relative to $\mathbb{R}_+$. For if $x \in [0,1)$ then $(x-\varepsilon, x+\varepsilon) \cap \mathbb{R}_+ \subseteq [0,1)$ for sufficiently small ε. In particular $(-\varepsilon, \varepsilon) \cap \mathbb{R}_+ = [0,\varepsilon) \subseteq [0,1)$.

(ii) Consider $\mathbb{N} \subseteq \mathbb{R}$. $\{1\}$ is open relative to $\mathbb{N}$. For $(1-\varepsilon, 1+\varepsilon) \cap \mathbb{N} = \{1\} \subseteq \{1\}$ for $\varepsilon < 1$.

(iii) Consider $X := \mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$. $U := (a,b) \times \{0\}$ is open relative to X. For if $x \in U$, then $B_\varepsilon(x) \cap X = (x-\varepsilon, x+\varepsilon) \times \{0\} \subseteq U$ for sufficiently small ε.



Remark 1. If it is clear we are using the relative topology of S, $B_\varepsilon(x)$ is often used in place of $B_\varepsilon \cap S$. For example in (i), $B_\varepsilon(0)$ would be $[0,\varepsilon)$. This is the case in the Correspondence notes.

Definition 2. A set $V \subseteq S$ is closed relative to S if for any sequence (x_n) in V that converges to $x \in S$ has $x \in V$.

Example 2. Consider $\mathbb{R}_{++} \subseteq \mathbb{R}$. $(0,1]$ is closed relative to $\mathbb{R}_{++}$, because $0 \notin \mathbb{R}_{++}$.

Remark 2. If it is clear we are using the relative topology of S, if (x_n) is a sequence in S such that $x_n \to x \notin S$, we sometimes say that (x_n) diverges. That is, convergence means convergence to a point in S. In the previous example, the sequence $(\frac{1}{n})$ would be said to diverge.

Remark 3. The following results about open and closed sets generalise to relatively open and relatively closed sets:

- A is relatively open in S iff $S \setminus A$ is relatively closed in S.
- The union of relatively open sets is also relatively open.
- The intersection of relatively closed sets is also relatively closed.
- The finite intersection of relatively open sets is also relatively open.
- The finite union of relatively closed sets is also relatively closed.

Remark 4. Some results don't generalise to the relative topology:

- Heine-Borel may not hold. That is, some sets that are bounded and relatively closed are not compact (in the sense of sequential or covering compactness). For example, $(0,1]$ is relatively closed in $\mathbb{R}_{++}$ but $\frac{1}{n}$ has no subsequence that converges in $(0,1]$.
- Cauchy sequences may not converge. For example, $(\frac{1}{n})$ is Cauchy but doesn't converge in the relative topology of R_{++}.

Remark 5. Compactness is not relative: a set is (sequentially or covering) compact iff it is compact in every relative topology.

Correspondences

Definition 3. We define the power set of X to be the set of all subsets of X, and denote it by 2^X.

Definition 4. A correspondence $F : X \rightrightarrows Y$, from a nonempty set X to another nonempty set Y, is a function from X to 2^Y.

(i) The domain of F is X.

(ii) The codomain of F is Y.

(iii) The range of F is $\bigcup\{F(x) \in 2^Y \mid x \in X\}$.

In our course, X and Y will always be subsets of Euclidean space.

Definition 5. A correspondence is p-valued if the sets it maps to all have property p.

Remark 6. We can think of functions as a singleton-valued correspondences, in the sense that every function with values $f(x)$ uniquely specifies a correspondence with values $\{f(x)\}$, and *vice versa*.

Section Exercise 1. Determine whether the following correspondences are closed-valued, compact-valued, singleton-valued, and/or convex-valued:

(i) A budget correspondence, $B : \mathbb{R}^{n+1}_{++} \rightrightarrows \mathbb{R}^n_+$, given by $B(p,w) := \{x \in \mathbb{R}^n_+ \mid p \cdot x \leq w\}$.
$p \cdot 0 \leq w$, so $0 \in B$. For fixed p, w the set $B(p,w)$ is the intersection of the closed half-space $p \cdot x \leq w$, with the nonnegative orthant $\mathbb{R}^n_+$. It is therefore the intersection of closed sets, and so closed itself. Given $p \gg 0$ and $x \geq 0$, it is also bounded, and so compact. If $p \cdot x \leq w$ and

$p \cdot y \leq w$, then $p \cdot (\alpha x + (1-\alpha) y) \leq \alpha w + (1-\alpha) w = w$, so it is convex. Therefore, B is a nonempty-valued, compact-valued and convex-valued correspondence.

(a) What happens if we let some of the prices be 0?
B is no longer bounded-valued, and thus no longer compact-valued.

(ii) A Walrasian demand correspondence, $X : \mathbb{R}^{n+1}_{++} \rightrightarrows \mathbb{R}^n_+$, given by $X(p, w) := \arg\max\{u(x) \mid x \in B(p, w)\}$

(a) ...without restrictions on u.
$X(p, w) \subseteq B(p, w)$, so X is bounded-valued. But it may be neither closed nor convex. Take, for example, the utility function $u : \mathbb{R}^2_+ \to \mathbb{R}$ given by

$$u(x_1, x_2) = \mathbf{1}_{\{x_1 \neq x_2\}}$$

Then $(0, \frac{1}{n})$ is a sequence that is (eventually) in $X(p, w)$, but the limit, $(0, 0) \notin X(p, w)$, so X is not closed-valued. Moreover, $(\varepsilon, 0)$ and $(0, \varepsilon)$ are in $X(p, w)$ for sufficiently small $\varepsilon > 0$, but $\frac{1}{2}(\varepsilon, 0) + \frac{1}{2}(0, \varepsilon) = (0, 0) \notin X(p, w)$, so it is not convex-valued either.

(b) ...where u is continuous.
Continuity of u and compactness of $B(p, w)$ imply that u attains a maximum on $B(p, w)$, so X is nonempty-valued. Let (x_i) be an arbitrary sequence in $X(p, w)$. Then $u(x_i) = u^*$ for all $i \in \mathbb{N}$, so by continuity of u, $u(\lim x_i) = \lim u(x_i) = u^*$. Therefore, X is closed-valued, and because it is bounded-valued, compact-valued.

(c) ...where u is continuous and quasiconcave.
Continuity implies that u attains a maximum on $B(p, w)$. Quasiconcavity implies $X(p, w) = \{x \in B(p, w) \mid u(x) \geq \max_{y \in B(p,w)} u(y)\}$ is convex, so X is convex-valued.

(d) ...where u is continuous and *strictly* quasiconcave.
X is singleton-valued. For if $x \neq x'$ and $u(x) = u(x') \geq u(y)$ for all $y \in B(p, w)$, then $u\left(\frac{1}{2}x + \frac{1}{2}x'\right) > u(x)$ by strict quasiconcavity, a contradiction.

(iii) An upper contour correspondence, $R : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^n_+$, given by

$$R(x) := \{y \in \mathbb{R}^n_+ \mid u(y) \geq u(x)\}$$

where u is continuous?
R is nonempty-valued, for $x \in R(x)$. R is closed-valued, for if $(y_i)_{i=1}^{\infty}$ is a convergent sequence in $R(x)$, then $u(y_i) \geq u(x)$ for all i. By continuity of u, this implies that $u(\lim_i y_i) \geq u(x)$, so $\lim_i y_i \in R(x)$.

Section Exercise 2. Consider the following static game

	L	R
U	1,1	0,0
D	0,1	1,1

where player 1 chooses the row and player 2 chooses the column. The ordered-pair entries are the payoffs, (v_1, v_2), to player 1 and player 2, respectively. Define the best response correspondence of player 1 $\text{BR}_1 : \{L, R\} \rightrightarrows \{U, D\}$ by

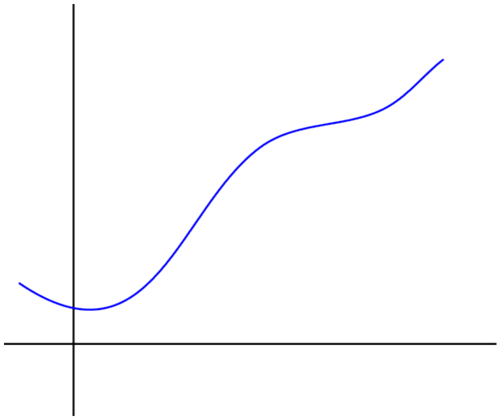
$$x \in \text{BR}_1(y) \iff v_1(x, y) \geq v_1(x', y) \text{ for } x' \in \{U, D\}$$

Define the best response correspondence of player 2 similarly. Which player has a singleton-valued best response correspondence?
Player 1. $\text{BR}_1(L) = \{U\}$ and $\text{BR}_1(R) = \{D\}$. In contrast, $\text{BR}_2(D) = \{L, R\}$.

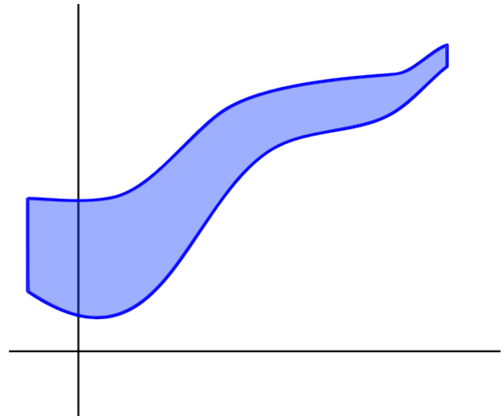
Definition 6. The graph of a function $f : X \to Y$ is the set of points $\{(x, y) \in X \times Y \mid y = f(x)\}$.

Definition 7. The graph of a correspondence $F : X \rightrightarrows Y$ is the set of points $\{(x, y) \in X \times Y \mid y \in F(x)\}$.

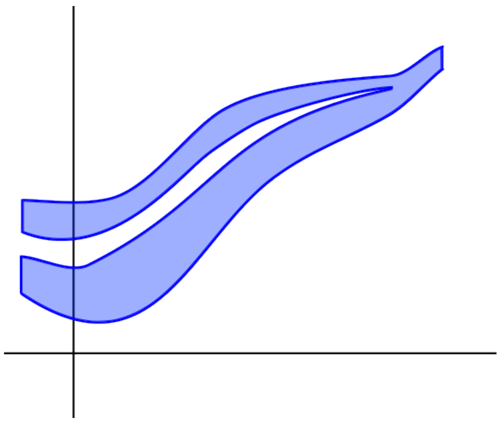
Remark 7. As with functions, graphs of correspondences $F : X \subseteq \mathbb{R} \rightrightarrows \mathbb{R}$ are easy to visualize. The difference is that for correspondences, a vertical line can intersect the graph more than once.



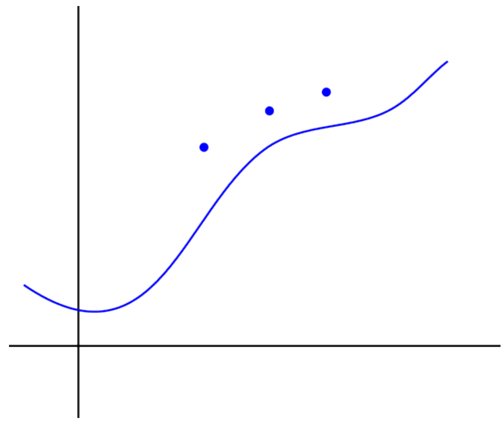
Singleton-valued



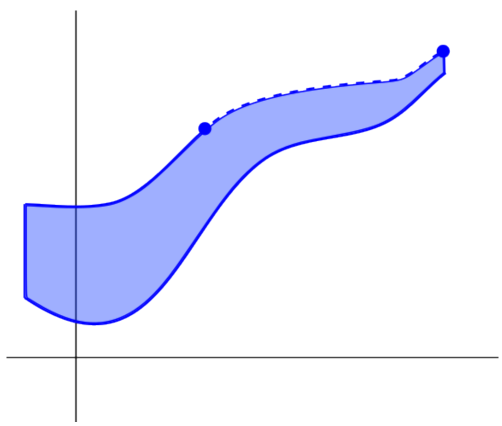
Compact- and convex-valued



Not convex-valued



Not convex-valued



Not closed-valued