ECON 6090

Problem Set 1

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Problem 1 Objecting and rational choice

(a) Consider a decision-maker i, deciding between alternatives x_5^i and x_0^i, which represent the plans that have five percent and zero percent deducted respectively. Define $\succ$ such that $x \succ y$ if i would choose x if given the option. In this model, we have that $x_5^i \succ x_0^i$ in Scenario I, which from our definition of preference relations implies that $x_5^i \succsim x_0^i$, and $x_0^i \not\succsim x_5^i$. However, we have in Scenario II that $x_0^i \succ x_5^i$, which implies that $x_0^i \succsim x_5^i$. However, this is a contradiction of the preferences implied earlier, so this decision-maker is not rational.

(Note that this model assumes that the decision-maker cannot be indifferent between the two options. This fits the empirical results, as it appears that the majority of people prefer not objecting. However, this may be a stronger assumption than is warranted.)

(b) Consider the following set of objects:

$$X = \{(x_0, o), (x_0, n), (x_5, o), (x_5, n)\}$$

where o denotes objecting and n denotes not objecting. Define the following preference relation over these alternatives, which mirrors the Lexicographic preference relation:

$$x \succeq x' \text{ if } x_2 = n \text{ and } x_2' = 0, \text{ or } x_2 = x_2' \text{ and } x_1 \geq x_1'$$

where we (arbitrarily) assume that $x_0 > x_5$. This rationalizes the choices made by the decision-maker, where $(x_0, n) \succeq (x_5, o)$ and $(x_5, n) \succeq (x_0, o)$. This preference relation is additionally complete and transitive over X – indeed, all relationships are strict and ordered, so we have

$$(x_0, n) \succ (x_5, n) \succ (x_0, o) \succ (x_5, o)$$

(c) These preferences are not rationalizable, as the observed preferences violate transitivity. Taking the new set of objects, and making no assumptions about the form of the revealed preference relation:

$$X = \{(x_0, o), (x_0, n), (x_5, o), (x_5, n), (x_{10}, o), (x_{10}, n), (x_{20}, o), (x_{20}, n)\}$$

our observed preferences are, in scenario order:

$$\begin{gathered}(x_5, n) \succ (x_0, o) \\ (x_0, n) \succ (x_5, o) \\ (x_5, o) \succ (x_{10}, n) \\ (x_{10}, n) \succ (x_{20}, o) \\ (x_{20}, o) \succ (x_0, n)\end{gathered}$$

We can construct the following chain:

$$(x_5, o) \succ (x_{10}, n) \succ (x_{20}, o) \succ (x_0, n) \succ (x_5, o)$$

which is a contradiction of transitivity. Since the revealed preferences are not transitive, they are not rationalizable.

Problem 2 $\succeq$ is not a rational preference relation. Consider $x = 2$, and $y = 3$. $x \not\succeq y$, as $2 \not\geq 6$, but $y \not\succeq x$, as $3 \not\geq 4$. Thus, $\succeq$ is not complete, and so is not rational.

Problem 3 $\succeq$ is a preference relation on X such that for some $x, y \in X$, $y \succ x$. We will examine $C^\star(\cdot, \succeq)$.

(a) For $B \ni x, y$, $C^\star(B, \succeq) = C^\star(B \setminus \{x\}, \succeq)$. To see why, note that the only possible difference between them would require $x \in C^\star(B, \succeq)$. However, from the definition of choice functions, that would require $x \succeq z \; \forall \, z \in B$, but $y \in B$ and $y \succ x \Rightarrow x \not\succeq y$. Thus, $x \notin C^\star(B, \succeq)$, so $C^\star(B, \succeq) = C^\star(B \setminus \{x\}, \succeq)$.

(b) For $B \ni x$ where $y \notin B$, it may be the case that $C^\star(B, \succeq) \neq C^\star(B \setminus \{x\}, \succeq)$. That would require that $x \succeq z \; \forall \; z \in B$, which would mean that $x \in C^\star(B, \succeq)$ and $x \notin C^\star(B \setminus \{x\}, \succeq)$. However, if $\exists \, z \in B$ s.t. $z \succ x$, it will be the case that $C^\star(B, \succeq) = C^\star(B \setminus \{x\}, \succeq)$.

Problem 4 $X = \{a, b, c\}$, and $(B, C(\cdot))$ is a choice structure where $B = \mathcal{P}(X)$, and $C(\{a, b, c\}) = \{a\}$. We can say that if $a \in A$, $C(A) = \{a\}$. This is because the fact that $a \in C(\{a, b, c\})$, and $b, c \notin C(\{a, b, c\})$ together imply that $a \succ b$ and $a \succ c$. Thus, for $A \in \{\{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$, $C(A) = \{a\}$. Also, trivially, $C(\{b\}) = \{b\}$, and $C(\{c\}) = \{c\}$, from the definition of the choice correspondence that $C(X) \subseteq X$ and $C(X) \neq \varnothing \; \forall \, X$. We can say nothing about $C(A)$ when $A = \{b, c\}$, as we have no information about whether $b \succeq c$ or $c \succeq b$.

Problem 5 (WARP $\Rightarrow$ Sen's β)

Proof. We have that $(\beta, C(\cdot))$ satisfies WARP. Take some $x, y \in A \subset B$, and assume that $x, y \in C(A)$ and $y \in C(B)$. These are the necessary conditions for Sen's β. By WARP, since $x, y \in A \cap B$, $x \in C(A)$, and $y \in C(B)$, $x \in C(B)$. Thus, since $x \in C(B)$ whenever $x, y \in A \subset B$, $x, y \in C(A)$, and $y \in C(B)$, $(\beta, C(\cdot))$ satisfies Sen's β. $\square$

Problem 6 From Proposition 2.16, we have that for strictly increasing f, if there exists u such that $x \succsim y \Leftrightarrow u(x) \geq u(y)$, then $x \succsim y \Leftrightarrow f(u(x)) \geq f(u(y))$. Thus, this is a valid transformation – if a bundle $x \in \mathbb{R}^n_+$ is preferred to $y \in \mathbb{R}^n_+$, so $x \succsim y$, then $v(x) \geq v(y)$, where $v(x) = f(u(x))$. This transformation will give the same demand as with the original utility function, in terms of bundles of items demanded. To see this, consider that the maximization problem is equivalent to finding an ideal bundle, $x^\star \in \operatorname{argmax}_{x \in \mathbb{R}^n_+} u(x) \equiv \operatorname{argmax}_{x \in \mathbb{R}^n_+} v(x)$. Since the set of maximizers of each function are the same, as $u(x) \geq u(y) \Leftrightarrow v(x) \geq v(y)$, the demand will be the same.

Problem 7 (Compensating Demand)

(a) First, note that by assumption the consumer's demand $x(p, w)$ is homogeneous of degree 0. Thus, the conditions of the law of compensated demand hold, and since the consumer's demand satisfies Walras' law, we have that for their new bundle of goods x'

$$(p' - p) \cdot (x' - x^\star) \leq 0 \Longrightarrow t(x'_1 - x^\star_1) \leq 0$$

Thus, since the tax is strictly positive, $x'_1 \leq x^\star_1$, so their consumption of good 1 weakly decreases.

(b) There are two possible cases here. In the first, $R = 0$ while $x^\star_1 > 0$, *i.e.*, the consumer's demand was simply not compensated. In this case, they will have a strictly lower budget constraint in good 1, so we will have that $x'_1 < x^\star_1$.

In the second case, $x^\star_1 = 0$. In this case, the consumer's demand was compensated and the law of compensated demand holds. We can say that

$$(p' - p) \cdot (x' - x^\star) \leq 0 \Rightarrow t(x'_1) \leq 0$$

Since $t > 0$ and $x_1' \geq 0$, we can say that $x_1' = 0$ and the consumer's demand will not change.

Problem 8 These preferences are convex. By Proposition 2.32, a utility function representing a preference relation $\succeq$ is quasiconcave if and only if $\succeq$ is convex. Taking the derivatives of u, we get that $u'(x) = 1 - 2x$, and $u''(x) = -2$. Since $u''(x) < 0 \; \forall \; x \in [0, 1]$, u is strictly concave, and by implication quasiconcave[1], and thus the preferences $\succeq$ that it represents are convex.

Problem 9 Note first that since $15 \cdot 50 + 9 \cdot 50 = 1,200 < 1,250$, the bundle x^0 is available in the consumer in year 1. Thus, their chosen bundle in year 1 must not have been available to them in year 0. Otherwise, their preferences would violate WARP. In other words, the bundles that violate WARP satisfy:

$$15x_1 + 9x_2 \leq 1,250 \qquad \text{and} \qquad 10x_1 + 10x_2 \leq 1,000$$

Assuming that Walras' Law holds, we have that any amount not spent on x_2 is spent on x_1, so these equations should hold with equality. Solving them, we get that the minimum number of x_2^1 that would not violate WARP is 41.67. If they choose a bundle that contains less x_2, it would have been attainable under the year 0 prices, so the choice of $(50, 50)$ in year 0 would violate WARP.

Problem 10 (Identifying whether consumers are better off)

(i) We consider the three cases:

  (a) $p_1 x_2 < w_1$ and $p_2 x_1 > w_2$: In this case, we can say that consumer 1 is better off than consumer 2. Specifically, we can see that the bundle x_2 is attainable under (p_1, w_1). Since x_2 is attainable, it must be that $x_1 \succeq x_2$.

  (b) $p_1 x_2 > w_1$ and $p_2 x_1 > w_2$: In this case, we cannot say whether either consumer is better off, as neither of their bundles are attainable to the other.

  (c) $p_1 x_2 < w_1$ and $p_2 x_1 < w_2$: These choices are inconsistent with rationality. To see why, note that $\exists \; \varepsilon_1 > 0$ s.t. $B_{\varepsilon_1}(x_2)$ is entirely contained in the feasible set. By local non-satiation, we have that $\exists \; x' \in B_{\varepsilon_1}$ s.t. $x' \succ x_2$. However, since consumer 1 chooses x_1, we have that $x_1 \succsim x' \succ x_2 \Rightarrow x_1 \succ x_2$. This implies that consumer 2's choice of x_2 violates rationality of their common preferences, as $x_2 \not\succsim x_1$ and x_1 is feasible. The same argument applies in reverse, so the choices violate rationality.

(ii) This argument makes sense. If we consider the location as another element of their preferences, it would violate WARP for consumer 2 to choose to live in their current location instead of (costlessly) moving to consumer 1's location and being able to attain their better bundle. This argument, and the fact that moving is costless, entirely undermines the assumption that their preferences are identical.

[1] Because quasiconcavity is a weaker condition. Tak stated this in Math Review, so I'm assuming we can take it as given, but I can provide a proof if you want.