Econ 6190 Problem Set 2

Fall 2024

1. [Hansen 4.9] Suppose that X_i are i.n.i.d. (independent but not necessarily identically distributed) with $\mathbb{E}[X_i] = \mu_i$ and $\text{var}[X_i] = \sigma_i^2$.

   (a) Find $\mathbb{E}[\bar{X}]$;

   (b) Find $\text{var}[\bar{X}]$.

2. [Mid term, 2022] Let $X \sim \text{N}(\mu, \sigma^2)$ for some unknown μ and **known** σ^2. Furthermore, suppose I believe that μ can only take two values, $\frac{1}{2}$ or $-\frac{1}{2}$, and I believe $P\{\mu = \frac{1}{2}\} = \frac{1}{2}$, and $P\{\mu = -\frac{1}{2}\} = \frac{1}{2}$. Now, I draw a single observation X_1 from the distribution of X, and it turns out $X_1 < 0$. Given that I observe $X_1 < 0$, what is my updated probability that $\mu = \frac{1}{2}$? That is, find $P\{\mu = \frac{1}{2} | X_1 < 0\}$. The following notations can be useful: $\Phi(t)$ is the cdf of a standard normal, and $\phi(t)$ is the pdf of a standard normal.

3. [Hansen, 5.2, 5.3] For the standard normal density $\phi(x)$, show that $\phi'(x) = -x\phi(x)$. Then, use integration by parts to show that $\mathbb{E}[Z^2] = 1$ for $Z \sim \text{N}(0, 1)$.

4. [Mid term, 2023] If X is normal with mean μ and variance σ^2, it has the following pdf

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right), \text{ for } x \in \mathbb{R}.$$

   Let X and Y be jointly normal with the joint pdf

$$f(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - 2\frac{\rho xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2}\right)\right), \text{ for } x, y \in \mathbb{R} \tag{1}$$

   where $\sigma_X > 0, \sigma_Y > 0$ and $-1 \leq \rho \leq 1$ are some constants.

   (a) Without using the properties of jointly normal distributions, show that the marginal distribution of Y is normal with mean 0 and variance σ_Y^2.

   (b) If you cannot work (a) out, assume it is true and move on. Derive the conditional distribution of X given $Y = y$. (Hint: it should be normal with mean $\frac{\sigma_X}{\sigma_Y}\rho y$ and variance $(1-\rho^2)\sigma_X^2$).

(c) Let $Z = \frac{X}{\sigma_X} - \frac{\rho}{\sigma_Y} Y$. Show Y and Z are independent. Clearly state your reasoning. (Hint: For this question, you can use the properties of jointly normal distributions.)

5. [Hansen 5.18, 5.19] Show that:

   (a) If $e \sim \mathrm{N}(0, I_n \sigma^2)$ and $\mathbf{H}'\mathbf{H} = I_n$, then $u = \mathbf{H}'e \sim \mathrm{N}(0, I_n \sigma^2)$.

   (b) If $e \sim \mathrm{N}(0, \Sigma)$ and $\Sigma = \mathbf{A}\mathbf{A}'$, then $u = \mathbf{A}^{-1}e \sim \mathrm{N}(0, I_n)$.

6. [Hansen 6.13] Let $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$. Find the covariance of $\hat{\sigma}^2$ and $\bar{X}$. Under what condition is this zero? [Hint: This exercise shows that the zero correlation between the numerator and the denominator of the t ratio does not always hold when the random sample is not from a normal distribution].

1. (a). $E\,\bar{X}_n = E\,\dfrac{X_1+X_2+\cdots+X_n}{n} = \dfrac{1}{n}\sum_{i=1}^{n} E X_i$

$= \dfrac{1}{n}\sum_{i=1}^{n} u_i$

(b). $Var(\bar{X}_n) = Var\left(\dfrac{1}{n}\sum_{i=1}^{n} X_i\right)$

$= \dfrac{1}{n^2}\cdot\sum_{i=1}^{n} Var(X_i)$

$= \dfrac{1}{n^2}\cdot\sum_{i=1}^{n} \sigma_i^2 .$

2. *Answer:* By the Bayes rule for events:

$$
\begin{aligned}
P\{\mu=\frac{1}{2}|X_1<0\} &= \frac{P\{\mu=\frac{1}{2},X_1<0\}}{P\{X_1<0\}} \\
&= \frac{P\{\mu=\frac{1}{2}\}P\{X_1<0|\mu=\frac{1}{2}\}}{P\{\mu=\frac{1}{2}\}P\{X_1<0|\mu=\frac{1}{2}\}+P\{\mu=-\frac{1}{2}\}P\{X_1<0|\mu=-\frac{1}{2}\}}.
\end{aligned}
$$

We know $P\{\mu=\frac{1}{2}\}=P\{\mu=-\frac{1}{2}\}=\frac{1}{2}$. It suffices to calculate $P\{X_1<0|\mu=\frac{1}{2}\}$ and $P\{X_1<0|\mu=-\frac{1}{2}\}$. Since $X_1\sim \mathrm{N}(\mu,\sigma^2)$, $\frac{X_1-\mu}{\sigma}\sim \mathrm{N}(0,1)$. Thus, $P\{X_1<0|\mu\}=P\{\frac{X_1-\mu}{\sigma}<\frac{-\mu}{\sigma}\}=\Phi(\frac{-\mu}{\sigma})$. Therefore,

$$
\begin{aligned}
P\{X_1<0|\mu=\frac{1}{2}\} &= \Phi(\frac{-\frac{1}{2}}{\sigma}), \\
P\{X_1<0|\mu=-\frac{1}{2}\} &= \Phi(\frac{\frac{1}{2}}{\sigma}).
\end{aligned}
$$

It follows

$$
\begin{aligned}
P\{\mu=\frac{1}{2}|X_1<0\} &= \frac{\frac{1}{2}\Phi(\frac{-\frac{1}{2}}{\sigma})}{\frac{1}{2}\Phi(\frac{-\frac{1}{2}}{\sigma})+\frac{1}{2}\Phi(\frac{\frac{1}{2}}{\sigma})} \\
&= \frac{1-\Phi(\frac{1}{2\sigma})}{1-\Phi(\frac{1}{2\sigma})+\Phi(\frac{1}{2\sigma})} = 1-\Phi(\frac{1}{2\sigma}).
\end{aligned}
$$

3.

a) derivative of the standard normal density:

The standard normal density function is given by $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.

Use the chain rule to find $\phi'(x)$,

$$\begin{aligned}\phi'(x) &= \frac{d}{dx}\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\right) \\ &= -x\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = -x\phi(x).\end{aligned}$$

So, we have $\phi'(x) = -x\phi(x)$.

b) use integration by parts to show that $\mathbb{E}[Z^2] = 1$:

For a standard normal random variable $Z \sim N(0,1)$, the expectation $\mathbb{E}[Z^2]$ is given by:

$$\mathbb{E}[Z^2] = \int_{-\infty}^{\infty} x^2\phi(x)\,dx.$$

Use integration by parts. Define $u(x) = x$ and $dv = x\phi(x)\,dx$, which implies:

$$\begin{aligned}du &= dx \\ v(x) &= -\phi(x) \quad \left(\text{since } \frac{d}{dx}(-\phi(x)) = x\phi(x)\right)\end{aligned}$$

Recall the integration by parts formula:

$$\int u\,dv = uv\Big|_{-\infty}^{\infty} - \int v\,du.$$

Substituting into the formula:

$$\int_{-\infty}^{\infty} x^2\phi(x)\,dx = -x\phi(x)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \phi(x)\,dx$$

The first term $-x\phi(x)\Big|_{-\infty}^{\infty}$ evaluates to 0, because as $x \to \infty$ or $x \to -\infty$, $x\phi(x) \to 0$. Thus, we are left with:

$$\mathbb{E}[Z^2] = \int_{-\infty}^{\infty} \phi(x)\,dx = 1$$

4. (a)

Answer:

$$
\begin{aligned}
f_Y(y) &= \int f(x,y)dx \\
&= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \int \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - 2\frac{\rho xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2}\right)\right) dx \\
&= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \int \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - 2\frac{\rho xy}{\sigma_X\sigma_Y} + \frac{\rho^2 y^2}{\sigma_Y^2} - \frac{\rho^2 y^2}{\sigma_Y^2} + \frac{y^2}{\sigma_Y^2}\right)\right) dx \\
&= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \int \exp\left(-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x}{\sigma_X} - \frac{\rho y}{\sigma_Y}\right)^2 + \frac{(1-\rho^2)\,y^2}{\sigma_Y^2}\right)\right) dx \\
&= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2}\frac{y^2}{\sigma_Y^2}\right) \int \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x}{\sigma_X} - \frac{\rho y}{\sigma_Y}\right)^2\right) dx \\
&= \frac{1}{2\pi\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2}\frac{y^2}{\sigma_Y^2}\right) \int \exp\left(-\frac{1}{2(1-\rho^2)}\left(t - \frac{\rho y}{\sigma_Y}\right)^2\right) dt (\textit{by change of variable}) \\
&= \frac{1}{\sqrt{2\pi}\sigma_Y} \exp\left(-\frac{1}{2}\frac{y^2}{\sigma_Y^2}\right) \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \int \exp\left(-\frac{1}{2(1-\rho^2)}(t - \frac{\rho y}{\sigma_Y})^2\right) dt \\
&= \frac{1}{\sqrt{2\pi}\sigma_Y} \exp\left(-\frac{1}{2}\left(\frac{y^2}{\sigma_Y^2}\right)\right),
\end{aligned}
$$

where the last equality follows as $\frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \int \exp\left(-\frac{1}{2(1-\rho^2)}(t - \frac{\rho y}{\sigma_Y})^2\right) dt = 1$, *since it is the integration of the pdf of a normal random variable with mean* $\frac{\rho y}{\sigma_Y}$ *and variance* $1-\rho^2$.

(b)

Answer:

$$
\begin{aligned}
f_{X|Y}(x|y) &= \frac{f(x,y)}{f_Y(y)} \\
&= \frac{\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - 2\frac{\rho xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2}\right)\right)}{\frac{1}{\sqrt{2\pi}\sigma_Y} \exp\left(-\frac{1}{2}\left(\frac{y^2}{\sigma_Y^2}\right)\right)} \\
&= \frac{1}{\sqrt{2\pi}\sigma_X\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - 2\frac{\rho xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2}\right) + \frac{1}{2}\left(\frac{y^2}{\sigma_Y^2}\right)\right) \\
&= \frac{1}{\sqrt{2\pi}\sigma_X\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - 2\frac{\rho xy}{\sigma_X\sigma_Y} + \frac{y^2\rho^2}{\sigma_Y^2}\right)\right) \\
&= \frac{1}{\sqrt{2\pi}\sigma_X\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x}{\sigma_X} - \frac{y\rho}{\sigma_Y}\right)^2\right) \\
&= \frac{1}{\sqrt{2\pi}\sigma_X\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)\sigma_X^2}\left(x - \frac{y\sigma_X\rho}{\sigma_Y}\right)^2\right)
\end{aligned}
$$

as required.

(c)

Answer: As Z is a linear combination of (X, Y), Y is also (trivially) a linear combination of (X, Y), and (X, Y) are jointly normal, it follows that (Y, Z) are jointly normal as well. We calculate the covariance between Z and Y:

$$\begin{aligned} Cov(Z, Y) &= Cov(\frac{X}{\sigma_X} - \frac{\rho}{\sigma_Y}Y, Y) = \frac{1}{\sigma_X}Cov(X, Y) - \frac{\rho}{\sigma_Y}Var(Y) \\ &= \frac{1}{\sigma_X}\sigma_X\sigma_Y\rho - \frac{\rho}{\sigma_Y}\sigma_Y^2 \\ &= \sigma_Y\rho - \rho\sigma_Y = 0. \end{aligned}$$

As Z and Y are jointly normal, $Cov(Z, Y) = 0$ implies that Z and Y are independent.

5.

(a). Note: H should be $n \times n$ matrix.

$$e = \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} \sim N(0, I_n 6^2)$$

since $u = H'e$, then each element in u is linear combination of e_i

u should also follow normal distribution

$$E u = E(H'e) = 0$$

$$\begin{aligned} Var\ u &= Var(H'e) = H'\ Var(e) \cdot H \\ &= 6^2 \cdot H'H = 6^2 I_n \end{aligned}$$

then $u \sim N(0, I_n 6^2)$

(b). same logic as above.

$u = A^{-1}e$ should follow normal distribution

$$Eu = E(A^{-1}e) = 0$$

$$\text{Var } u = \text{Var}(A^{-1}e) = A^{-1} \text{ Var } e \, (A^{-1})'$$

$$= A^{-1} \cdot A \, A' \cdot (A^{-1})' = I_n$$

#.

6. $\text{cov}(\bar{X}, \hat{\sigma}^2)$

$$= E((\bar{X} - u + u)\hat{\sigma}^2) - E(\bar{X}) E\hat{\sigma}^2$$

$$= E((\bar{X} - u)\hat{\sigma}^2) + u E\hat{\sigma}^2 - E(\bar{X}) E\hat{\sigma}^2$$

$$= E[(\bar{X} - u)\hat{\sigma}^2]$$

$$= E\left\{\left[\frac{1}{n}\sum_{i=1}^{n}(X_i - u)\right] \cdot \left[\frac{1}{n}\sum_{i=1}^{n}[(X_i - u)^2 - (u - \bar{X}_n)^2]\right]\right\}$$

we can check $E(X_i - u)(X_j - u)^2 = 0 \quad (i \neq j)$

$$\frac{1}{n^2} E \sum_{i=1}^{n}(X_i - u)\left[(X_i - u)^2 - n(u - \bar{X}_n)^2\right]$$

$$= \frac{1}{n^2} E \sum_{i=1}^{n}\left[(X_i - u)^3 - n^2(\bar{X}_n - u)^3\right]$$

$$= \frac{1}{n^2}\left[\sum_{i=1}^{n} E(X_i - u)^3 - n^2 E\left(\frac{1}{n}\sum_{i=1}^{n}(X_i - u)\right)^3\right]$$

$$= \frac{1}{n^2}\left[n \cdot E(X - u)^3 - \frac{1}{n^2} \cdot n^2 E(X - u)^3\right]$$

$$= \frac{n-1}{n^2} E(X - u)^3$$

$\text{cov}(\bar{X}, \hat{\sigma}^2) = 0$ if the third central moment $= 0$.

Check more details in section 3 notes.