ECON 6090 - Solutions to PS2

Feiyu Wang & Yuxuan Ma

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Exercise 2

a.

$$\max_{x,l\geq 0} \quad (x^{\frac{1}{2}}+l)^2$$
$$\text{s.t. } px \leq w-l$$

b. Since the utility is strictly increasing on x for any l, budget constraint must be binding, i.e. $w-l=px$.

So we have the problem

$$\max_{x\geq 0} \quad (x^{\frac{1}{2}}+w-px)^2$$
$$\text{s.t. } px \leq w$$

Then we use KKT condition to solve the problem (using normal Lagrangian and check corner solution is also good)

$$L=(x^{\frac{1}{2}}+l)^2+\lambda(w-px)$$
$$\frac{\partial L}{\partial x}=2(x^{\frac{1}{2}}+l)(\frac{1}{2}x^{-\frac{1}{2}}-p)-\lambda p=0$$
$$\frac{\partial L}{\partial \lambda}=w-px\geq 0 \qquad \lambda\geq 0$$
$$\frac{\partial L}{\partial \lambda}\lambda=0$$

Case 1: $\lambda=0, w-px\geq 0$

$$\Rightarrow 2(x^{\frac{1}{2}}+w-px)(\frac{1}{2}x^{-\frac{1}{2}}-p)=0$$
$$\Rightarrow x=\frac{1}{4p^2}$$
$$u(x)=(\frac{1}{4p}+w)^2$$
$$w-px=w-\frac{1}{4p}\geq 0 \Leftrightarrow w\geq\frac{1}{4p}$$

Case 2: $\lambda>0, w-px=0$

$$\Rightarrow x=\frac{w}{p}$$
$$u(x)=\frac{w}{p}$$

We know the optimal utility level in case 1 is always higher than case 2, but case 1 only valid when $w \geq \frac{1}{4p}$. So, we have the Walrasian demand:

$$\text{When } w - \frac{1}{4p} \geq 0, \quad x(p,w) = \frac{1}{4p^2}, \quad l(p,w) = w - \frac{1}{4p}$$
$$\text{When } w - \frac{1}{4p} < 0, \quad x(p,w) = \frac{w}{p}, \quad l(p,w) = 0$$

c.

$$\text{When } w - \frac{1}{4p} \geq 0, \quad V(p,w) = (\frac{1}{4p} + w)^2$$
$$\text{When } w - \frac{1}{4p} < 0, \quad V(p,w) = \frac{w}{p}$$

d.

$$V(p,w) = (\frac{1}{4p} + w)^2$$
$$\text{Use } V(p, e(p,u)) = u$$
$$\Rightarrow (\frac{1}{4p} + e(p,u))^2 = u$$
$$e(p,u) = u^{\frac{1}{2}} - \frac{1}{4p}$$
$$h(p,u) = \frac{\partial e(p,u)}{\partial p} = \frac{1}{4p^2}$$

Exercise 3

a.

$$h_i(p,u) = \frac{\partial e(p,u)}{\partial p_i} = g(u)\frac{\partial r(p)}{\partial p_i}$$
$$e(p, V(p,w)) = w$$
$$V(p,w) = g^{-1}(\frac{w}{r(p)})$$
$$x_i(p,w) = h_i(p, V(p,w)) = \frac{\partial r(p)}{\partial (p_i)} \frac{w}{r(p)}$$

b. Walras Law: $\sum p_i x_i(p,w) = w$

$$\Rightarrow \sum p_i \frac{\partial r(p)}{\partial p_i} \frac{w}{r(p)} = w$$
$$\Rightarrow r(p) = \sum p_i \frac{\partial r(p)}{\partial p_i}$$

So, we don't need any further assumptions.

c.

$$
\begin{aligned}
&\text{Aggregate Demand for good } i \\
&= \sum_{n=1}^{M} x_i(p, w_n) \\
&= \sum_{n=1}^{M} \frac{\partial r(p)}{\partial p_i} \frac{w_n}{r(p)} \\
&= \frac{\partial r(p)}{\partial p_i} \frac{\bar{w}}{r(p)} \text{ ,where } \bar{w} = \sum_{n=1}^{M} w_n
\end{aligned}
$$

Exercise 4

a. Firstly, since $e(p,u)$ is HOD 1 on p, we need to have $\alpha + \beta = 1$.

Secondly, we need $e(p,u)$ is concave on p, thus $\alpha, \beta \in [0,1]$.

b.

$$
\begin{aligned}
e(p, V(p,w)) &= w \\
\Rightarrow V(p,w) &= \frac{w}{p_1^{\alpha} p_2^{\beta}} \\
h_1(p,u) &= \frac{\partial e(p,u)}{\partial p_1} = \alpha u p_1^{\alpha-1} p_2^{\beta} \\
h_2(p,u) &= \beta u p_1^{\alpha} p_2^{\beta-1}
\end{aligned}
$$

$$\text{Then we have} \quad x_1(p,w) = h_1(p, V(p,w)) = \alpha \frac{w}{p_1}, \quad x_2(p,w) = \beta \frac{w}{p_2}$$

c.

$$
\begin{aligned}
x_1(p,w) &= -\frac{\partial V(p,w)/\partial p_1}{\partial V(p,w)/\partial w} \\
&= \alpha \frac{w}{p_1} \\
x_2(p,w) &= \beta \frac{w}{p_2}
\end{aligned}
$$

d. Denote $p_0 = (1,1)$ and $p_1 = (16,16)$

$$u_0 := V(p_0, w) = 512$$

$$u_1 := V(p_1, w) = 32$$

$$CV = e(p_1, u_1) - e(p_1, u_0) = 512 - 512 \times 16 = -7680$$

$$EV = e(p_0, u_1) - e(p_0, u_0) = 32 - 512 = -480$$

The absolute value of CV is higher because there is positive income effect. CV is more reasonable, because it is by definition the amount of money compensated to consumer after price change such that they are indifferent.

Exercise 5

a.

$$e(p,u) = min \quad p_1x_1 + p_2x_2 \quad \text{s.t. } 2ln(x_1) + 2ln(x_2) \geq u$$

$$e^*(p,u^*) = min \quad p_1x_1 + p_2x_2 \quad \text{s.t. } x_1x_2 \geq u^*$$

$$\Leftrightarrow 2ln(x_1) + 2ln(x_2) \geq 2lnu^*$$

$$\Rightarrow e^*(p, exp\frac{u}{2}) = min \quad p_1x_1 + p_2x_2 \quad \text{s.t. } 2ln(x_1) + 2ln(x_2) \geq u$$

$$\Rightarrow e^*(p, exp\frac{u}{2}) = e(p,u)$$

b. It is compensated price change, by WARP we know that $\Delta x_i \leq 0$ and consumer get weakly higher utility. So, as long as consumer does not choose the same bundle before and after the price change, we have $\Delta x_i < 0$ and consumer get higher utility.

c. Suppose that $\exists$ $u(x)$, optimal consumption path $\{c_t\}$ and t_0 s.t. $c_{t_0+1} > c_{t_0}$. Then we can find another consumption path $\{c_t^*\}$, s.t.

$c_t^* = c_t$, for $t \neq t_0$ or $t_0 + 1$

$c_{t_0}^* = c_{t_0+1}$ and $c_{t_0+1}^* = c_{t_0}$

Then $\sum \beta^t u(c_t^*) - \sum \beta^t u(c_t) = (\beta^t - \beta^{t+1})(u(c_{t_0+1}) - u(c_t)) > 0$

Contradict with the assumption that $\{c_t\}$ is optimal.