Econ 6190 Problem Set 9

Fall 2024

1. Let $X \sim \text{binomial}(5, \theta)$ with θ unknown. Consider testing $\mathbb{H}_0 : \theta = \frac{1}{2}$ versus $\mathbb{H}_1 : \theta > \frac{1}{2}$.

   (a) Consider test alpha that rejects $\mathbb{H}_0$ if and only if all "successes" are observed. Derive the power function of this test. Calculate its type I error. Express its type II error as a function of θ where $\theta > \frac{1}{2}$.

   (b) Consider an alternative test beta that rejects $\mathbb{H}_0$ if we observe $X = 3, 4$, or 5. Write down the power function of this test. Calculate its type I error. Express its type II error as a function of θ where $\theta > \frac{1}{2}$.

   (c) Between tests alpha and beta, which test has a smaller type I error? Which test has a smaller type II error? Which test would you prefer?

2. Take the model $X \sim \mathrm{N}(\mu, \sigma^2)$ with σ^2 unknown. A sample of size $n = 4$ yields $\sum_{i=1}^4 X_i = 40$, $\sum_{i=1}^4 \left(X_i - \bar{X}\right)^2 = 48$, where $\bar{X}$ is the sample average.

   (a) Propose a test for testing $\mathbb{H}_0 : \mu = 9$ and $\mathbb{H}_1 : \mu \neq 9$ given significance value $\alpha = 5\%$. What is the critical value? Can you reject the null? Draw a graph of the distribution of your statistic if the null hypothesis is correct and indicate the rejection region.

   (b) Do the same for $\mathbb{H}_0 : \mu = 7$ and $\mathbb{H}_1 : \mu > 7$ given significance value $\alpha = 5\%$.

3. Take the model $X \sim \mathrm{N}(\mu, 4)$. We want to test the null hypothesis $\mathbb{H}_0 : \mu = 20$ against $\mathbb{H}_1 : \mu > 20$. A sample of $n = 16$ independent realizations of X was collected, and the sample mean $\bar{X} = 20.5$.

   (a) Propose a test with size α equal to 1%. What is the condition for rejecting $\mathbb{H}_0$ for this test?

   (b) What is the p value of this test?

   (c) What is the condition for rejecting $\mathbb{H}_0$ with $\alpha = 1\%$ if we increase the size of the sample to $n = 25$?

   (d) We want a test with power 90% if $\mu = 21$. What is the size of the sample n needed for that? Explain briefly how n affects the power of the test.

   (e) Now consider the two-sided test $\mathbb{H}_0 : \mu = 20$ against $\mathbb{H}_1 : \mu \neq 20$. Write down the power function of the test if $\mu = 21$. Compare with (d). Do you need a larger or smaller n in order to achieve 90% power?

4. [Hansen 13.11] You have two samples (Madison and Ann Arbor) of monthly rents paid by n individuals in each sample. You want to test the hypothesis that the average rent in the two cities is the same. Construct an appropriate test.

5. [Hansen 13.13] You design a statistical test of some hypothesis $\mathbb{H}_0$ which has asymptotic size 5% but you are unsure of the approximation in finite samples. You run a simulation experiment on your computer to check if the asymptotic distribution is a good approximation. You generate data which satisfies $\mathbb{H}_0$. On each simulated sample, you compute the test. Out of $B = 50$ independent trials you find 5 rejections and 45 acceptances.

   (a) Based on the $B = 50$ simulation trials, what is your estimate $\hat{p}$ of p, the probability of rejection?

   (b) Find the asymptotic distribution for $\sqrt{B}(\hat{p} - p)$.

   (c) Test the hypothesis that $p = 0.05$ against $p \neq 0.05$. Does the simulation evidence support or reject the hypothesis that the size is 5%?

6. One very striking abuse of hypothesis testing is to choose size α **after** seeing the data and to choose them in such a way as to force rejection (or acceptance) of a null hypothesis. To see what the **true** Type I and Type II error probabilities of such a procedure are, calculate size and power of the following two trivial tests:

   (a) Always reject $\mathbb{H}_0$, no matter what data are obtained. (equivalent to the practice of choosing the α level to force rejection of $\mathbb{H}_0$)

   (b) Always accept $\mathbb{H}_0$, no matter what data are obtained. (equivalent to the practice of choosing the α level to force acceptance of $\mathbb{H}_0$)

7. [Final exam, 2021 fall] Suppose $X \sim \mathrm{N}(\mu, \sigma^2)$ where both μ and σ^2 are unknown. We hope to use a random sample $\{X_i, i = 1 \dots n\}$ drawn from X to test hypothesis: $\mathbb{H}_0 : \mu = \mu_0$ for some $\mu_0 \in \mathbb{R}$ against $\mathbb{H}_1 : \mu \neq \mu_0$.

   (a) Let $\beta = (\mu, \sigma^2)$. Write down the log likelihood of β under $\mathbb{H}_0$.

   (b) The unconstrained MLE of β is $\hat{\beta} = (\bar{X}_n, \hat{\sigma}^2)$, where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$. Derive the likelihood ratio statistic LR_n for testing $\mathbb{H}_0 : \mu = \mu_0$ vs $\mathbb{H}_1 : \mu \neq \mu_0$. Simplify as much as you can.

   (c) Show the likelihood ratio test based on $LR_n > c$ for some c is equivalent to $|T| > b$ for some b, where $T = \frac{\bar{X}_n - \mu_0}{\sqrt{\frac{\hat{\sigma}^2}{n}}}$.

Q1

(a) The power function of test alpha is

$$\pi_\alpha(\theta) = P\{X = 5\} = \theta^5.$$

Thus probability of type I error is $\pi_\alpha(\theta|\theta = \frac{1}{2}) = \left(\frac{1}{2}\right)^5 = 0.0313$. Probability of Type II error is $1 - \pi_\alpha(\theta|\theta > \frac{1}{2}) = 1 - \theta^5$. When $\theta > \frac{1}{2}$, $\theta^5 < 0.0313$. Therefore Type II error of test alpha is quite large.

(b) The power function of test beta is

$$\begin{aligned}\pi_\beta(\theta) &= P\{X = 3, 4 \text{ or } 5\} \\ &= \theta^5 + \begin{pmatrix} 5 \\ 4 \end{pmatrix} \theta^4(1-\theta) + \begin{pmatrix} 5 \\ 3 \end{pmatrix} \theta^3(1-\theta)^2.\end{aligned}$$

Thus probability of type I error is $\pi_\beta(\theta|\theta = \frac{1}{2}) = \left(\frac{1}{2}\right)^5 + \begin{pmatrix} 5 \\ 4 \end{pmatrix} \left(\frac{1}{2}\right)^5 + \begin{pmatrix} 5 \\ 3 \end{pmatrix} \left(\frac{1}{2}\right)^5 = 0.5$. And probability of type II error is

$$1 - \pi_\beta(\theta|\theta > \frac{1}{2}) = 1 - \theta^5 - \begin{pmatrix} 5 \\ 4 \end{pmatrix} \theta^4(1-\theta) - \begin{pmatrix} 5 \\ 3 \end{pmatrix} \theta^3(1-\theta)^2,$$

which is smaller than probability of type II error of test alpha.

(c) Test alpha has a smaller Type I error, but test beta has a smaller type II error. In order to choose between the two test, a researcher must further specify the quantitative tradeoff between the two errors.

Q2

Note $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i = 10$, $s^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2 = \frac{48}{3} = 16$, $se(\bar{X}) = \frac{s}{\sqrt{n}} = 2$

(a) The test statistic, is given by

$$t = \frac{\bar{X} - 9}{se(\bar{X})} \sim t_{n-1} \text{ under } H_0$$

For the two-sided alternative we reject if $|t| > t_{3,(1-0.025)} = t_{3,0.975}$, where $t_{3,0.975}$ is the 0.975-th quantile of the t_3 distribution. From statistical table, we can find $t_{3,0.975} = 3.18$. Hence the critical value is 3.18 and we reject if $|t| > 3.18$ ($t > 3.18$ or $t < -3.18$). We can not reject the null since

$$t = \frac{10-9}{2} = \frac{1}{2}$$

and $|t| < 3.18$.

(b) The test statistic, is given by

$$t = \frac{\bar{X} - 7}{se(\bar{X})} \sim t_{n-1} \text{ under } H_0$$

For the one-sided alternative we reject if $t > t_{3,1-0.05} = t_{3,0.95}$. From statistical tables, we can find $t_{3,0.95} = 2.35$. Since $t = \frac{10-7}{2} = 1.5 < 2.35$, we can not reject H_0.

Q3

(a) The test statistic is

$$T = \frac{\bar{X} - 20}{\text{sd}(\bar{X})} \sim N(0,1) \text{ under } H_0$$
$$\text{with } \text{sd}(\bar{X}) = \frac{\sigma}{\sqrt{n}}$$

we reject H_0 if $T > z_{1-0.01}$, where $z_{1-0.01} = z_{0.99}$ is the 0.99-th quantile of standard normal distribution. Looking from statistical tables yields $z_{0.99} = 2.33$. In this case, $\text{sd}(\bar{X}) = \sqrt{\frac{4}{16}} = \frac{1}{2}$, i.e., $t = \frac{20.5-20}{\frac{1}{2}} = 1 < 2.33$. So we do not reject H_0 with significance level 1%.

(b) The p-value of the testis defined by

$$P(T > t|H_0) = 1 - \Phi(1) = 1 - 0.8413 = 0.1587$$

(c) The only thing that changes is the $\text{sd}(\bar{X}) = \sqrt{\frac{4}{25}} = \frac{2}{5}$, i.e., we reject H_0 if

$$T = \frac{\bar{X} - 20}{\frac{2}{5}} > 2.33.$$

In this case, $t = \frac{20.5-20}{\frac{2}{5}} = \frac{5}{4} < 2.33$. So we do not reject H_0 with significance level 1%.

(d) With $\mathbb{H}_0 : \mu = 20$ against $\mathbb{H}_1 : \mu > 20$, the rejection rule at the 1% level of significance is given by:

$$\text{reject } H_0 \text{ if } T = \frac{\bar{X} - 20}{\frac{\sigma}{\sqrt{n}}} > 2.33$$

The power at $\mu = 21$ is 90% if

$$P\{\text{reject } H_0|\mu = 21\} = 0.9$$
$$\Longleftrightarrow P\left\{\frac{\bar{X} - 20}{\frac{\sigma}{\sqrt{n}}} > 2.33|\mu = 21\right\} = 0.9$$

Note when $\mu = 21$, $\frac{\bar{X}-21}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$. Therefore

$$P\left\{\frac{\bar{X} - 20}{\frac{2}{\sqrt{n}}} > 2.33|\mu = 21\right\} = P\left\{\frac{\bar{X} - 21}{\frac{2}{\sqrt{n}}} > 2.33 - \frac{1}{\frac{2}{\sqrt{n}}}\right\}$$
$$= 1 - \Phi\left(2.33 - \frac{1}{\frac{2}{\sqrt{n}}}\right)$$

Therefore, n needs to satisfy

$$
\begin{aligned}
&1-\Phi\left(2.33-\frac{1}{\frac{2}{\sqrt{n}}}\right)=0.90\\
\Longleftrightarrow &\Phi\left(2.33-\frac{1}{\frac{2}{\sqrt{n}}}\right)=0.1
\end{aligned}
$$

Looking at the normal table, $\Phi(-1.28)=0.10$, (because $\Phi(1.28)=0.90$), so we require

$$
\begin{aligned}
&2.33-\frac{1}{\frac{2}{\sqrt{n}}}=-1.28\\
\Longleftrightarrow &\sqrt{n}=7.22\\
\Longleftrightarrow &n=(7.22)^2\approx 53
\end{aligned}
$$

The power of the test is increasing in n. As we increase n, the precision of our estimator is improving and we start rejecting the null earlier for the same level of significance. That also means that for the same level of significance we start rejecting the null earlier when the null in fact is false, i.e., larger power.

(e) A two sided test with $\mu=21$ as alternative is less powerful than one-sided test. To get the same power we therefore expect the sample size to be even bigger. Here is the formal reasoning. With $\mathbb{H}_0:\mu=20$ against $\mathbb{H}_1:\mu\neq 20$, the rejection rule at the 1% level of significance is given by:

$$
\text{reject } H_0 \text{ if } T=\left|\frac{\bar{X}-20}{\frac{\sigma}{\sqrt{n}}}\right|>z_{1-0.005}=2.57
$$

the power function at $\mu=21$ is

$$
\begin{aligned}
&P\left\{\left|\frac{\bar{X}-20}{\frac{\sigma}{\sqrt{n}}}\right|>2.57|\mu=21\right\}\\
=&P\left\{\frac{\bar{X}-20}{\frac{\sigma}{\sqrt{n}}}>2.57,\text{or }\frac{\bar{X}-20}{\frac{\sigma}{\sqrt{n}}}<-2.57|\mu=21\right\}\\
=&P\left\{\frac{\bar{X}-21}{\frac{\sigma}{\sqrt{n}}}>2.57-\frac{1}{\frac{\sigma}{\sqrt{n}}},\text{or }\frac{\bar{X}-21}{\frac{\sigma}{\sqrt{n}}}<-2.57-\frac{1}{\frac{\sigma}{\sqrt{n}}}|\mu=21\right\}\\
=&1-\Phi\left(2.57-\frac{1}{\frac{2}{\sqrt{n}}}\right)+\Phi\left(-2.57-\frac{1}{\frac{2}{\sqrt{n}}}\right)
\end{aligned}
$$

To achieve same 90% power, n needs to satisfy

$$
1-\Phi\left(2.57-\frac{1}{\frac{2}{\sqrt{n}}}\right)+\Phi\left(-2.57-\frac{1}{\frac{2}{\sqrt{n}}}\right)=0.90,
$$

and n is approximately 60.

On the other hand, you can draw a graph to illustrate that for the same n

$$
1-\Phi\left(2.57-\frac{1}{\frac{2}{\sqrt{n}}}\right)+\Phi\left(-2.57-\frac{1}{\frac{2}{\sqrt{n}}}\right)\text{(power of two sided test)}<1-\Phi\left(2.33-\frac{1}{\frac{2}{\sqrt{n}}}\right)\text{(power of one sided test)}
$$

Q4

Let $\{X_{ai}\}_{i=1}^n$ and $\{X_{bi}\}_{i=1}^n$ be iid sample for rents at Ann Arbor and Madison, respectively. Let $\mu_a = \mathbb{E}[X_a]$ and $\mu_b = \mathbb{E}[X_b]$. We want to test null hypothesis

$$\mathbb{H}_0 : \mu_a = \mu_b, \text{ vs } \mathbb{H}_1 : \mu_a \neq \mu_b$$

Suppose $\sigma_a^2 = \text{var}(X_a) < \infty$, $\sigma_b^2 = \text{var}(X_b) < \infty$. Also denote $\sigma_{ab} = \text{Cov}(X_a, X_b)$.

By central limit theorem,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n [(X_{ai} - X_{bi}) - (\mu_a - \mu_b)] \xrightarrow{d} \text{N}(0, \text{var}(X_a - X_b)),$$

where $\text{var}(X_a - X_b) = \text{var}(X_a) + \text{var}(X_b) - 2\text{Cov}(X_a, X_b)$. Note $\text{var}(X_a - X_b)$ can be estimated by

$$\hat{V} = \frac{1}{n} \sum_{i=1}^n \left((X_{ai} - X_{bi}) - \left(\frac{1}{n} \sum_{i=1}^n (X_{ai} - X_{bi}) \right) \right)^2$$

which can be shown to be a consistent estimator of $\text{var}(X_a - X_b)$ by imposing suitable law of large numbers.

Therefore, under $\mathbb{H}_0$ and invoking continuous mapping theorem,

$$T = \frac{\frac{1}{n} \sum_{i=1}^n (X_{ai} - X_{bi})}{\sqrt{\frac{\hat{V}}{n}}} \xrightarrow{d} \text{N}(0, 1)$$

Thus at size α, an asymptotic t-test is given by

$$\begin{aligned} \text{accept } \mathbb{H}_0 \text{ if } |T| &\leq Z_{1-\frac{\alpha}{2}} \\ \text{reject } \mathbb{H}_0 \text{ if } |T| &> Z_{1-\frac{\alpha}{2}} \end{aligned}$$

where $Z_{1-\frac{\alpha}{2}}$ is the $\left(1 - \frac{\alpha}{2}\right)$ $-$th quantile of $\text{N}(0, 1)$.

Q5.

(a) Probability of rejection p is estimated by

$$\hat{p} = \frac{\text{number of rejections}}{\text{number of trials}} = \frac{5}{50} = 0.1$$

(b) For each test $i = 1 \dots B$, let

$$X_i = \begin{cases} 1 & \text{if } \mathbb{H}_0 \text{ is rejected} \\ 0 & \text{if } \mathbb{H}_0 \text{ is accepted} \end{cases}.$$

Then $\hat{p} = \frac{1}{B} \sum_{i=1}^n X_i$, and $p = \mathbb{E}X_i$. Central limit theorem yields

$$\sqrt{B}(\hat{p} - p) \xrightarrow{d} \text{N}(0, \text{var}(X)),$$

where $\text{var}(X) = p(1 - p)$.

(c) Based on (b), asymptotic variance of $\hat{p}$ can be estimated by $\hat{p}(1 - \hat{p})$. Asymptotic t-statistic for testing $\mathbb{H}_0 : p = 0.05$ against $\mathbb{H}_1 : p \neq 0.05$ is given by

$$T = \sqrt{B}\frac{\hat{p} - 0.05}{\hat{p}(1-\hat{p})} \xrightarrow{d} \mathrm{N}(0,1)$$

under $\mathbb{H}_0$. Thus at size 5%, an asymptotic t-test is given by

$$\begin{aligned} \text{accept } \mathbb{H}_0 \text{ if } |T| \leq Z_{0.975} \\ \text{reject } \mathbb{H}_0 \text{ if } |T| > Z_{0.975} \end{aligned}$$

where $Z_{0.975} = 1.96$ is the $0.975-$th quantile of $\mathrm{N}(0,1)$. Since the realization of T is

$$\sqrt{B}\frac{\hat{p} - 0.05}{\hat{p}(1-\hat{p})} = \sqrt{50}\frac{0.1 - 0.05}{0.1(1-0.1)} = 1.18.$$

Thus we accept $\mathbb{H}_0$.

Q6.

(a) Since the test always reject $\mathbb{H}_0$, the size of the test (probability of type I error) is 1 and power is also 1.

(b) Since the test always accept $\mathbb{H}_0$, the size of the test (probability of type I error) is 0 but power is also 0.

Q7. Suppose now $X \sim \mathrm{N}(\mu, \sigma^2)$ where both μ and σ^2 are **unknown**. We hope to use a random sample $\{X_i, i = 1 \ldots n\}$ drawn from X to test hypothesis: $\mathbb{H}_0 : \mu = \mu_0$ for some $\mu_0 \in \mathbb{R}$ against $\mathbb{H}_1 : \mu \neq \mu_0$.

(a) Let $\beta = (\mu, \sigma^2)$. Write down the loglikelihood of β under $\mathbb{H}_0$.

Answer: The loglikelihood under $\mathbb{H}_0$ is

$$\sup_{\sigma^2>0} \ell_n(\mu_0, \sigma^2) = \ell_n(\mu_0, \tilde{\sigma}^2),$$

where $\tilde{\sigma}^2 = \arg\max_{\sigma^2>0} \ell_n(\sigma^2, \mu_0)$ is the MLE estimator in part (a). Thus the loglikelihood under $\mathbb{H}_0$ is

$$\begin{aligned}\ell_n(\mu_0, \tilde{\sigma}^2) &= -\sum_{i=1}^{n} \left[\frac{1}{2}\log(2\pi\tilde{\sigma}^2) + \frac{(X_i - \mu_0)^2}{2\tilde{\sigma}^2}\right] \\ &= -\frac{1}{2}n\left[\log(2\pi\tilde{\sigma}^2) + 1\right].\end{aligned}$$

(b) We know the unconstrained MLE of β is $\hat{\beta} = (\bar{X}_n, \hat{\sigma}^2)$, where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$. Based on this and your answer to (b)-(i), write down the likelihood ratio statistic LR_n for testing $\mathbb{H}_0 : \mu = \mu_0$ vs $\mathbb{H}_1 : \mu \neq \mu_0$.
Answer: The loglikelihood under of $\hat{\beta}$ is

$$\begin{aligned}\ell_n(\bar{X}_n, \hat{\sigma}^2) &= -\sum_{i=1}^{n} \left[\frac{1}{2}\log(2\pi\hat{\sigma}^2) + \frac{(X_i - \bar{X}_n)^2}{2\hat{\sigma}^2}\right] \\ &= -\frac{n}{2}\left[\log(2\pi\hat{\sigma}^2) + 1\right].\end{aligned}$$

Thus

$$LR_n = 2\left(\ell_n(\bar{X}_n, \hat{\sigma}^2) - \ell_n(\mu_0, \tilde{\sigma}^2)\right) = n\log(\frac{\tilde{\sigma}^2}{\hat{\sigma}^2}).$$

(c) Show the likelihood ratio test based on $LR_n > c$ for some c is equivalent to $|T| > b$ for some b, where $T = \frac{\bar{X}_n - \mu_0}{\sqrt{\frac{\hat{\sigma}^2}{n}}}$.
Answer: Note rejecting $LR_n > c$ for some c is equivalent to rejecting when

$$n\frac{\tilde{\sigma}^2 - \hat{\sigma}^2}{\hat{\sigma}^2} > b^2$$

for some $b > 0$. Further, $\tilde{\sigma}^2 - \hat{\sigma}^2 = (\bar{X}_n - \mu_0)^2$. Thus

$$n\frac{\tilde{\sigma}^2 - \hat{\sigma}^2}{\hat{\sigma}^2} = \frac{n(\bar{X}_n - \mu_0)^2}{\hat{\sigma}^2} = T^2.$$

Hence rejecting when

$$n\frac{\tilde{\sigma}^2 - \hat{\sigma}^2}{\hat{\sigma}^2} > b$$