Module 1 answer key

1. Why can we write "*the*" least upper bound? (Formally, prove that $\sup S$ is unique: if β and β' both satisfy the definition, then $\beta = \beta'$.)
   Solution: Suppose for the sake of contradiction and without loss of generality that $\beta < \beta'$, $\beta = \sup S, \beta' = \sup S$. Since $\beta < \beta' = \sup S$, there exists some $a \in S$ such that $a > \beta$. However, this means that β cannot be an upper bound.

2. Prove or disprove: If $\sup S$ exists, then $\sup S \in S$.
   Solution: False. Consider the set $\{x \in \mathbb{R} : x < 2\}$, which can also be written $(-\infty, 2)$.

3. Let $S \subset \mathbb{R}$ be nonempty and bounded. Prove that $\inf S \leq \sup S$. What can you say if $\inf S = \sup S$?
   Solution: Since $S \neq \emptyset$, there must exist some element in S. By definition of infinum,
   $$\forall s \in S,\ s \geq \inf S$$
   $$\forall s \in S,\ s \leq \sup S$$
   The two inequalities imply $\inf S \leq \sup S$. Note that if $\inf S = \sup S$ then $\inf S = s = \sup S$ and $S = \{s\}$.

4. Recall the formal definition of maximum and minimum of a set (don't look them up—model your definitions on those of supremum and infimum). Prove or disprove: Every set (in $\mathbb{R}$) has a maximum. Every *bounded* set has a maximum.
   Solution: False; consider the following bounded set: $S = \{x \in \mathbb{R} : 1 < x < 2\}$. Let's prove that this set does not have a maximum. Suppose $s \in S$ is the maximum of the set S. Then, $s < 2$. Take $\bar{s} = \frac{s+2}{2}$. Clearly, $\bar{s} < 2$ and, therefore, $\bar{s} \in S$; at the same time, $\bar{s} > s$, therefore, s can't be the maximum of S. Although S does not have the maximum, it still has the supremum. In particular, $\sup S = 2$.

5. Prove or disprove: If $S \subseteq \mathbb{R}$ has a maximum $\max S$, then $\max S = \sup S$.
   Solution: True. If $m = \max S$, then for all $a \in S$, $a \leq m$ by the definition of maximum, so m is an upper bound. There cannot exist another upper bound $u < m$ because $m \in S$, and, hence, at least one of the elements of S (this element is m) would be greater than u. Thus, m is the least upper bound. In fact, maximum is often defined so that $m = \max S$ if and only if $m = \sup S$ and $m \in S$.

6. Let S and T be nonempty and bounded subsets of $\mathbb{R}$. Prove or disprove: $\sup(S \cup T) = \max\{\sup S, \sup T\}$.
   Solution: True. Without loss of generality (WLOG), let $\sup S \geq \sup T$. Then for any $s \in S$, we have $\sup S \geq s$ by definition, and for any $t \in T$, we have $\sup S \geq \sup T \geq t$. Therefore, for any x in $S \cup T$, we have

$\sup S \geq x$. Thus, $\sup S$ is an upper bound and we need only show that $\sup S$ is minimal among upper bounds for $S \cup T$. Now, assume that there exists an upper bound $u < \sup S$ for $S \cup T$. Since u is an upper bound for $S \cup T$, it is also an upper bound for S. However, $u < \sup S$ contradicts the definition of $\sup S$.

7. Prove that Proposition 8 and Proposition 9 are equivalent: Proposition 8 follows from Proposition 9 and vice versa.
Solution: Suppose that $\mathbb{N}$ is bounded. Then, by completeness, it has the supremum. Denote $s = \sup \mathbb{N}$. Since, s is the supremum, $s - 1$ is not an upper bound. Hence, there exist $m \in \mathbb{N}$ such that $m > s - 1$, otherwise $s - 1$ would be an upper bound. Rearranging terms, we get that $m + 1 > s$. But because $m \in \mathbb{N}$, $m + 1 \in \mathbb{N}$. But then $m + 1 \leq s$ by the definition of supremum. Contradiction. $\mathbb{N}$ is unbounded in $\mathbb{R}$.

8. Prove or disprove: If $a > 0$, then there exists an $n \in N$ such that $\frac{1}{n} < a < n$.
Solution: True. Take $b = 1$ and apply Archimedean property: $\exists n \in \mathbb{N},\ na > 1 \Rightarrow a > \frac{1}{n}$. By previous exercise, $\mathbb{N}$ is unbounded, hence, $\exists m \in \mathbb{N},\ m > a$. Take $k = \max\{n, m\}$. Then, $\frac{1}{k} \leq \frac{1}{n} < a < m \leq k$.

9. Prove or disprove: If $a < b$, then there exist infinitely many rationals between a and b.
Solution: True. Suppose for the sake of contradiction, there exist only n rationals between a and b, ordered $a, r_1, r_2, ...r_n, b$ (we know that $n \geq 1$). Then by the density property there exists another rational number $a < r_0 < r_1$. However, then there would be $n + 1$ rationals between a and b.

10. According to a strict interpretation of the definition of supremum and infimum, what are $\sup \emptyset$ and $\inf \emptyset$ (where $\emptyset$ is the empty set)?
Solution: Since there are no elements in the empty set, the statement "for all $e \in \emptyset$, $e < (or >)\ r$" for any $r \in \mathbb{R}$ is vacuously true. Thus, every number in $\mathbb{R}$ is an upper (lower) bound. Thus, $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$.

11. Prove or disprove: If a sequence has a limit, then the limit is unique.[1]
Solution: True. Suppose for the sake of contradiction that $x_n \to L, L'$ where $L \neq L'$. WLOG, assume that $L \geq L'$. For any $\epsilon > 0$, for sufficiently large n, $|x_n - L| < \epsilon$ and $|x_n - L'| < \epsilon$. Thus, we have $|x_n - L| + |x_n - L'| < 2\epsilon$. By the triangle inequality, this implies $|L - L'| = L - L' < 2\epsilon$. However, note that this must hold for *any* $\epsilon > 0$. Take $\epsilon = \frac{L-L'}{2}$, then we get that $L - L' < L - L'$. Contradiction.

12. Find the limit (if they exist) of the following sequences, or show that they do not exist.

    (a) $(a_n)_n = \left(\frac{1}{n}\right)_n$

[1] *Hint:* recall the *triangle inequality*: $|a - b| \leq |a - c| + |c - b|$, for all $a, b, c \in \mathbb{R}$.

(b) $(b_n)_n = ((-1)^n)_n$

(c) $(c_n)_n = \left((-1)^{2n}\right)_n$

Solution:
(a) This converges to zero. For any $\epsilon > 0$, pick $N = \lceil \frac{1}{\epsilon} \rceil$. Then for any $n > N$, we have $n > N \geq \frac{1}{\epsilon} \Rightarrow \epsilon n > 1 \Rightarrow \epsilon > \frac{1}{n} > 0 > -\varepsilon$. Thus, $|\frac{1}{n} - 0| < \epsilon$.
(b) No; pick $\epsilon = 1$. Since we have $b_n = 1$ for even n, the limit must be in $(0, 1)$. But since we have $b_n = -1$ for odd n, the limit must be in $(-1, 0)$.
(c) Yes; this sequence is a constant sequence where every term equals 1. Its limit is 1. Notice, $|c_n - 1| = |1 - 1| = 0 < \epsilon$ for any $\epsilon > 0$ and any N you pick.

13. Prove or disprove: If $x_n \to x$ and $y_n \to y$, then $(x_n + y_n)_n$ converges to $x + y$.
Solution: True. For any ϵ, for sufficiently large n, we have $|x_n - x| < \frac{\epsilon}{2}$ and $|y_n - y| < \frac{\epsilon}{2}$. This gives us $|x_n - x| + |y_n - y| < \epsilon$. Using the triangle inequality, we have $|(x_n + y_n) - (x + y)| = |x_n - x + y_n - y| \leq |x_n - x| + |y_n - y| < \epsilon$

14. Prove or disprove: a sequence (x_n) converges to x if and only if there exists $\varepsilon > 0$ such that all terms x_i are contained in $(x - \varepsilon, x + \varepsilon)$.
Solution: (a) ($\Rightarrow$) True. Take $\bar{\varepsilon} > 0$. Since (x_n) converges, there exists N such that $\forall n > N$ it is true that $x_n \in (x - \bar{\varepsilon}, x + \bar{\varepsilon})$. At the same time there is only a finite number of sequence points that have an index smaller than N. Hence, $\bar{x} = \max\{x_1, x_2, \ldots, x_N\}$ and $\underline{x} = \min\{x_1, x_2, \ldots, x_N\}$ exist. Therefore, the entire sequence is bounded by $\max\{\bar{x}, x + \bar{\varepsilon}\}$ from above and by $\min\{\underline{x}, x - \bar{\varepsilon}\}$ from below. So, obviously we can find such ε such that all points of the sequnce will lie inside $(x - \varepsilon, x + \varepsilon)$.
(b) ($\Leftarrow$) False. Take $(x_n) = ((-1)^n)_n$. Take $x = 0$ and $\varepsilon = 2$, then all points of the sequence are contained in $(-2, 2)$, but this sequence does not converge to 0.

15. Prove or disprove: a sequence (x_n) converges to x if and only if for all $\varepsilon > 0$ all but finitely many terms x_i are contained in $(x - \varepsilon, x + \varepsilon)$.
Solution: True. (a) ($\Rightarrow$) True. If all but finely many terms are contained in $(x - \epsilon, x + \epsilon)$, then there must be a maximum term x_N such that for all $x_n \notin (x - \epsilon, x + \epsilon)$, $n \leq N$. Thus, for all $n > N$, $|x_n - x| < \epsilon$.
(b) ($\Leftarrow$) True. There exists some N such that for all $n > N$, $|x_n - x| < \epsilon$. Since there are an infinite number of $n > N$, there are an infinite number of terms such that $x_n \in (x - \epsilon, x + \epsilon)$.

16. Prove or disprove: a sequence (x_n) converges to x if and only if for all $\varepsilon > 0$ infinitely many terms are contained in $(x - \varepsilon, x + \varepsilon)$.
Solution: False. (a) ($\Leftarrow$) False. Consider $x_n = (-1)^n$ and take $x = 1$.
(b) ($\Rightarrow$) True. There exists some N such that for all $n > N$, $|x_n - x| < \epsilon$. Since there are an infinite number of $n > N$, there are an infinite number of terms such that $x_n \in (x - \epsilon, x + \epsilon)$.

17. Prove or disprove: a sequence (x_n) converges to x if and only if for all $\varepsilon > 0$ infinitely many terms are contained in $(x-\varepsilon, x+\varepsilon)$, and x is the only number with this property.
Solution: (a) ($\Rightarrow$) True. If (x_n) converges, then there exists some N such that for all $n > N$, $|x_n - x| < \epsilon$. Since there is an infinite number of $n > N$, there are an infinite number of terms such that $x_n \in (x-\epsilon, x+\epsilon)$. Suppose x is not the only point with such property and there exists x' such that for all $\varepsilon' > 0$ there infinitely many points contained in $(x'-\epsilon', x'+\epsilon')$. WLOG, assume that $x' > x$. Then, take $\varepsilon = \varepsilon' = \frac{x'-x}{2}$. In this case $B_\varepsilon(x) \cap B_{\varepsilon'}(x') = \emptyset$. We know that since (x_n) converges to x for all points but finitely many are contained in $(x-\epsilon, x+\epsilon)$. Hence, only a finite number of points could be in $B_{\varepsilon'}(x')$. Contradiction.
(b) ($\Leftarrow$) False. Take $x_n = \begin{cases} n, & \text{if } n \text{ is odd;} \\ 0, & \text{if } n \text{ is even.} \end{cases}$

18. Prove or disprove: If a series does not converge, then it diverges to either $+\infty$ or $-\infty$.
Solution: False. Consider $x_n = (-1)^n$.

19. Prove or disprove: Let (x_n) diverge to $+\infty$ and $y_n \to y > 0$ (y can be finite or $+\infty$). Then $\lim x_n y_n$ exists (and is ...?).
Solution: Since (x_n) diverges to $+\infty$, then for any $M > 0$, and some $\epsilon < y$, there exists N_1 such that for all $n > N_1$, $x_n > \frac{M}{y-\epsilon}$. Since (y_n) converges to $y > 0$, there exists N_2 such that for all $n > N_2$, we have $|y_n - y| < \epsilon \Leftrightarrow -\epsilon + y < y_n < \epsilon + y$,$y_n > -\epsilon + y$. Thus, for all $n > \max\{N_1, N_2\}$, we have $x_n y_n > \frac{M}{y-\epsilon} \cdot (y-\epsilon) = M$.

20. Prove or disprove: Let (x_n) diverge to $+\infty$ and $y_n \to 0$. Then $\lim x_n y_n$ exists (and is ...?).
Solution: In the case where $y = 0$, we cannot say anything for certain. For example, if $x_n = n$ and $y_n = \frac{1}{n}$, $x_n y_n \to 1$. However, if we set $x_n = n^2$ and keep $y_n = \frac{1}{n}$, this diverges. If we set $x_n = n$ and $y_n = 0$, then $x_n y_n \to 0$.

21. Prove or disprove: Every bounded sequence is convergent.
Solution: False. Consider $x_n = (-1)^n$.

22. Prove or disprove: Every convergent sequence (with a finite limit) is bounded.
Solution: True; this is the “only if” direction of exercise 14. Take $\bar{\varepsilon} > 0$. Since (x_n) converges, there exists N such that $\forall n > N$ it is true that $x_n \in (x-\bar{\varepsilon}, x+\bar{\varepsilon})$. At the same time there is only a finite number of sequence points that have an index smaller than N. Hence, $\bar{x} = \max\{x_1, x_2, \ldots, x_N\}$ and $\underline{x} = \min\{x_1, x_2, \ldots, x_N\}$ exist. Therefore, the entire sequence is bounded by $\max\{\bar{x}, x+\bar{\varepsilon}\}$ from above and by $\min\{\underline{x}, x-\bar{\varepsilon}\}$ from below. So, obviously we can find such ε such that all points of the sequnce will lie inside $(x-\varepsilon, x+\varepsilon)$.

23. Complete the following: A sequence is both nondecreasing and nonincreasing if and only if it is $\ldots$.
Solution: ...it is constant. If for all n, $x_n \leq x_{n+1}$ and $x_n \geq x_{n+1}$, then $x_n = x_{n+1}$.

24. Prove or disprove: If a sequence converges, then every subsequence converges (to the same limit).
Solution: True. If x_n converges to x, then for any ϵ, there exists N such that for every $n > N$, $|x_n - x| < \epsilon$. Since we got a subsequence s_m by deleting some terms of x_n, then this must also hold for every $m > N$.

25. Prove or disprove: if a sequence is bounded, then every subsequence is bounded.
Solution: True. If a sequence is bounded, then its set of values $X = \{x_n : n \in \mathbb{N}\}$ is bounded and its supremum $\sup X$ and infinum $\inf X$ exist. If every element of a sequence is less than $\sup X$, then every element of a subsequence must be as well, because they are constructed from original elements of the sequence. And vice versa, if every element of the sequence is greater or equal to $\inf X$, then every element of the subsequence is greater or equal to $\inf X$ as well.

26. Prove or disprove: if a sequence is unbounded, then every subsequence is unbounded.
Solution: False. Take $x_n = \begin{cases} n, & \text{if } n \text{ is odd;} \\ 0, & \text{if } n \text{ is even.} \end{cases}$. $(x_{n_k})_k = 0$ is an obvious subsequence.

27. Prove or disprove: if a sequence is unbounded, then it has a subsequence which is bounded.
Solution: False. Consider the sequence $x_n = n$. Since this diverges to infinity, any subsequence must as well.

28. Prove or disprove: Referring to the previous proof, if $\max S_0$ does not exist then neither do $\max S_n$, for all $n \geq 1$.
Solution: True. Suppose the opposite is true: $\max S_0$ does not exist, but there is some S_n such that it has $\bar{x} = \max S_n$. Denote $S = S_0 \backslash S_n = \{x_0, x_1, ..., x_{n-1}\}$. Since S_0 does not have a maximum, there exists $\bar{\bar{x}} \in S$ such that $\bar{\bar{x}} > \bar{x}$ (otherwise $\bar{x} = \max S_0$). Since S is finite, $x = \max S$ exists. Then, $x \geq \bar{\bar{x}} \geq \bar{x}$, but that means that x is a maximum of S_0, since it is the biggest element in $S = S_0/S_n$ and is bigger than the biggest element in S_n.

29. In the second part of the proof, can you replace $\min\{m \in \mathbb{N} : x_m = \max S_{(n_k)+1}\}$ with $\max\{m \in \mathbb{N} : x_m = \max S_{(n_k)+1}\}$?
Solution: No. It might not exist. Consider $(x_n) = 1$. Then maximum exists for all S_n and is equal to 1. However, $\max\{m \in \mathbb{N} : x_m = 1\} = \infty$.

30. Prove or disprove: If (x_n) is a sequence, there exists an $M \in \mathbb{N}$ such that $\limsup x_n = \sup\{x_n : n \geq M\}$.
Solution: False. Take any strictly decreasing sequence x_n that converges to some point x. $\limsup x_n = x$ as well, by prop 30. But for any N, $\sup\{x_n | n \geq N\}$ is x_N and $x_N > x$ by assumption that x_n is strictly decreasing.

31. Replace $\star$ with an appropriate symbol, then prove: For any sequences (x_n), (y_n),
$$\limsup(a_n + b_n) \ \star \ \limsup a_n + \limsup b_n$$
provided the right hand side is not of the form $\infty + (-\infty)$ (which is undefined).
Solution: Replace $\star$ with "less than or equal to". For an example where the inequality is strict, let $(a_n) = (1, -1, 1, -1, ...)$ and let $(b_n) = (-1, 1, -1, 1, ...)$. Then $\limsup(a_n + b_n) = 0$. $\limsup a_n + \limsup b_n = 2$.
To prove the inequality in general: note that it suffices to prove that $\sup\{a_n + b_n | n \geq M\} \leq \sup\{b_n | n \geq M\} + \sup\{b_n | n \geq M\} \ \forall M$.[2]
To prove this, note that $\{a_n + b_n | n \geq M\}$ is a subset of $\{a_n + b_m | n, m \geq M\}$, so $\sup\{a_n + b_n | n \geq M\} \leq \sup\{a_n + b_m | n, m \geq M\}$.
Now if we show that $\sup\{a_n + b_m | n, m \geq M\} = \sup\{a_n | n \geq M\} + \sup\{b_n | n \geq M\}$, we are done.
Let $A = \{a_n | n \geq M\}$ and $B = \{b_n | n \geq M\}$. Then $A + B = \{a + b | a \in A$ and $b \in B\} = \{a_n + b_m | n, m \geq M\}$. So we have to prove $\sup(A + B) = \sup(A) + \sup(B)$, which has been proved in HW1, additional problem 1a).

32. Consider the following non-theorem: *Let $x_n \to x \geq 0$ and (y_n) be any sequence. Then* $\limsup x_n y_n = x \limsup y_n$. Disprove this, then identify a tiny change to the assumptions that makes it true (but don't prove it).
Solution: A counter example would be $x_n = 1/n$ and $y_n = n$. Either the assumption that $x > 0$ or the assumption that y_n is a bounded sequence would make the statement true.

[2]The fact that showing this suffices follows from two even more basic facts: if (x_k) and (y_k) converge, then (1) if $x_k \leq y_k$ for all n, then $\lim x_k \leq \lim y_k$, (2) $\lim(x_k + y_k) = \lim x_k + \lim y_k$. We apply the first fact to the sequences $x_k = \sup\{a_n + b_n | n \geq k\}$ and $y_k = \sup\{a_n | n \geq k\} + \sup\{b_n | n \geq k\}$, for all k. And we apply the second fact to the sequence $x_k = \sup\{a_n | n \geq k\}$ and $y_k = \sup\{a_n | n \geq k\} + \sup\{b_n | n \geq k\}$.