Econ 6190 Problem Set 7

Fall 2024

1. Let $\{X_1 \dots X_n\}$ be a sequence of i.i.d random variables with mean μ and variance σ^2. Let $\hat{\mu} = \frac{1}{n}\sum_{i=1}^n X_i$, and $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \hat{\mu})^2$.

   (a) Suppose $\mathbb{E}X_i^2 < \infty, i = 1, \dots n$. Show $\hat{\sigma}^2 \xrightarrow{p} \sigma^2$ as $n \to \infty$.

   (b) Imposing additional assumptions if necessary, find the asymptotic distribution of

   $$\sqrt{n}(\hat{\sigma}^2 - \sigma^2)$$

   by using delta method. Carefully state your results.

2. [Hansen 8.10] Let $X \sim U[0, b]$ and $M_n = \max_{i \le n} X_i$, where $\{X_i, i = 1 \dots n\}$ is a random sample from X. Derive the asymptotic distribution using the following the steps.

   (a) Calculate the distribution $F(x)$ of $U[0, b]$.

   (b) Show

   $$Z_n = n(M_n - b) = n\left(\max_{1 \le i \le n} X_i - b\right) = \max_{1 \le i \le n} n\left(X_i - b\right).$$

   (c) Show that the cdf of Z_n is

   $$G_n(x) = P\{Z_n \le x\} = (F(b + \frac{x}{n}))^n.$$

   (d) Derive the limit of $G_n(x)$ as $n \to \infty$ for $x < 0$. [Hint: Use $\lim_{n\to\infty}(1 + \frac{x}{n})^n = e^x$]

   (e) Derive the limit of $G_n(x)$ as $n \to \infty$ for $x \ge 0$.

   (f) Find the asymptotic distribution of Z_n as $n \to \infty$.

1. (a) Note: $\hat{\mu} \xrightarrow{p} \mu = \mathbb{E}X_i$ directly by Chebyshev's Law of Large numbers since we assumed $\mathbb{E}X_i^2 < \infty, i = 1, \dots n$. Moreover,

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - (\hat{\mu})^2$$

Since $\mathbb{E}X_i^2 < \infty$, it follows by Khinchin's Law of Large numbers $\frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{p} \mathbb{E}X_i^2$. We've shown $\hat{\mu} \xrightarrow{p} \mu$. It follows by continuous mapping theorem that

$$\hat{\sigma}^2 \xrightarrow{p} \mathbb{E}X_i^2 - [\mathbb{E}X_i]^2 = \text{var}(X) = \sigma^2$$

(b) Note we can write $\sigma^2 = \mathbb{E}X^2 - (\mathbb{E}X)^2 = h(\mathbb{E}X^2, \mathbb{E}X)$, where $h(a,b) = a - b^2$ is a smooth function of both a and b. Similarly, write $\hat{\sigma}^2 = h(\hat{\mu}_2, \hat{\mu}_1)$, where $\hat{\mu}_2 = \frac{1}{n}\sum\limits_{i=1}^{n} X_i^2$, $\hat{\mu}_1 = \frac{1}{n}\sum\limits_{i=1}^{n} X_i$. Thus by Taylor expansion:

$$\sqrt{n}(\hat{\sigma}^2 - \sigma^2) = \begin{pmatrix} \frac{\partial}{\partial a}h(a,b) \mid \begin{pmatrix} a = \tilde{\mu}_2 \\ b = \tilde{\mu}_1 \end{pmatrix} \\ \frac{\partial}{\partial b}h(a,b) \mid \begin{pmatrix} a = \tilde{\mu}_2 \\ b = \tilde{\mu}_1 \end{pmatrix} \end{pmatrix} \sqrt{n} \begin{pmatrix} \frac{1}{n}\sum\limits_{i=1}^{n}(X_i^2 - \mathbb{E}X^2) \\ \frac{1}{n}\sum\limits_{i=1}^{n}(X_i - \mathbb{E}X) \end{pmatrix},$$

where $\begin{pmatrix} \tilde{\mu}_2 \\ \tilde{\mu}_1 \end{pmatrix}$ lie on the line between $\begin{pmatrix} \hat{\mu}_2 \\ \hat{\mu}_1 \end{pmatrix}$ and $\begin{pmatrix} \mathbb{E}X^2 \\ \mathbb{E}X \end{pmatrix}$.

Now assuming $\mathbb{E}\left\|\begin{pmatrix} X^2 \\ X \end{pmatrix}\right\|^2 < \infty$, which requires $\mathbb{E}X^4 < \infty$, it follows by multivariate CLT that

$$\sqrt{n} \begin{pmatrix} \frac{1}{n}\sum\limits_{i=1}^{n}(X_i^2 - \mathbb{E}X^2) \\ \frac{1}{n}\sum\limits_{i=1}^{n}(X_i - \mathbb{E}X) \end{pmatrix} \xrightarrow{d} \text{N}(0, \text{var}\left(\begin{pmatrix} X^2 \\ X \end{pmatrix}\right)).$$

Also note $\begin{pmatrix} \hat{\mu}_2 \\ \hat{\mu}_1 \end{pmatrix} \xrightarrow{p} \begin{pmatrix} \mathbb{E}X^2 \\ \mathbb{E}X \end{pmatrix}$ under assumption $\mathbb{E}X^4 < \infty$, so

$$= \begin{pmatrix} \frac{\partial}{\partial a}h(a,b) \mid \begin{pmatrix} a = \tilde{\mu}_2 \\ b = \tilde{\mu}_1 \end{pmatrix} \\ \frac{\partial}{\partial b}h(a,b) \mid \begin{pmatrix} a = \tilde{\mu}_2 \\ b = \tilde{\mu}_1 \end{pmatrix} \end{pmatrix} \xrightarrow{p} \begin{pmatrix} \frac{\partial}{\partial a}h(a,b) \mid \begin{pmatrix} a = \mathbb{E}X^2 \\ b = \mathbb{E}X \end{pmatrix} \\ \frac{\partial}{\partial b}h(a,b) \mid \begin{pmatrix} a = \mathbb{E}X^2 \\ b = \mathbb{E}X \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 \\ -2\mathbb{E}X \end{pmatrix}.$$

It follows by continuous mapping theorem that

$$\sqrt{n}(\hat{\sigma}^2 - \sigma^2) \xrightarrow{d} \text{N}\left(0, \begin{pmatrix} 1 \\ -2\mathbb{E}X \end{pmatrix}' \text{var}\left(\begin{pmatrix} X^2 \\ X \end{pmatrix}\right)\begin{pmatrix} 1 \\ -2\mathbb{E}X \end{pmatrix}\right),$$

where

$$\text{var}\left(\begin{pmatrix} X^2 \\ X \end{pmatrix}\right) = \begin{pmatrix} \text{var}(X^2) & \text{cov}(X^2, X) \\ \text{cov}(X, X^2) & \text{var}(X) \end{pmatrix}.$$

It can be further verified that

$$\begin{pmatrix} 1 \\ -2\mathbb{E}X \end{pmatrix}' \text{var}\left(\begin{pmatrix} X^2 \\ X \end{pmatrix}\right)\begin{pmatrix} 1 \\ -2\mathbb{E}X \end{pmatrix} = \text{var}(X^2) - 4\text{cov}(X, X^2)\mathbb{E}X + 4\text{var}(X)\left(\mathbb{E}X\right)^2$$

which simplifies to

$$V_{\sigma^2} = \text{var}\left[(X - \mathbb{E}X)^2\right].$$

Such verification is algebraically tedious and not required.

2. (a) This is a uniform distribution, so

$$F(x) = P[X \leq x] = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{b} & \text{if } 0 \leq x \leq b \\ 1 & \text{if } x > b. \end{cases}$$

(b) Since both n and b are constants,

$$\max_{1 \leq i \leq n} n\,(X_i - b) = n \max_{1 \leq i \leq n} (X_i - b) = n \max_{1 \leq i \leq n} (X_i) - b = Z_n$$

as required

(c)

$$\begin{aligned} G_n(x) =& P\{Z_n \leq x\} \\ =& P\left\{n\left(\max_{1 \leq i \leq n} X_i - b\right) \leq x\right\} \\ =& P\left\{\max_{1 \leq i \leq n} X_i \leq \frac{x}{n} + b\right\} \\ =& P\left\{X_1 \leq \frac{x}{n} + b, \ldots X_n \leq \frac{x}{n} + b\right\} \\ =& \left[P\left\{X_i \leq \frac{x}{n} + b\right\}\right]^n \text{ (by i.i.d assumption)} \\ =& \left[F(\frac{x}{n} + b)\right]^n \end{aligned}$$

as required

(d) Note $X_i \sim U[0, b]$. If $x < 0$, $\frac{x}{n} + b < b$, hence

$$G_n(x) = \left[\frac{\frac{x}{n} + b}{b}\right]^n = \left[1 + \frac{x}{nb}\right]^n \to e^{\frac{x}{b}}$$

as $n \to \infty$.

(e) If $x \geq 0$, $\frac{x}{n} + b \geq b$, hence

$$G_n(x) = [1]^n = 1.$$

(f) Based on results from (d) and (e), we can conclude

$$G_n(x) \to \begin{cases} e^{\frac{x}{b}} & x < 0 \\ 1 & x \geq 0 \end{cases}$$

as $n \to \infty$.