ECON 6130: Problem set 1

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Please upload your problem set on Canvas, in the Assignement section. You may work in groups, but you must turn in your own answers. Actively working on the assignments is *absolutely essential* for your understanding of the course material.

Problem 1. *Bob arrives on a deserted island with some amount of coconuts $e_0 > 0$. There are no other coconuts on the island but Bob can plant, in each period, some coconut seeds. If he plants b_{t-1} coconut seeds in period $t-1$, he receives $r_t b_{t-1}$* new *coconuts in period t (he can still use the original ones). Initially, there are no coconuts in the ground ($b_{-1} = 0$). Bob values consumption streams according to the utility function:*

$$U(c) = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

with $\beta \in (0,1)$, $u' > 0$ and $u'' < 0$.

1. Write Bob's maximization problem. Carefully define all the objects.
2. Write Bob's first order conditions.
3. Write conditions such that the agent's consumption is constant every period. Suppose that these conditions hold from now on.
4. Solve for the optimal consumption as a function of e_0 and r. How many coconuts does Bob have at the beginning of any period?
5. From your expression for consumption, compute $\lim_{r \to \infty} c$. Is the limit finite? Why? Which assumption should we relax to have a more realistic result?

Problem 2. *Consider a consumer who lives forever and has preferences over consumption streams representable by*

$$u(c) = \sum_{t=0}^{\infty} \beta^t U(c_t) \tag{1}$$

where U is the Constant Relative Risk Aversion (CRRA) utility function

$$U(c_t) = \frac{c_t^{1-\sigma} - 1}{1 - \sigma}$$

where $\sigma \geq 0$ is a parameter

1. Show that

$$\lim_{\sigma \to 1} U(c_t) = \ln(c_t)$$

2. The Arrow-Pratt measure of relative risk aversion is

$$\frac{-cU''(c)}{U'(c)}.$$

Show that the CRRA utility function has a constant Arrow-Pratt coefficient of relative risk aversion equal to σ.

3. The intertemporal elasticity of substitution is defined as

$$-\frac{\partial \ln(c_{t+1}/c_t)}{\partial \ln(U'(c_{t+1})/U'(c_t))}.$$

Show that the CRRA utility function has a constant intertemporal elasticity of substitution equal to $1/\sigma$.

4. Show that the CRRA utility function is strictly increasing, strictly concave and satisfies the Inada condition.

5. Define the marginal rate of substitution between consumption at any two dates t and $t+s$ as

$$MRS(c_{t+s}, c_t) = \frac{\frac{\partial u(c)}{\partial c_{t+s}}}{\frac{\partial u(c)}{\partial c_t}}.$$

The function u is said to be homothetic if $MRS(c_{t+s}, c_t) = MRS(\lambda c_{t+s}, \lambda c_t)$ for all $\lambda > 0$ and c. Show that if U is of CRRA form, then u is homothetic.

6. Suppose the consumer faces the following Arrow-Debreu budget constraint

$$\sum_{t=0}^{\infty} p_t c_t \leq y \tag{2}$$

where y is the income the consumer receives from selling her endowment. Let $\{\hat{c}_t\}_{t=0}^{\infty}$ denote the solution to the consumer problem of maximizing 1 with respect to 2. Show that an identical consumer with income $\tilde{y} = \lambda y$ with $\lambda > 0$, facing the same prices would choose a consumption sequence $\{\tilde{c}_t\}_{t=0}^{\infty}$ satisfying

$$\tilde{c}_t = \lambda \hat{c}_t$$

7. Suppose the consumer maximizes 1 subject to a sequence of budget constraints of the form

$$c_t + \frac{a_{t+1}}{1 + r_{t+1}} = e_t + a_t.$$

Derive the consumption Euler equation relating the MRS between consumption at time t and $t+1$ to the interest rate r_{t+1}.

8. Suppose that consumption is growing at a constant rate of growth rate g, that is, $c_t = (1+g)^t c_0$. We will later call such a situation a balanced growth path. Use the consumption Euler equation to derive a relationship between the growth rate g and the (constant) real interest rate r along the balanced growth path.

9. Suppose that growth g increases. What happens to the interest rate r? How does your answer, quantitatively, depend on the size of the intertemporal elasticity of substitution $1/\sigma$. Give some intuition for your answer.

10. More generally, use the Euler equation to show that if u is homothetic, then constant consumption growth is consistent with a constant interest rate (that is, if u is homothetic, then a balanced growth path with constant interest rate exist.) Note that the 'only if' statement is also true, but I am not asking you to prove it.

Another period utility function that is sometimes used in applications (because in particular circumstances closed-form solutions are obtainable) is the Constant Absolute Risk Aversion (CARA) utility function

$$U(c_t) = -e^{-\gamma c_t}$$

with $\gamma > 0$.

1. Define the Arrow-Pratt coefficient of absolute risk aversion as

$$\gamma(c) = -\frac{U''(c)}{U'(c)}$$

Show that the CARA utility function has a constant absolute risk aversion coefficient, but increasing relative risk aversion coefficient.

2. If U is of CARA form, is u homothetic? If yes, prove it, if no, show that it is not.

3. Let U be of CARA form and let $\{\hat{c}_t\}_{t=0}^{\infty}$ denote the solution to the consumer problem of maximizing 1 with respect to 2. Consider again an identical consumer with income $\tilde{y} = \lambda y$ with $\lambda > 1$ and let $\{\tilde{c}_t\}_{t=0}^{\infty}$ denote this consumer's optimal consumption choice. Find $\tilde{c}_t$ in terms of $\hat{c}_t$, λ, y and $\{p_t\}_{t=0}^{\infty}$. You may ignore the nonnegativity constraints on consumption for this question. The answer to this question shows that with CARA utility the units in which we measure consumption matter for the resulting consumption allocations.