ECON 6170 Section 1

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  Monday 4.15-5.15pm
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Introduction

Sections will primarily focus on reviewing difficult questions from the most recent problem set, and difficult topics from the most recent lectures. I will also include original "Section Exercises" that are similar in style to problem-set and exam questions. Where possible, these will include actual past exam questions.

Exercise 7. Let S and T be nonempty and bounded subsets of $\mathbb{R}$. TFU:

$$\sup(S \cup T) = \max\{\sup S, \sup T\}$$

Solution. True. WLOG, let $\sup S \geq \sup T$. Then for any $s \in S$, we have $\sup S \geq s$. Moreover, for any $t \in T$, we have $\sup S \geq \sup T \geq t$. Therefore, for any x in $S \cup T$, we have $\sup S \geq x$. Thus, $\sup S$ is an upper bound for $S \cup T$ and we need only show that it is the least such upper bound. Let u be an arbitrary upper bound for $S \cup T$. Then u must also be an upper bound for S. But then, by definition, $\sup S \leq u$. Since u is an arbitrary upper bound of $S \cup T$, this means that $\sup S$ is the least upper bound of $S \cup T$.

Section Exercise 1. Use proof by induction and Exercise 7 to show that if $\{S_i \mid i = 1, 2, \ldots, N\}$, $N \in \mathbb{N}$ is a collection of nonempty and bounded subsets of $\mathbb{R}$, then

$$\sup\left(\bigcup_{i=1}^{N} S_i\right) = \max\{\sup S_i \mid i = 1, \ldots, N\}$$

[1] Changed to avoid clash with Tak's OH.

Solution.
This is trivially true for $N = 1$, and Exercise 7 proves it for $N = 2$. Suppose that it's also true for some natural number $N - 1 \geq 2$. That is,

$$\sup\left(\bigcup_{i=1}^{N-1} S_i\right) = \max\{\sup S_i \mid i = 1, \ldots, N-1\} \tag{1}$$

We want to show that it must then be true for N. Write

$$\begin{aligned}
\sup\left(\bigcup_{i=1}^{N} S_i\right) &= \sup\left(\bigcup_{i=1}^{N-1} S_i \cup S_N\right) \\
&= \max\left\{\sup\left(\bigcup_{i=1}^{N-1} S_i\right), \sup S_N\right\} \\
&= \max\left\{\max\{\sup S_i \mid i = 1, \ldots, N-1\}, \sup S_N\right\} \\
&= \max\{\sup S_i \mid i = 1, \ldots, N\}
\end{aligned}$$

where the second equality uses Exercise 7, and the third equality uses our induction hypothesis (1).[2]

Problem 1. Let A and B be nonempty subsets of $\mathbb{R}$. Define $A + B := \{a + b \mid a \in A \text{ and } b \in B\}$, and define $A - B$ similarly. Show the following:

1. $\sup(A + B) = \sup(A) + \sup(B)$
2. $\sup(A - B) = \sup(A) - \inf(B)$

Solution.
1. Suppose $x \in A + B$. Then $x = a + b$ for some $a \in A, b \in B$, implying $a \leq \sup A$ and $b \leq \sup B$. Thus $x \leq \sup A + \sup B$. This implies $\sup A + \sup B$ is an upper bound of $A + B$, and so $\sup(A + B) \leq \sup A + \sup B$.

Conversely, say $\sup A + \sup B > \sup(A + B)$.

First, assume that both A and B are bounded above. Then $\sup A > \sup(A + B) - \sup B$ implying that there exists an $a \in A$ such that $a > \sup(A + B) - \sup B$. Therefore, $\sup B > \sup(A + B) - a$. It follows that there must exist some $b \in B$ such that $b > \sup(A + B) - a$. This implies that there exists $a \in A$ and $b \in B$ such that $a + b > \sup(A + B)$, contradicting the definition of $\sup(A + B)$.

Now, suppose that one of A or B has no upper bound. WLOG, say $\sup A = \infty$. Then, because $B \neq \emptyset$, we have $\sup B > -\infty$, and so $\sup A + \sup B = \infty$. Furthermore, if for all $M \in \mathbb{R}$, we can find an $a \in A$ such that $a > M$, then, fixing some $b \in B$, we can find some $a + b \in A + B$ such that $a + b > M + b$. Because M is arbitrary, this proves unboundedness of $A + B$ above, so $\sup A + B = \infty$.

(Skip this in section.) 2. Define $-B := \{-x \in \mathbb{R} \mid x \in B\}$.

[2] The fourth equality uses $\max\{x_1, \ldots, x_K\} = \max\{\max\{x_1, \ldots, x_{K-1}\}, x_K\}$. I consider this obvious enough not to warrant proof, but it can be proven by an induction argument using the definition of a maximum.

First, suppose that B is bounded below, or equivalently, $-B$ is bounded above. This implies that $A - B = A + (-B)$ and $\sup(A-B) = \sup A + \sup(-B)$ by part 1. The supremum of $-B$ is defined by $\sup(-B) \geq -x$ for all $x \in B$ and $\sup(-B) \leq m$, for all upper bounds, m, of $-B$. Equivalently, $-\sup(-B) \leq x$ for all $x \in B$ and $-\sup(-B) \geq -m$, for all lower bounds, $-m$, of B. But this is just the definition of the infimum of B, so $\inf B = -\sup(-B)$. Thus, $\sup(A - B) = \sup A - \inf B$.

Now, suppose B is unbounded below. Then $\inf B = -\infty$, so $-\inf B = -(-\infty) = \infty$. Therefore $\sup A - \inf B = \infty$. Because B is unbounded below, for any $M \in \mathbb{R}$, we can find some $b \in B$ such that $b < -M$. Equivalently, $-b > M$. Then, fixing some $a \in A$, we can find an $a - b \in A - B$ such that $a - b > a + M$. This proves unboundedness of $A - B$ above, so $\sup(A - B) = \infty$.

Remark 1. Note that we can only say there exists $a \in A$ satisfying $\sup A - \epsilon < a < \sup A$ if $\sup A$ is finite. If $\sup A$ were infinite, we would be saying $\infty < a < \infty$, which doesn't make sense.

Remark 2. If A was empty and B unbounded above, we would have $\sup A + \sup B = -\infty + \infty$, which is undefined.[3] Hence, the nonemptiness restriction in the question.

Section Exercise 2. Let (a_n) and (b_n) be two sequences and define $\sup x_n := \sup\{x_n \mid n \in \mathbb{N}\}$. Prove that $\sup(a_n + b_n) \leq \sup a_n + \sup b_n$, and give an example to show that the inequality may hold strictly. Compare with the previous problem.

Let $x_k = a_k + b_k$ for some k. Then $x_k \leq \sup a_n + \sup b_n$. This implies $\sup a_n + \sup b_n$ is an upper bound of $(x_n) = (a_n + b_n)$, so $\sup(a_n + b_n) \leq \sup a_n + \sup b_n$. The reverse inequality does not hold. Consider the sequences $(a_n) = (-1, 1, -1, 1, -1, \dots)$ and $(b_n) = (1, -1, 1, -1, 1, \dots)$, which have

$$\sup(a_n + b_n) = \sup 0 = 0 < 2 = \sup a_n + \sup b_n$$

The key difference from the previous problem is that sequence addition is defined for corresponding entries, giving $\sup(a_n + b_n) = \sup\{a_n + b_n \mid n \in \mathbb{N}\}$. Whereas set addition[4] entails addition of each element of one set with every element of the other. If we added the sequences as sets of values, we would get $\sup\{a_n + b_m \mid n, m \in \mathbb{N}\}$, which is potentially larger than the previous expression.

Problem 2. Let A and B be nonempty sets, and let $f : A \times B \to \mathbb{R}$ be some real valued function.

1. Show that
$$\sup_{a \in A} \inf_{b \in B} f(a, b) \leq \inf_{b \in B} \sup_{a \in A} f(a, b).$$

2. Give an $f : [0, 1]^2 \to \mathbb{R}$ for which the above inequality is strict.

Note: For a real valued function, f, on a nonempty set, S, $\sup_{x \in S} f(x) \equiv \sup\{f(x) | x \in S\}$.

Remark 3. Note that $\sup_{a \in A} f(a, b) = \sup\{f(a, b) \mid a \in A\}$ depends on b. However, if we plug in a specific b, it is unique. Therefore, we can think of $\sup_{a \in A} f(a, b)$ as a function of b, call it $g : B \to \mathbb{R}$. Then $\inf_b \sup_a f(a, b) = \inf_b g(b) = \inf\{g(b) \mid b \in B\}$.

[3] Intuitively, this means that the limit of the sum of two sequences, one diverging to ∞ and the other diverging to $-\infty$ could equal any number in $\mathbb{R} \cup \{\infty, -\infty\}$ or may not even exist, depending on the particular sequences being added.

[4] Formally, Minkowski addition.

Solution to problem:

1. Suppose

$$\sup_{a\in A}\inf_{b\in B} f(a,b) > \inf_{b\in B}\sup_{a\in A} f(a,b)$$

Then there must exist $\bar{a}\in A$ such that

$$\inf_{b\in B} f(\bar{a},b) > \inf_{b\in B}\sup_{a\in A} f(a,b) \tag{2}$$

(For otherwise $\sup_{a\in A}\inf_{b\in B} f(a,b)$ would not be the supremum of $\{\inf_{b\in B} f(a,b) \mid a\in A\}$). But (2) is false as

$$f(\bar{a},b)\in\{f(a,b)\mid a\in A\}$$

so

$$\sup_{a\in A} f(a,b)\geq f(\bar{a},b) \text{ for all } b\in B$$

hence[5]

$$\inf_{b\in B}\sup_{a\in A} f(a,b)\geq \inf_{b\in B} f(\bar{a},b)$$

This proves

$$\sup_{a\in A}\inf_{b\in B} f(a,b)\leq \inf_{b\in B}\sup_{a\in A} f(a,b)$$

by contradiction.

2. There are many possible counterexamples.[6] For example, if $f\colon [0,1]^2\to\mathbb{R}$ is given by $f(a,b)=(a-b)^2$ then

$$\sup_{a\in A}(a-b)^2=\begin{cases}(1-b)^2 & \text{if } b<\frac{1}{2} \quad (\text{set } a=1)\\ b^2 & \text{if } b\geq\frac{1}{2} \quad (\text{set } a=0)\end{cases}$$

This implies $\inf_{b\in B}\sup_{a\in A}(a-b)^2=(1/2)^2=1/4$. On the other hand,

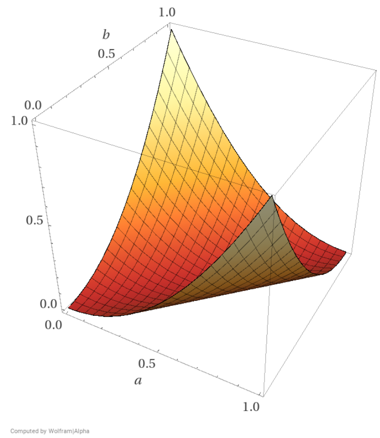
$$\inf_{b\in B}(a-b)^2=0 \text{ for all } a\in A \qquad (\text{set } b=a)$$

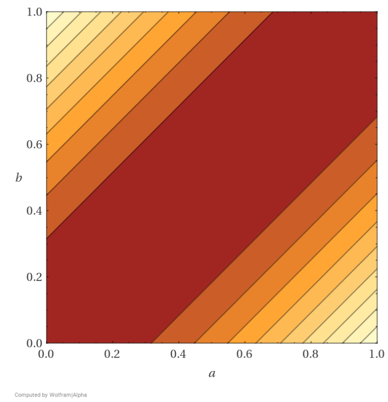
Therefore,

$$\sup_{a\in A}\inf_{b\in B}(a-b)^2=0<1/4=\inf_{b\in B}\sup_{a\in A}(a-b)^2$$

[5]Lemma: $g(x)\leq h(x)$ for all $x\in X$ implies $\inf_{x\in X} g(x)\leq \inf_{x\in X} h(x)$. Proof: Suppose not. Then $\inf g(x) > \inf h(x)$, so there exists $\overline{x}$ such that $\inf g(x)>h(\overline{x})$. But then $g(\overline{x})>h(\overline{x})$, a contradiction.

[6]$f(x,y)=\mathbf{1}\{x=y\}$ is a simple one.





From the remark, $\sup_{a \in A} f(a,b)$ is a function of b, and $\inf_{b \in B} f(a,b)$ is a function of a. We know that $f(a,b) = (a-b)^2$ is minimised with respect to b by choosing $b = a$. Graphically, this is the 45° line through the origin on the contour plot. Whereas $f(a,b)$ is maximised with respect to a by choosing $a = \mathbf{1}\{a < 1/2\}$. On the contour plot, this is a vertical line between $(0,1)$ and $(0,\frac{1}{2})$, and a second vertical line between $(1,\frac{1}{2})$ and $(1,0)$.