Problem Set 9

Due: TA Discussion, 1 November 2023.

1 Exercises from class notes

From "5. Differentiation.pdf".

Exercise 18. Let $F : \mathbb{R}^4 \to \mathbb{R}^2$. Suppose the conditions for the implicit function theorem are satisfied at all points and that $F(x_1^*, x_2^*, y_1^*, y_2^*) = 0$. Let $h = (h_1, h_2)$ denote the implicitly defined function of (x_1, x_2) for the relation $F(x_1, x_2, y_1, y_2) = (0, 0)$ near $(x_1^*, x_2^*, y_1^*, y_2^*)$. Give explicit formulae for $\frac{\partial h_i}{\partial x_j}$ for $i, j \in \{1, 2\}$.

Exercise 19. Prove the Inverse Function Theorem. **Hint:** An inverse function of $f : X \to Y$, f^{-1}, satisfies following equation:

$$\mathbf{y} - f\left(f^{-1}(\mathbf{y})\right) \equiv 0.$$

Thus, we can think of $\mathbf{x} = f^{-1}(\mathbf{y})$ as being implicitly defined via the expression above.

From "6. Optimisation.pdf".

Exercise 3 Prove the following: Suppose f is $\mathbf{C}^2$ on X, where $\text{int}(X)$ is convex, and that f is concave. Fix $\mathbf{x}^* \in \text{int}(X)$. The following are equivalent:

(i) $\nabla f(\mathbf{x}^*) = \mathbf{0}$.

(ii) f has a local maximum at $\mathbf{x}^*$.

(iii) f has a global maximum at $\mathbf{x}^*$.

Hint: Use Proposition 14 from "5. Differentiation.'

2 Additional Exercises

Exercise 1. Consider the equality-constrained optimisation problem from class notes:

$$\max_{\mathbf{x} \in \mathbb{R}^d} f(\mathbf{x}) \text{ s.t. } h(\mathbf{x}) = \mathbf{0}, \tag{1}$$

where $h(\cdot) = (h_k(\cdot))_{k=1}^K$, and functions $f : \mathbb{R}^d \to \mathbb{R}$ and $h_k : \mathbb{R}^d \to \mathbb{R}$ for each $k \in \{1, \ldots, K\}$ are all $\mathbf{C}^1$. Define a function $\mathcal{L} : \mathbb{R}^d \times \mathbb{R}^K \to \mathbb{R}$ as

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\mu}) = f(\mathbf{x}) + \sum_{k=1}^{K} \mu_k h_k(\mathbf{x}). \tag{2}$$

Let

$$S := \left\{ (\mathbf{x}, \boldsymbol{\mu}) \in \mathbb{R}^d \times \mathbb{R}^K : \nabla \mathcal{L}(\mathbf{x}, \boldsymbol{\mu}) = \mathbf{0} \right\}.$$

and define S_X as the project of S onto the first d components of S; i.e.,

$$S_X := \left\{ \mathbf{x} \in \mathbb{R}^d : \exists \boldsymbol{\mu} \in \mathbb{R}^K, (\mathbf{x}, \boldsymbol{\mu}) \in S \right\}.$$

Now consider the following problem:

$$\max_{\mathbf{x} \in S_X} f(\mathbf{x}). \tag{3}$$

(i) Show that if a the problem (1) attains a global maximum at some $\mathbf{x}^* \in \mathbb{R}^d$ such that $h_k(\mathbf{x}^*) = 0$ for all $k \in \{1, \ldots, K\}$, *and* the constraint qualification under equality constraints holds at $\mathbf{x}^*$, then a $\mathbf{x}^\circ \in S_X$ that solves (3) is also a global maximum.

(ii) Show that (3) is equivalent to

$$\max_{(\mathbf{x}, \boldsymbol{\mu}) \in \mathbb{R}^d \times \mathbb{R}^K} \mathcal{L}(\mathbf{x}, \boldsymbol{\mu}). \tag{4}$$

Remark 1. The function $\mathcal{L}(\mathbf{x}, \boldsymbol{\mu})$ in (2) is called the *Lagrangian* of the problem (1). The solution to (4) is called the *solution to the Lagrangian*.

Exercise 2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $h : \mathbb{R}^2 \to \mathbb{R}$ be defined as $f(x, y) := -y$ and $h(x, y) := y^3 - x^2$, respectively. Consider the problem of maximising f with respect to $(x, y) \in \mathbb{R}^2$ such that $h(x, y) = 0$. Show that the unique solution to the constrained problem is at $(0, 0)$. Show that the constraint qualification under equality constraints is violated at $(0, 0)$ and that there does not exist a $\mu \in \mathbb{R}$ that satisfies

$$\nabla f(\mathbf{x}^*) + \sum_{k=1}^{K} \mu_k^* \nabla h_k(\mathbf{x}^*) = \mathbf{0}_{1 \times K}.$$

Exercise 3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to$ be defined as $f(x, y) := \frac{1}{3}x^3 - \frac{3}{2}y^2 + 2x$ and $g(x, y) := x - y$. Consider the problem of maximising f with respect to $(x, y) \in \mathbb{R}^2$ such that $g(x, y) = 0$. Show that the constraint qualification under equality constraints holds everywhere. Solve for (x^*, y^*, μ^*)'s that solve (4). Are these solutions to (1)?

Exercise 4. What do Exercises 2 and 3 above tell you about solving (1) via (4)?