4. Correspondences

Takuma Habu*
takumahabu@cornell.edu

26th August 2024

1 Correspondences

Many economic problems have more than one "answer" at a time: multiple solutions of a constrained optimisation problem, multiple equilibria of a game or a market, etc. A function from parameters of the problem to its answer cannot capture this multiplicity, since functions are by definition single-valued. We thus need a "multi-valued" function, which we formally define as a correspondence.

Definition 1. A *correspondence* F from a nonempty set X to a nonempty set Y, denoted[1]

$$F : X \rightrightarrows Y,$$

maps every element in X to not-necessarily unique elements in Y.[2] Equivalently, a correspondence F is a function from X to 2^Y (i.e., set of all subsets of Y). Given a correspondence $F : X \rightrightarrows Y$, X is referred to as the *domain* and Y as the *codomain,* and its *range* is given by

$$F(X) := \bigcup_{x \in X} F(x).$$

Definition 2. A correspondence $F : X \rightrightarrows Y$ is *closed-valued* if $F(x)$ is closed for all $x \in X$; it is *compact-valued* if $F(x)$ is compact for all $x \in X$; it is *convex-valued* if $F(x)$ is convex for all $x \in X$; it is *nonempty-valued* if $F(x) \neq \varnothing$ for all $x \in X$.

Definition 3. The *graph* of a correspondence $F : X \rightrightarrows Y$, denoted $\mathrm{gr}(F)$, is the set of points $\{(x, y) \in X \times Y : y \in F(x)\}$. (Compare to the graph of a function.)

Example 1 (Budget correspondence)**.** Consider a correspondence $B : \mathbb{R}^{d+1}_{++} \rightrightarrows \mathbb{R}^d_+$ defined as

$$B(\mathbf{p}, m) := \left\{\mathbf{x} \in \mathbb{R}^d_+ : \mathbf{p} \cdot \mathbf{x} \leq m\right\}.$$

This correspondence gives the the budget set given any strictly positive prices $\mathbf{p} \in \mathbb{R}^d_{++}$ and income $w \in \mathbb{R}_{++}$.

*Thanks to Giorgio Martini, Nadia Kotova and Suraj Malladi for sharing their lecture notes, on which these notes are heavily based.

[1] You may also see $F : X \Rightarrow Y$ as well as $F : X \twoheadrightarrow Y$.

[2] Recall that a function $f : X \to Y$ maps every element of X to a unique element in Y.

Example 2 (Solution correspondence)**.** Given a function $f : X \to Y$ and some set $G \subseteq X$,

$$\max_{x \in G} f(x) := \max \{ f(x) : x \in \Gamma \},$$
$$\arg\max_{x \in G} f(x) := \left\{ x \in X : x \in \Gamma,\ f(x) = \max_{y \in G} f(y) \right\}.$$

Let $\Gamma : \Theta \rightrightarrows X$ be a correspondence and $f : X \to \mathbb{R}$ be a function given some set Θ and X. Consider the following constrained maximisation problem

$$\max_{x \in \Gamma(\theta)} f(x),$$

where $\Gamma(\cdot)$ is a constraint correspondence. We can define a solution correspondence $X^* : \Theta \rightrightarrows X$ as

$$X^*(\theta) := \arg\max_{x \in \Gamma(\theta)} f(x).$$

2 Continuity of correspondences

Throughout, we will assume that $X \subseteq \mathbb{R}^d$ and $Y \subseteq \mathbb{R}^q$ are Euclidean spaces (with $d, q \in \mathbb{N}$). Recall that a function $f : X \subseteq \mathbb{R}^d \to Y \subseteq \mathbb{R}^q$ is continuous if and only if $f^{-1}(O)$ is open for any open subset O of Y (recall problem set 4). To extend this definition to correspondences, we must first decide how to define the inverse image of a set under F. Given a subset $S \subseteq Y$, note that

$$\begin{aligned} f^{-1}(S) &\equiv \{ x \in X : f(x) \in S \} \\ &= \{ x \in X : \{ f(x) \} \subseteq S \} && (1) \\ &= \{ x \in X : \{ f(x) \} \cap S \neq \varnothing \}. && (2) \end{aligned}$$

The expressions (1) and (2) represent two views on what it means by "$f(x) \in S$": Expression (1) interprets it to mean all $y \in F(x)$ is in S, while (2) interprets it to mean that at least one $y \in F(x)$ exists that belongs in S.

If we follow (1), we get the *upper inverse image of* F:

$$F^{-1}(S) := \{ x \in X : F(x) \subseteq S \} \ \forall S \subseteq Y.$$

If we follow (2), we get the *lower inverse image of* F*:*

$$F_{-1}(S) := \{ x \in X : F(x) \cap S \neq \varnothing \} \ \forall S \subseteq Y.$$

Above suggests there are at least two "genuine" ways (in the sense that the definition reduces to the usual continuity of functions when F is single-valued) of defining continuity. The first way, called *upper hemicontinuity*, captures the idea that moving slightly away from a point $x \in X$ does not cause $F(x)$ to become "much larger." The second way, called *lower hemicontinuity*, captures the idea that moving slightly away from a point $x \in X$ does not cause $F(x)$ to become "much smaller."

Definition 4. A correspondence $F : X \subseteq \mathbb{R}^d \rightrightarrows Y \subseteq \mathbb{R}^q$ is

▷ *upper hemi-continuous* at $x \in X$ if, for any open subset $O \subseteq Y$ such that $F(x) \subseteq O$, there exists $\epsilon > 0$ such that $F(B_\epsilon(x)) \subseteq O$.

▷ *lower hemi-continuous* at $x \in X$ if, for any open subset $O \subseteq Y$ such that $F(x) \cap O \neq \varnothing$, there exists $\epsilon > 0$ such that $F(z) \cap O \neq \varnothing$ for all $z \in B_\epsilon(x)$.

The correspondence F is upper (resp. lower) hemi-continuous if it is upper (resp. lower) continuous at all $x \in X$.

Proposition 1. *$F : X \subseteq \mathbb{R}^d \rightrightarrows Y \subseteq \mathbb{R}^q$ is upper hemi-continuous (resp. lower hemi-continuous) if and only if $F^{-1}(O)$ (resp. $F_{-1}(O)$) is open for every open $O \subseteq Y$.*

Proof. (Upper hemi-continuity) Suppose $F^{-1}(O)$ is open for every open $O \subseteq Y$. Fix some $x \in X$ and take any open $O \subseteq Y$ such that $F(x) \subseteq O$ (i.e., $x \in F^{-1}(O)$). Then, by the hypothesis, $F^{-1}(O)$ is open and so there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq F^{-1}(O)$. This, in turn, implies that $F(B_\epsilon(x)) \subseteq F(F^{-1}(O)) = O$. Hence, F is upper hemi-continuous. Conversely, suppose F is upper hemi-continuous. Choose some open $O \subseteq Y$. Pick some $x \in F^{-1}(O)$. Then, $F(x) \subseteq O$. To show that $F^{-1}(O)$ is open, we need to show that there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq F^{-1}(O)$. To that end, because F is upper hemi-continuous, there exists $\epsilon > 0$ such that $F(B_\epsilon(x)) \subseteq O$. This implies that $B_\epsilon(x) = F^{-1}(F(B_\delta(x))) \subseteq F^{-1}(O)$.

(Lower hemi-continuity) Suppose $F_{-1}(O)$ is open for every open $O \subseteq Y$. Fix some $x \in X$ and take any open $O \subseteq Y$ such that $F(x) \cap O \neq \varnothing$ (i.e., $x \in F_{-1}(O)$). Then, by hypothesis, $F_{-1}(O)$ is open so that there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq F_{-1}(O)$. This, in turn, implies that $F(B_\epsilon(x)) \subseteq F(F_{-1}(O)) = O$. Conversely, suppose F is lower hemi-continuous. Choose some open $O \subseteq Y$. Pick some $x \in F_{-1}(O)$. Then, $F(x) \cap O \neq \varnothing$. We want to show that $F_{-1}(O)$ is open. Because F is lower hemi-continuous, there exists $\epsilon > 0$ such that $F(z) \cap O \neq \varnothing$ for all $z \in B_\epsilon(x)$. That is, $B_\epsilon(x) \subseteq F_{-1}(O)$. ∎

Remark 1. A correspondence $F : X \subseteq \mathbb{R}^d \rightrightarrows Y \subseteq \mathbb{R}^q$ is called *singleton-valued* if $F(x)$ is a singleton for all $x \in X$. In this case, we can define a function $f : X \to Y$ via $f(x) \in F(x)$ for all $x \in X$. Now consider the definition of hemi-continuity when F is singleton-valued. Upper hemi-continuity at $x \in X$ says that, for any open subset $O \subseteq Y$ such that $F(x) = \{f(x)\} \subseteq O$ meaning $f(x) \in O$, there exists $\epsilon > 0$ such that $f(B_\epsilon(x)) \subseteq O$. Lower hemi-continuity at $x \in X$ says that, for any open subset $O \subseteq Y$ such that $F(x) \cap O = \{f(x)\} \cap O \neq \varnothing$ meaning $f(x) \in O$ there exists $\epsilon > 0$ such that $F(z) \cap O = \{f(z)\} \cap O \neq \varnothing$ for all $z \in B_\epsilon(x)$ meaning $f(B_\epsilon(x)) \subseteq O$. Therefore, we see that the two definitions of hemi-continuity leads to the same condition that is equivalent to the continuity of function f at x.[3] Then, problem set 4 tells us that if a correspondence is singleton-valued, then both upper and lower hemi-continuity implies continuity of f.

Proposition 2. *Let $F : X \subseteq \mathbb{R}^d \rightrightarrows Y \subseteq \mathbb{R}^q$ be a correspondence.*

(i) F is upper hemi-continuous at $x \in X$ if, for any sequence $(x_n)_n$ in X and any sequence $(y_n)_n$ in Y such that $x_n \to x$ and $y_n \in F(x_n)$ for all $n \in \mathbb{N}$, there exists a subsequence of $(y_n)_n$ that converges to a point in $F(x)$. If F is compact-valued, then the converse is also true.

[3] To see how this condition implies continuity of f at x, fix $r > 0$. Let $A_\epsilon(f(x))$ be an open ball centred at $f(x)$ with radius $\epsilon > 0$. Since $A_\epsilon(f(x))$ is an open subset of Y that contains $f(x)$, there exists $\delta > 0$ such that $f(B_\delta(x)) \subseteq A_\epsilon(f(x))$. In other words, there exists $\delta > 0$ such that for any $x' \in X$ such that $\|x' - x\| < \delta$, we have $\|f(x') - f(x)\| < \epsilon$. Hence, f is continuous at x.

(ii) *F is lower hemi-continuous at $x \in X$ if and only if, for any sequence $(x_n)_n$ in X with $x_n \to x$ and $y \in F(x)$, there exists a sequence $(y_n)_n$ in Y and $N \in \mathbb{N}$ such that $y_n \to y$ and $y_n \in F(x_n)$ for all $n > N$.*

Remark 2. In words, (when the correspondence is compact-valued) upper hemi-continuity is the property that any sequence in the correspondence converges to a limit in the correspondence. Lower hemi-continuity is the property that any point in the correspondence (which need not be compact-valued) can be reached by a sequence in a correspondence. In characterising lower hemi-continuity using sequences, we do not require $y_n \in F(x_n)$ for all $n \in \mathbb{N}$ to account for the possibility that $F(x_n)$ may be empty.[4]

Remark 3. Some textbooks have different definitions and different assumptions about the domain and codomain. I adopt the definition from Efe Ok's textbook, *Real Analysis with Economic Applications* while (i) imposing that the domain and codomain are always Euclidean spaces, and (ii) allowing the correspondence to take the "value" $\varnothing$ (i.e., a correspondence from X to Y is a function $X \to 2^Y$ and not $X \to 2^Y \backslash \{\varnothing\}$).

Definition 5. A correspondence $F : X \subseteq \mathbb{R}^d \rightrightarrows Y \subseteq \mathbb{R}^q$ is *closed* at $x \in X$ if, for all sequences $(x_n)_n$ in X and $(y_n)_n$ in Y such that $x_n \to x$, $y_n \to y \in Y$ and $y_n \in F(x_n)$ for all $n \in \mathbb{N}$, we have $y \in F(x)$. The correspondence F has the *closed graph property* if F is closed at all $x \in X$.

Remark 4. In words, F is closed at $x \in X$ if some points in the image of points nearby x (i.e., points around $F(x')$ where x' is near x) concentrate around a particular point y in Y, that point y must be contained in the image of x (i.e., $F(x)$).

Proposition 3. *Let $F : X \subseteq \mathbb{R}^d \rightrightarrows Y \subseteq \mathbb{R}^q$ be a correspondence. If F is closed at $x \in X$, then $F(x)$ is a closed set.*

Proof. Take any convergent sequence $(y_n)_n$ in $F(x)$ such that $y_n \to y \in Y$. We must show that $y \in F(x)$ Let $(x_n)_n$ be a constant sequence in X by defining $x_n = x$ for all $n \in \mathbb{N}$. Then, $x_n \to x$, $y_n \to y$ and $y_n \in F(x_n) = F(x)$ for all $n \in \mathbb{N}$. Because F is closed at x, it follows that $y \in F(x)$. ■

Exercise 1 (PS6)**.** Give an example of a correspondence $F : X \rightrightarrows Y$ such that $F(x)$ is closed for some $x \in X$ but F is not closed at x.

Proposition 4. *A correspondence $F : X \subseteq \mathbb{R}^d \rightrightarrows Y \subseteq \mathbb{R}^q$ has the closed graph property if and only if $\text{gr}(F)$ is closed in the product space $X \times Y$.*[5]

Proof. Take a sequence $(x_n)_n$ in X and sequence $(y_n)_n$ in Y such that $x_n \to x$, $y_n \to y \in Y$ and $y_n \in F(x_n)$ for all $n \in \mathbb{N}$. Because $\text{gr}(F)$ is closed in $X \times Y$, and $(x_n, y_n) \to (x, y)$, we have $(x, y) \in \text{gr}(F)$; i.e., $y \in F(x)$ so that F is closed at $x \in X$. Conversely, suppose that F has the closed graph property. To show that $\text{gr}(F)$ is closed, it suffices to show that any convergent sequence $(x_n, y_n)_n$ in $\text{gr}(F)$ converges to a point in $\text{gr}(F)$. Then take any sequence $(x_n, y_n) \to (x, y) \in X \times Y$. Then, $x_n \to x$ and $y_n \to y$ (why?) and that $(x_n, y_n) \in \text{gr}(F)$ implies $y_n \in F(x_n)$ for all $n \in \mathbb{N}$. Because F has the closed graph property, we must have $y \in F(x)$; i.e., $(x, y) \in \text{gr}(F)$. ■

[4] If we had required F to be nonempty-valued, then we could have required that $y_n \in F(x_n)$ for all $n \in \mathbb{N}$ in characterising lower hemi-continuity using sequences.

[5] Recall that $X \subseteq \mathbb{R}^d$ and $Y \subseteq \mathbb{R}^q$ are equipped with metrics $\|\cdot\|_d$ and $\|\cdot\|_q$ respectively. We can use $\|\cdot\|_{d+q}$ as the metric for $X \times Y$.

It is often easier to verify whether F has the closed graph property rather than checking whether F is upper hemi-continuous directly.

Proposition 5. *Let $F : X \rightrightarrows Y$ be a correspondence.*

(i) If F has the closed graph property and Y is compact, then F is upper hemi-continuous.

(ii) If F is upper hemi-continuous and closed-valued, then F has the closed graph property.

Proof. (i) Fix some $x \in X$. We will use the sequential characterisation of upper hemicontinuity. Consider a sequence $(x_n)_n$ in X that converges to $x \in X$ and a sequence $(y_n)_n$ in Y such that $y_n \in F(x_n)$ for all $n \in \mathbb{N}$. Since $y_n \in Y$ and Y is (sequentially) compact, there exists a subsequence of $(y_n)_n$, $(y_{n_k})_k$, that converges to some $y \in Y$. Because F has the closed graph property, in particular, F is closed at x. Thus, we must have $y \in F(x)$.

(ii) Suppose F is upper hemi-continuous and closed-valued. Consider sequences $(x_n)_n$ in X and $(y_n)_n$ in Y such that $x_n \to x$, $y_n \to y \in Y$ and $y_n \in F(x_n)$ for all $n \in \mathbb{N}$. We wish to show that $y \in F(x)$. By way of contradiction, suppose $y \notin F(x)$. Since $F(x)$ is closed, $Y \backslash F(x)$ is open. Thus, there exists $\epsilon > 0$ such that $B_\epsilon(y) \subseteq Y \backslash F(x)$. This means that $\|y - y'\| \geq \epsilon > 0$ for all $y' \in F(x)$. Define $\epsilon^* \coloneqq \inf\{\|y - y'\| : y' \in F(x)\} > 0$ and

$$T \coloneqq \left\{ z \in Y : \inf\left\{\|z - y'\| : y' \in F\left(x\right)\right\} \leq \frac{\epsilon^*}{2} \right\}.$$

Let us show that T is closed. We first show that the function $f(y) \coloneqq \inf\{\|y - y'\| : y' \in F(x)\}$ is continuous. To see this, for each $z, z' \in Y$,

$$f\left(z\right) = \inf\left\{\|z - y'\| : y' \in F\left(x\right)\right\} \leq \inf\left\{\|z - z'\| + \|z' - y'\| : y' \in F\left(x\right)\right\} = \|z - z'\| + f\left(z'\right)$$

and similarly,

$$f\left(z'\right) = \inf\left\{\|z' - y'\| : y' \in F\left(x\right)\right\} \leq \inf\left\{\|z' - z\| + \|z - y'\| : y' \in F\left(x\right)\right\} = \|z - z'\| + f\left(z\right).$$

Thus,

$$\left|f\left(z\right) - f\left(y\right)\right| \leq \|z - z'\| \ \ \forall z, z' \in Y$$

so that f is continuous. Now, take any sequence $(z_n)_n$ in T that converges to some $z \in Y$. We wish to show that $z \in T$. Since $z_n \in T$ for each $n \in \mathbb{N}$ and f is continuous,

$$f\left(z_n\right) \leq \frac{\epsilon^*}{2} \ \forall n \in \mathbb{N} \Rightarrow f\left(z\right) = \lim_{n \to \infty} f\left(z_n\right) \leq \frac{\epsilon^*}{2}.$$

Hence, $z \in T$; i.e., T is closed. By construction $F(x) \subseteq \operatorname{int}(T)$ and $y \notin T$. But since F is upper hemi-continuous at x, there exists $\delta > 0$ such that $F(B_\delta(x)) \subseteq \operatorname{int}(T)$. That is, there exists $N \in \mathbb{N}$ such that $y_n \in F(x_n) \subseteq \operatorname{int}(T)$ for all $n \geq N$. But then since T is a closed set and $y_n \to y$, we must have $y \in T$; a contradiction. ■

Corollary 1. *Suppose Y is compact and let $F : X \rightrightarrows Y$ be a compact-valued correspondence. Then, F has the closed graph property if and only if F is upper hemi-continuous.*

Proof. Since Y is compact, if F has the closed graph property, by Proposition (5), F is upper hemi-continuous. Conversely, if F is upper hemi-continuous, then F has a closed graph property because F being compact-valued implies that F is also closed-valued. ■

Exercise 2 (PS6)**.** TFU: If a correspondence $F : X \rightrightarrows Y$ is upper hemi-continuous, then $F(x)$ is closed for every $x \in X$.

Exercise 3 (PS6)**.** Are following correspondences upper hemi-continuous and/or lower hemi-continuous?

$$F(x) = \begin{cases} \{4-x, 2-x\}, & \text{if } x < 2, \\ [2-x, 4-x], & \text{if } 2 \leq x \leq 3, \\ \{x-3\}, & \text{if } x > 3. \end{cases}$$

$$G(x) = \begin{cases} \{4-x, 2-x\}, & \text{if } x < 2, \\ [3-x, 5-x], & \text{if } 2 \leq x \leq 3, \\ \{x-3\}, & \text{if } x > 3. \end{cases}$$

Proposition 6. *Let $F : X \subseteq \mathbb{R}^d \rightrightarrows Y \subseteq \mathbb{R}$ be a nonempty-valued correspondence. If F is compact-valued and upper hemi-continuous, then $\max F(x)$ is upper semi-continuous and $\min F(x)$ is lower semi-continuous.*

Proof. Fix any $x \in X$, since F is compact-valued, $F(x)$ is compact and so $\max F(x) = \sup F(x)$ and $\min F(x) = \inf F(x)$. Define $f^*, f_* : X \to \mathbb{R}$ pointwise as $f^*(x) \coloneqq \max F(x)$ and $f_*(x) \coloneqq \min F(x)$. To show that f^* is upper semi-continuous, we show that for any $\alpha \in \mathbb{R}$, $\{x \in X : f^*(x) < \alpha\}$ is open (in X). Observe that

$$F^{-1}((-\infty, \alpha)) = \{x \in X : F(x) \subseteq (-\infty, \alpha)\} = \{x \in X : \sup F(x) < \alpha\} = \{x \in X : f^*(x) < \alpha\},$$

Since $(-\infty, \alpha) \subseteq \mathbb{R}$ is open and F is upper hemi-continuous, $F^{-1}((-\infty, \alpha))$ is open (Proposition 1). Now let $(x_n)_n$ be a convergence sequence such that $x_n \to x_0 \in X$. Fix $\epsilon > 0$. Then, since x_0 belongs to an open set $\{x \in X : f^*(x) < f^*(x_0) + \epsilon\}$, there is $\delta > 0$ such that $B_\delta(x_0) \subseteq \{x \in X : f^*(x) < f^*(x_0) + \epsilon\}$; i.e.,

$$f^*(x) \leq f^*(x_0) + \epsilon \ \forall x \in X : |x - x_0| < \delta.$$

Hence, f^* is upper semi-continuous.

To show that f_* is lower semicontinuous, it suffices to show that, fixing $\alpha \in \mathbb{R}$, $\{x \in X : f_*(x) > \alpha\}$ is open (in X). Observe that

$$F^{-1}((\alpha, \infty)) = \{x \in X : F(x) \subseteq (\alpha, \infty)\} = \{x \in X : \inf F(x) > \alpha\} = \{x \in X : f_*(x) > \alpha\},$$

Since $(\alpha, \infty) \subseteq \mathbb{R}$ is open and F is upper hemi-continuous, $F^{-1}((\alpha - \infty))$ is open (Proposition 1). Now, let $(x_n)_n$ be a convergence sequence such that $x_n \to x_0 \in X$. Fix $\epsilon > 0$. Then, since x_0 belongs to an open set $\{x \in X : f_*(x) > f_*(x_0) - \epsilon\}$, there is $\delta' > 0$ such that $B_{\delta'}(x_0) \subseteq \{x \in X : f_*(x) > f_*(x_0) - \epsilon\}$; i.e.,

$$f_*(x) \geq f(x_0) - \epsilon \ \forall x \in X : |x - x_0| < \delta'.$$

Hence, f_* is lower semi-continuous. ■

Remark 5. In the proof above, we showed that a function $f : S \subseteq \mathbb{R} \to \mathbb{R}$ is upper semi-continuous (resp. lower semi-continuous) if $\{x \in X : f(x) < \alpha\}$ (resp. $\{x \in X : f(x) > \alpha\}$ is open for all $\alpha \in \mathbb{R}$. The converse, in fact is true, so that we've in fact given another characterisation of semi-continuity. This also gives us that f is upper semi-continuous (resp. lower semi-continuous) if (and only if) $\{x \in X : f(x) \geq \alpha\}$ (resp. $\{x \in X : f(x) \leq \alpha\}$ is closed for all $\alpha \in \mathbb{R}$.

3 Berge's theorem of the maximum

Definition 6. A correspondence is *continuous* if it is both upper hemi-continuous and lower hemi-continuous.

We are now ready to state one of the most important theorems in optimisation.

Theorem 1 (Berge)**.** *Suppose $\Theta \subseteq \mathbb{R}^d$ and $X \subseteq \mathbb{R}^q$ and let $\Gamma : \Theta \rightrightarrows X$ be a nonempty-valued and compact-valued correspondence that is continuous at some $\theta_0 \in \Theta$, and $f : X \times \Theta \to \mathbb{R}$ be continuous. Define $f^* : \Theta \to \mathbb{R}$ and $X^* : \Theta \rightrightarrows X$ by*

$$f^*(\theta) \coloneqq \max_{x \in \Gamma(\theta)} f(x, \theta), \ X^*(\theta) \coloneqq \arg\max_{x \in \Gamma(\theta)} f(x, \theta). \tag{3}$$

Then,

(i) X^* is nonempty-valued, compact-valued, upper hemi-continuous at θ_0, and closed at θ_0.

(ii) f^* is continuous at θ_0.

Let us make slightly stronger assumptions before we interpret.

Corollary 2. *Suppose $\Theta \subseteq \mathbb{R}^d$ and $X \subseteq \mathbb{R}^q$ and let $\Gamma : \Theta \rightrightarrows X$ be a nonempty-valued, compact-valued, continuous correspondence, and $f : X \times \Theta \to \mathbb{R}$ be continuous. Define f^* and X^* as in (3). Then,*

(i) X^* is a nonempty-valued, compact-valued, upper hemi-continuous correspondence.

(ii) f^* is continuous.

Think of f as an objective function in an optimisation problem (e.g., utility), where x is the variable that we are maximising with respect to (e.g., consumption) and θ is a parameter (e.g., prices). Then, Γ represent a set of constraints on x that can depend upon the parameter θ (e.g., budget constraint). Thus, inter alia, the Maximum Theorem tells us the following:

- ▷ A solution to the optimisation problem exists; e.g., solution to consumer's maximisation problem exists.
- ▷ The maximised objective function varies continuously with the parameter θ; e.g., maximised utility is a continuous function of prices.
- ▷ Given parameter θ, the set of maximisers are compact and continuous; e.g., given prices, set of optimal consumption bundles are compact and the set does not expand much when prices change.

Let us break down the proof of Theorem 1 into several Lemmata.

Existence is guaranteed by the Weierstrass Extreme Value Theorem as we demonstrate below.

Lemma 1. *X^* and f^* are well-defined.*

Proof. For every $\theta \in \Theta$, $\Gamma(\theta)$ is nonempty and compact, and f is continuous so that by the Extreme Value theorem, $X^*(\theta) \neq \varnothing$. Hence, $X^*(\theta)$ and $f^*(\theta) = f(x^*, \theta)$ for any $x^* \in X^*(\theta)$ are well-defined (i.e., nonempty). ■

Lemma 2. *X^* is compact-valued.*

Proof. We wish to show that $X^*(\theta)$ is compact for all $\theta \in \Theta$. By definition, $X^*(\theta) \subseteq \Gamma(\theta)$ for each $\theta \in \Theta$ and since Γ is compact-valued, $\Gamma(\theta)$ is compact. Since closed subsets of compact sets are compact (why?), it suffices to show that $X^*(\theta)$ is closed for every $\theta \in \Theta$. To that end, fix an arbitrary $\theta \in \Theta$. Take any convergent sequence $(x_n)_n$ in $X^*(\theta)$. We wish to show that $x_n \to x \in X^*(\theta)$; i.e., $x \in \Gamma(\theta)$ and $f(x, \theta) = f^*(\theta)$. Since $(x_n)_n$ is a sequence in $\Gamma(\theta)$ and $\Gamma(\theta)$ is closed, $x_n \to x \in \Gamma(\theta)$. Since $f(x_n, \theta) = f^*(\theta)$ for all $n \in \mathbb{N}$ and $f(\cdot, \theta)$ is continuous, it follows that $f(x_n, \theta) \to f(x, \theta) = f^*(\theta)$. Thus, $x \in X^*(\theta)$ as we wanted. ■

The following makes it clear that we need the full continuity of f (i.e., both upper and lower semicontinuity of f) in both arguments (x, θ).

Lemma 3. *X^* is closed at θ_0.*

Proof. Next, we show that X^* is closed at θ_0. Take any sequence $(\theta_n)_n$ in Θ and a sequence $(x_n)_n$ in X such that $\theta_n \to \theta_0$ and $x_n \to x_0$, and suppose $x_n \in X^*(\theta_n)$ for all $n \in \mathbb{N}$ (i.e., $x_n \in \Gamma(\theta_n)$ and $f(x_n) = f^*(\theta_n)$ for all $n \in \mathbb{N}$). We must show that $x_0 \in X^*(\theta_0)$; i.e., $x_0 \in \Gamma(\theta_0)$ and $f(x_0, \theta_0) = f^*(\theta_0)$.

By way of contradiction, suppose that $x_0 \notin X^*(\theta_0)$. Since Γ is compact-valued (and thus closed-valued) and Γ is upper hemi-continuous, By Proposition 5, Γ has a closed graph; i.e., $x_0 \in \Gamma(\theta_0)$. Hence, $x_0 \notin X^*(\theta_0)$ if and only if $f(x_0, \theta_0) \neq f^*(\theta_0)$. By definition of f^*, this means that there exists $y_0 \in \Gamma(\theta_0)$ such that $f^*(\theta_0) = f(y_0, \theta_0) > f(x_0, \theta_0)$. By lower hemicontinuity of Γ (Proposition 2), since we have $x_n \to x_0$ and $y_0 \in \Gamma(\theta_0)$, we can find a sequence $(y_n)_n$ in X such that $y_n \to y_0$ and $y_n \in \Gamma(\theta_n)$ for each $n \in \mathbb{N}$.

Claim 1. For sufficiently large $n \in \mathbb{N}$, $f(y_n, \theta_n) > f(x_n, \theta_n)$.

Proof. Define $\epsilon > 0$ such that $f(y_0, \theta_0) - f(x_0, \theta_0) > \epsilon$. Since f is continuous, in particular, it is upper semicontinuous. Thus, by the ϵ-δ criterion, there exists $\delta > 0$ such that

$$f(x, \theta) \leq f(x_0, \theta_0) + \epsilon \;\forall\, (x, \theta) \in X \times \Theta : \|(x, \theta) - (x_0, \theta_0)\| < \delta.$$

Since $x_n \to x_0$ and $\theta_n \to \theta_0$, $(x_n, \theta_n) \to (x_0, \theta_0)$ (in $X \times \Theta$) so there exists $N_1 \in \mathbb{N}$ such that

$$\|(x, \theta) - (x_0, \theta_0)\| < \delta \;\forall n > N_1 \Rightarrow f(x_0, \theta_0) + \epsilon \geq f(x_n, \theta_n) \;\; \forall n > N_1.$$

By the choice of ϵ, $f(y_0, \theta_0) > f(x_0, \theta_0) + \epsilon$, and so we have

$$f(y_0, \theta_0) > f(x_0, \theta_0) + \epsilon \geq f(x_n, \theta_n) \;\; \forall n > N_1.$$

Hence,

$$f(y_0, \theta_0) > \sup\{f(x_n, \theta_n) : n > N_1,\ n \in \mathbb{N}\} =: s.$$

Pick any $e > 0$ such that $f(y_0, \theta_0) - s > e$. Since f is continuous, in particular, it is lower semicontinuous. Then, a similar argument as to above shows that there exists $N_2 \in \mathbb{N}$ such that

$$f(y_n, \theta_n) \geq f(y_0, \theta_0) - e > s\ \forall n > N_2.$$

Then, define $N \coloneqq \max\{N_1, N_2\}$, so that

$$f(y_n, \theta_n) > s \geq f(x_n, \theta_n)\ \ \forall n > N. \qquad \blacksquare$$

Since $y_n \in \Gamma(\theta_n)$ for all $n \in \mathbb{N}$, the claim contradicts the fact that $x_n \in X^*(\theta_n)$ for all $n \in \mathbb{N}$; i.e., we must have $x_0 \in X^*(\theta_0)$. $\blacksquare$

Lemma 4. *Suppose $Z \subseteq \mathbb{R}^d$ and $Y \subseteq \mathbb{R}^q$ and let $F_1, F_2 : Z \rightrightarrows Y$ with $F_1(z) \cap F_2(z) \neq \varnothing$ for all $z \in Z$. Define $F : Z \rightrightarrows Y$ by $F \coloneqq F_1 \cap F_2$. If F_1 is compact-valued and upper hemi-continuous at $z_0 \in Z$, and if F_2 is closed at z_0, then F is upper hemi-continuous at z_0.*

Proof. We will show that F is upper hemi-continuous at z_0 using part (i) of Proposition 2. To that end, take an arbitrary sequences $(z_n)_n$ in Z and $(y_n)_n$ in Y such that $z_n \to z_0$ and $y_n \in F(z_n) = F_1(z_n) \cap F_2(z_n)$ for all $n \in \mathbb{N}$. We want to show that there exists a subsequence of $(y_n)_n$ that converges to some $y_0 \in F(z_0)$. Since F_1 is compact-valued and upper hemi-continuous and $y_n \in F_1(z_n)$ for all $n \in \mathbb{N}$, by Proposition 2, there exists a subsequence $(y_{n_k})_k$ of $(y_n)_n$ that converges to $y_0 \in F_1(z_0)$. It remains to show that $y_0 \in F_2(z_0)$. Since $y_{n_k} \to y_0$, $y_{n_k} \in F_2(z_{n_k})$ for all $k \in \mathbb{N}$, and $z_n \to z_0$, that F_2 is closed at z_0 implies $y_0 \in F_2(z_0)$. That is, $y_0 \in F_1(z_0) \cap F_2(z_0) = F(z_0)$. $\blacksquare$

Lemma 5. *X^* is upper hemi-continuous.*

Proof. Since $\Gamma(\theta) \cap X^*(\theta) = X^*(\theta) \neq \varnothing$ for all $\theta \in \Theta$, $\Gamma(\theta)$ is compact-valued and upper hemi-continuous. Moreover, by Lemma 3, $X^*(\theta)$ is closed at x_0. Then, by lemma 4 (letting $F_1 = \Gamma$ and $F_2 = X^*$), we have that $X^*(\theta) = \Gamma(\theta) \cap X^*(\theta)$ is upper hemi-continuous. $\blacksquare$

Lemma 6. *f^* is continuous at θ_0.*

Proof. To show that f^* is continuous at θ, we will show that, for any sequence $(\theta_n)_n$ in Θ such that $\theta_n \to \theta_0$, we have $f^*(\theta_n) \to f^*(\theta_0)$. Since $(f^*(\theta_n))_n$ is a sequence in $\mathbb{R}$, it has a subsequence, say $(f^*(\theta_{n_k}))_k$ such that $f^*(\theta_{n_k}) \to \limsup_{n\to\infty} f^*(\theta_n)$ (why?). Pick any $x_{n_k} \in X^*(\theta_{n_k})$ so that $f^*(\theta_{n_k}) = f(x_{n_k}, \theta_{n_k})$ for each $k \in \mathbb{N}$. Since X^* is compact-valued and upper hemi-continuous at x, by Proposition 2, there exists a subsequence of $(x_{n_k})_k$, say $(x_{n_{k_\ell}})_\ell$, that converges to a point $x_0 \in X^*(\theta_0)$. By continuity of f, then

$$f^*\left(\theta_{n_{k_\ell}}\right) = f\left(x_{n_{k_\ell}}, \theta_{n_{k_\ell}}\right) \to f(x_0, \theta_0) = f^*(\theta_0).$$

Since $(f^*(\theta_{n_k}))_k$ is convergent, every subsequence of $(f^*(\theta_{n_k}))_k$ must also converge to the same limit. Thus, above shows that $f^*(\theta_0) = \limsup_{n\to\infty} f^*(\theta_n)$. An analogous argument shows that $f^*(\theta_0) = \liminf_{n\to\infty} f^*(\theta_n)$. Thus, it follows that $f^*(\theta_0) = \lim_{n\to\infty} f^*(\theta_n)$. $\blacksquare$

Proof of Theorem **??**). Part (i) of the theorem follows from Lemma 2, Lemma 3 and Lemma 5. Part (ii) of the theorem follows from Lemma 6. ■

Exercise 4 (PS6)**.** Prove that the budget correspondence is continuous. What does the Berge's maximum theorem tell you about the consumer's problem when the agent's utility function is continuous?

We will see how adding convexity to the corollary above gives us some additional properties on X^* and f^*. A correspondence $\Gamma : \Theta \rightrightarrows X$ is *convex-valued* if $\Gamma(\theta)$ is convex for all $\theta \in \Theta$. Γ has a convex graph if its graph, defined as

$$\mathrm{gr}\,(\Gamma) \coloneqq \{(\theta, x) \in \Theta \times X : x \in \Gamma\,(\theta)\}\,,$$

is convex.

Theorem 2 (Maximum Theorem with concavity)**.** *Let $\Theta \subseteq \mathbb{R}^d$ and $X \subseteq \mathbb{R}^q$ and let $\Gamma : \Theta \rightrightarrows X$ be a nonempty-valued, compact-valued, continuous correspondence and $f : X \times \Theta \to \mathbb{R}$ be continuous. Define F^* and X^* as in (3).*

(i) If $f(\cdot, \theta)$ is concave on X for each $\theta \in \Theta$ and Γ is convex-valued, then X^* is a convex-valued, upper hemi-continuous correspondence.

(ii) If $f(\cdot, \theta)$ is strictly concave on X for each $\theta \in \Theta$ and Γ is convex-valued, then X^* is a continuous function.

(iii) If f is concave on $X \times \Theta$ and Γ has a convex graph, then f^* is a concave function and X^* is a convex-valued, upper hemi-continuous correspondence.

(iv) If f is strictly concave on $X \times \Theta$ and Γ has a convex graph, then f^* is a strictly concave function and X^* is a continuous function.

Proof. (i) The fact that X^* is upper hemi-continuous follows from the corollary to the Maximum Theorem. Thus, it remains to show that X^* is convex-valued; i.e., $X^*(\theta)$ is convex for all $\theta \in \Theta$. Fix some $\theta \in \Theta$. Choose $x_1, x_2 \in X^*(\theta)$ and define $x_\alpha \coloneqq \alpha x_1 + (1-\alpha)x_2$ for some $\alpha \in (0, 1)$. Since $\Gamma(\theta)$ is convex, $x_\alpha \in \Gamma(\theta)$. Then,

$$\begin{aligned}
f\,(x_\alpha, \theta) &= f\,(\alpha x_1 + (1-\alpha)\,x_2, \theta) \\
&\geq \alpha f\,(x_1, \theta) + (1-\alpha)\,f\,(x_2, \theta) \\
&= \alpha f^*\,(\theta) + (1-\alpha)\,f^*\,(\theta) \\
&= f^*\,(\theta)\,,
\end{aligned}$$

where the inequality follows from the concavity of $f(\cdot, \theta)$. By definition of f^*, we must have $x_\alpha \in X^*(\theta)$.

(ii) If f is strictly concave, then we would obtain $f(x_\alpha, \theta) > f^*(\theta)$, which is a contradiction, unless $x_1 \neq x_2$, Hence, $X^*(\theta)$ must be unique. Recall that upper hemi-continuous single-valued correspondence are continuous functions.

(iii) Suppose $\theta_1, \theta_2 \in \Theta$ and define $\theta_\alpha \coloneqq \alpha\theta_1 + (1-\alpha)\theta_2$ for some $\alpha \in (0,1)$. Pick $x_1 \in X^*(\theta_1) \subseteq \Gamma(\theta_1)$, $x_2 \in X^*(\theta_2) \subseteq \Gamma(\theta_2)$ and define $x_\alpha \coloneqq \alpha x_1 + (1-\alpha)x_2$. Since $\theta_1 \in \Gamma(\theta_1)$ and $\theta_2 \in \Gamma(\theta_2)$ and Γ has a convex graph, we must have $x_\alpha \in \Gamma(\theta_\alpha)$.

$$\begin{aligned} f^*(\theta_\alpha) &= f^*(\alpha\theta_1 + (1-\alpha)\theta_2) \\ &\geq f(x_\alpha, \theta_\alpha) \\ &= f(\alpha x_1 + (1-\alpha)x_2, \alpha\theta_1 + (1-\alpha)\theta_2) \\ &\geq \alpha f(x_1, \theta_1) + (1-\alpha) f(x_2, \theta_2) \\ &= \alpha f^*(\theta_1) + (1-\alpha) f^*(\theta_2), \end{aligned}$$

where the first inequality follows from the fact that x_α is feasible but not necessarily optimal at θ_α, and the second inequality follows from the concavity of f on $X \times \Theta$. Thus, we have shown that f^* is concave in θ.

(iv) If f is strictly concave on $X \times \Theta$, the second inequality in the expression above is strict, which implies that f^* is strictly concave. ■

Theorem 3 (Maximum Theorem under quasiconcavity)**.** *Let $\Theta \subseteq \mathbb{R}^d$ and $X \subseteq \mathbb{R}^q$ and let $\Gamma : \Theta \rightrightarrows X$ be a nonempty-valued, compact-valued, continuous correspondence, and $f : X \times \Theta \to \mathbb{R}$ be continuous. Define f^* and X^* as in (3).*

(i) If $f(\cdot, \theta)$ is quasiconcave on X for each $\theta \in \Theta$ and Γ is convex-valued, then X^* is a convex-valued, upper hemi-continuous correspondence.

(ii) If $f(\cdot, \theta)$ is strictly quasiconcave on X for each $\theta \in \Theta$ and Γ is convex-valued, then X^* is a continuous function.

Proof. (i) Following the same argument as in part (i) of the proof of the previous theorem, we have

$$\begin{aligned} f(x_\alpha, \theta) &= f(\alpha x_1 + (1-\alpha)x_2, \theta) \\ &\geq \min\{f(x_1, \theta), f(x_2, \theta)\} = f^*(\theta). \end{aligned}$$

Therefore, we must again have $x_\alpha \in X^*(\theta)$. (ii) follows from the fact that strictly quasiconcave functions have a unique maximum. ■