PS 4

1. Suppose an investor with initial wealth w_0 has a payoff function of the form

$$u(w) = -\exp -r_a w$$

with $r_a > 0$. There are two investment opportunities: a risk-free asset and a risky asset whose payoff is normally distributed with mean μ and variance σ^2. The investor can allocate her wealth between the two opportunities. What share of her wealth should go into the risky asset?

Solution: The return to investing share s of wealth in the risky asset is a random variable

$$\tilde{w} = (1-s)w_0 + sw_0\tilde{x}$$

where $\tilde{x}$ is distributed $N(\mu, \sigma^2)$. Thus the return to investing s in the risky asset is normally distributed with mean $(1 - s + s\mu)w_0$ and variance $s^2 w_0^2 \sigma^2$.

You can integrate to get expected utility, or notice that the moment generating function of an $N(\alpha, \beta^2)$ random variable $\tilde{z}$ is $E\{\exp t\tilde{z}\} = \exp t\alpha + t^2\beta^2/2$. With this fact you can see that the log of EU for $\tilde{y}$ is quadratic in s, and so the maximum is easily computed. Notice that for small enough r_a, $s > 1$. You can make the problem a little trickier by supposing that individuals can borrow at some price $p > 1$. Work out in this case how s varies with c. What other comparative statics questions are interesting?

2. Suppose that $\succeq$ satisfies the Savage axioms with state space S and outcome space X, and suppose that it has an SEU representation with payoff function u and belief distribution μ. Prove that for every non-null event A the preference order σ_A has an SEU representation. What is it?

Solution: . The axioms and definition say that $g \succ_A g$ iff for any h, $f|Ah \succ g|Ah$. That is,

$$\int_A u(f)d\mu + \int_{A^c} u(h)d\mu > \int_A u(g)d\mu + \int_{A^c} u(h)d\mu$$

that is,

$$\int_A u(f)d\mu > \int_A u(g)d\mu.$$

Dividing both sides by $\mu(A)$ shows that this is equivalent to

$$\int_A u(f)d\mu(\,\cdot\,|A) > \int_A u(g)d\mu(\,\cdot\,|A).$$

The Savage axioms imply not just the existence of an SEU representation, but also that conditional preferences are represented by expected payoff with conditional beliefs.

3. Let M denote the right triangle in the plane with vertices $x = (0,1)$, $y = (0,0)$, and $z = (1,0)$. Each $m \in M$ can be written uniquely as $\alpha_m x + (1-\alpha_m)(\beta_m y + (1-\beta_m)z)$. Hint: Draw some pictures to see how this works.). Define the mixture operators

$$m \otimes_\lambda n = \begin{cases} z & \text{if } m = n = z \text{ or } m = z \text{ and} \\ & \lambda = 1 \text{ or } n = z \text{ and } \lambda = 0, \\ & \\ (\lambda\alpha_m + (1-\lambda)\alpha_n)x+ & \text{otherwise.} \\ \big(1-(\lambda\alpha_m + (1-\lambda)\alpha_n)\big)y & \end{cases}$$

(a) Show that this is a mixture space.

(b) Suppose the preference relation satisfies axioms A1-3 for mixture spaces. Describe what indifference sets must look like.

4. Random variable X is distributed with density $f(x) = x^{-6/5}/5$ and Y is distributed with density $g(x) = x^{-3/2}/2$.

(a) Which is bigger with respect to first-order stochastic dominance?

Solution: The CDF of X is $F(x) = 1 - x^{-1/5}$. The CDF of Y is $F(x) = 1 - x^{-1/2}$. So Y stochastically dominates X.

(b) Suppose a decision-maker maximized expected utility with payoff function $u(x) = \sqrt{x}$. Which does he prefer?

Solution: Neither. Utility is unbounded for both.

5. Suppose an expected utility maximizer faces a decision problem in which there are two states of nature and three choices, $a_1, \ldots, a_3$. Utility payoffs are described in the following table: The true probability distribution is $p = (p_1, p_2)$, where p_s is the probability of

	s_1	s_2
a_1	0	-8
a_2	-10	0
a_3	-4	-3

state s.

(a) The DM does not know p, and believes that it is equally likely that $p_1 = 1/4$ and $p_1 = 3/4$. Given these *a priori* beliefs about the models, what probability does she assign to the event s_1?

Solution: $p(s_1) = (1/2)(1/4) + (1/2)(3/4) = 1/2.$

(b) Which a_i will she choose?

Solution:

$$EU(a_1) = -4 \qquad EU(a_2) \qquad = -5 \qquad EU(a_3) = -7/2$$

so a_3 is EU maximizing.

(c) Before she chooses, she is told that the previous draw from the current distribution was s_1. Draws are independent, and her *a priori* belief, as before, is that the models are equally likely. What will she choose?

Solution: The probability of model 1 given a first draw of s_1 is

$$\Pr\{p_1|s_1\} = \Pr\{p_1\&s_1\}/\Pr\{s_1\} = (1/8)/(1/2) = 1/4.$$

EUs for a_1, a_2 and a_3 are now $-24/4$, $-10/4$ and $-13/4$, respectively. So a_2 is best.

(d) Suppose instead that she is told that s_2 was drawn. What will she choose?

Solution: $\Pr\{p_1\} = 3/4$. a_1 is optimal, and $EU(a_1) = -1/2$.

(e) How much is it worth to her, in utility terms, to know the value of the last draw (given that her prior beliefs are that both modes are equally likely). (Hint: In part (c) you computed her expected utility if she is told s_2. In part (b) you computed her expected utility if she is told s_1. Before you are told anything, you have beliefs about how likely you are to be told s_1 and s_2. So you can compute your expected expected utility [this is not a typo; it really is "expected expected utility"] before you are told anything. From this, you can compute the value of information—the value of knowing the value of the last draw. This notion of value of information is widely used.)

Solution: The *ex ante* probability of drawing each s_i is $1/2$. The *ex ante* EU of choosing after drawing is $(1/2)(-5/2) + (1/2)(-1/2) = -6/4$. The EU of choosing without information is $-7/2$, so the value of information is the difference, which is 2.

6. In the three-color Ellsberg paradox, which of Savage's axioms P1-5 (not 6 or 7) fail to hold?