ECON6190 Section 8

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<u>Motivation</u> convenient symbol for random variables and random vectors which converge in probability to zero or are stochastically bounded.

DEF Non Stochastic Orders

For nonstochastic sequence x_n and a_n

① (small-oh) $x_n = o(a_n)$ if $\frac{x_n}{a_n} \to 0$ as $n \to \infty$

↳ "as $n \to \infty$, x_n is eventually small compared to a_n"

② (big-oh) $x_n = O(a_n)$ if $\exists$ finite scalars (M, N_M) s.t. $\forall n \geq N_M$, $\left|\frac{x_n}{a_n}\right| < M$.

↳ x_n and a_n may be of the same order, but $\frac{x_n}{a_n}$ is always bounded for sufficiently large n.

Important special case arise $a_n = (1, \ldots 1)$, then

① $x_n = o(1)$ if $x_n \to 0$ as $n \to \infty$.

② $x_n = O(1)$ if $\limsup_{n \to \infty} x_n < \infty$ x_n is bounded uniformly in n.

DEF Stochastic Orders

For stochastic sequence x_n and nonstochastic sequence a_n,

① (small oh-p) $x_n = o_p(a_n)$ if $\frac{x_n}{a_n} \xrightarrow{P} 0$.

↳ " x_n is of smaller stochastic order than a_n"

② (big oh-p) $x_n = O_p(a_n)$ if $\forall \varepsilon > 0$, $\exists M_\varepsilon$, $N_{\varepsilon,M}$ s.t.

$$P\left(\left|\frac{x_n}{a_n}\right| > M_\varepsilon\right) \leq \varepsilon, \quad \forall n \geq N_{\varepsilon,M}.$$

↳ " x_n is of the same stochastic order as a_n".

Again, consider special case $a_n = \{1, \ldots 1\}$.

① $x_n = o_p(1)$ if $x_n \xrightarrow{P} 0$ → " x_n vanishes"

② $X_n = O_p(1)$ if $\forall \varepsilon > 0$, $\exists$ constant M_ε s.t. $\limsup_{n\to\infty} P(|X_n| > M_\varepsilon) \le \varepsilon$.

↳ " X_n is stochastically bounded".

Note: If X_n is stochastically bounded, this doesn't imply a deterministic upper bound on X_n, even for large n.

What is means is X_n can't take arbitrarily high values with non-vanishing probability.

Notes: $O_p(1)$ is a weaker notion than $o_p(1)$, in the sense that

$X_n = o_p(1) \Rightarrow X_n = O_p(1)$, not the reverse.

- A "typical" instance of $X_n = O_p(1)$ is that $X_n \xrightarrow{d} Z$, where Z is some known random variable.

  ex. If X_n is a studentized test statistic, let $Z \sim \mathcal{N}(0,1)$.

  → very large values of Z are possible but very unlikely.

- Another important use of the term is "$\sqrt{n}$-consistent".

  ⤳ An estimation error $(\hat{\theta} - \theta_0)$ is said to be of stochastic order $n^{-1/2}$. If $\hat{\theta} - \theta_0 = O_p(n^{-1/2}) \iff \sqrt{n}(\hat{\theta} - \theta_0) = O_p(1)$.

- By Chebyshev inequality, we can show $(\hat{\theta} - \theta_0) = O_p\left(\sqrt{MSE(\hat{\theta})}\right)$

  Proof. For each $\varepsilon > 0$, pick $M_\varepsilon = \left(\frac{1}{\varepsilon}\right)^{1/2}$.

  By chebyshev inequality:

$$P\left(\left|\frac{\hat{\theta} - \theta_0}{\sqrt{MSE(\hat{\theta})}}\right| > M_\varepsilon\right) = P\left(|\hat{\theta} - \theta_0| > \overbrace{\sqrt{MSE(\hat{\theta})}\, M_\varepsilon}^{\delta'}\right)$$

$$\le \frac{E[(\hat{\theta} - \theta_0)^2]}{\underbrace{MSE(\hat{\theta}) M_\varepsilon^2}_{\frac{1}{\varepsilon}}\ \delta'^2} = \varepsilon, \quad \forall n \ge N_{\varepsilon, M}$$

MSE$(\hat{\theta})$

∎

Algebra of stochastic orders

① If $X_n = O_p(a_n)$, $Y_n = O_p(b_n)$, then

- $X_nY_n = O_p(a_nb_n)$
- $X_n + Y_n = O_p(\max\{a_n, b_n\})$

② Replace O with o everywhere in ①, still holds

③ If $X_n = O_p(a_n)$, $Y_n = o_p(b_n)$, then $X_nY_n = o_p(a_nb_n)$

④ If $X_n = O_p(a_n)$, and $\frac{a_n}{b_n} \to 0$, then $X_n = o_p(b_n)$

$$o_p(1) + o_p(1) = o_p(1)$$
$$o_p(1) + O_p(1) = O_p(1)$$
$$O_p(1) + O_p(1) = O_p(1)$$
$$o_p(1)\,o_p(1) = o_p(1)$$
$$o_p(1)\,O_p(1) = o_p(1)$$
$$O_p(1)\,O_p(1) = O_p(1)$$

8.* Let $\{X_1 \dots X_n\}$ and $\{Y_1 \dots Y_n\}$ be mutually independent sequences of iid random variables, such that $\mathbb{E}(X_i) = 0$, $\text{var}(X_i) = 1$, $\mathbb{E}(Y_i) = 3$, $\text{var}(Y_i) = 2$. Let $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$, $\bar{Y} = \frac{1}{n}\sum_{i=1}^n Y_i$, $\overline{XY} = \frac{1}{n}\sum_{i=1}^n X_iY_i$. Find the orders in probability (the sharpest results possible) of:

(a) $\bar{X}$,

(b) $\bar{Y}$,

(c) $(\bar{X})^2$,

(d) $(\bar{Y})^3$,

(e) $\overline{XY}$,

(f) $\bar{X} + 2$.

(a) Since $E[X_i] = 0 < \infty$, iid, by WLLN, $\bar{X} \xrightarrow{P} E[X_i] = 0$

$\Rightarrow \bar{X} = o_p(1)$

To find the sharpest stochastic order, want to find the rate of convergence

$$\bar{X} - 0 = \bar{X} = O_p\left(\sqrt{MSE(\bar{X})}\right)$$
$$= O_p\left(\sqrt{\underbrace{(\text{bias}(\bar{X}))^2}_{=0 \text{ b/c } \bar{X} \text{ is unbiased}} + \underbrace{\text{var}(\bar{X})}_{\sigma^2/n}}\right)$$
$$= O_p\left(\sqrt{\frac{\sigma^2}{n}}\right)$$

σ^2 is a finite constant

$$= O_p\left(\frac{\sigma}{\sqrt{n}}\right) = O_p\left(\frac{1}{\sqrt{n}}\right)$$

$\Rightarrow \bar{X} = O_p\left(\frac{1}{\sqrt{n}}\right)$

(b) $\bar{Y}$ $E[Y_i] = 3$, $var(Y_i) = 2$.

By WLLN, $\bar{Y} \xrightarrow{P} E[Y_i] = 3 \neq 0$

$\Rightarrow \bar{Y} = O_p(1)$

or equivalently , $\bar{Y} \xrightarrow{P} 3 \Rightarrow \bar{Y} = 3 + o_p(1)$

$$= \underbrace{O_p(1)}_{\text{bounded}} + \underbrace{o_p(1)}_{\to 0}$$

$$= O_p(1)$$

(c) $(\bar{X})^2$

From (a) , $\bar{X} = O_p\left(\frac{1}{\sqrt{n}}\right)$

$(\bar{X})^2 = \bar{X} \cdot \bar{X} = O_p\left(\frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{n}}\right) = O_p\left(\frac{1}{n}\right)$

↑ by O_p algebra

(d) $(\bar{Y})^3$

From (b) , $\bar{Y} = O_p(1)$

$(\bar{Y})^3 = \bar{Y} \cdot \bar{Y} \cdot \bar{Y} = O_p(1 \cdot 1 \cdot 1) = O_p(1)$

(e) $\overline{XY}$

Since X_i, Y_i mutually independent, $E[XY] = \overbrace{E[X]}^{0}\,\overbrace{E[Y]}^{3} = 0$

$$var(XY) = E[(XY)^2] - (\underbrace{E[XY]}_{0})^2$$

$$= E[X^2 Y^2]$$

$$\overset{\text{ind}}{=} E[X^2]E[Y^2]$$

$$= \left(var(X) + (E[X])^2\right)\left(var(Y) + (E[Y])^2\right) < \infty$$

By chebyshev inequality, $\rightarrow \overline{XY} - E[XY] = O_p\left(\sqrt{MSE(\overline{XY})}\right)$

$$\overline{XY} - \underbrace{E[XY]}_{0} = O_p\left(\frac{1}{\sqrt{n}}\right)$$

$$\Rightarrow \overline{XY} = O_p\left(\frac{1}{\sqrt{n}}\right)$$

(f) $\bar{X} + Z$

From (a), $\bar{X} = O_P(\frac{1}{\sqrt{n}})$

$\bar{X} + Z = O_P(\frac{1}{\sqrt{n}}) + O_P(1) = O_P(\max\{\frac{1}{\sqrt{n}}, 1\}) = O_P(1)$.