2021 Final Solutions for 2024 Students

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December 13, 2024

Note that this was a take-home exam (due to Covid).

Exercise 1.

(a) False. Let $f, g : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) := \mathbf{1}\{x \geqslant 0\} \cdot (-x)$$

and

$$g(x) := \mathbf{1}\{x \leqslant 0\} \cdot x$$

Then $(f + g)(x) = -|x|$, which is not quasi-convex.

(b) False. $\mathrm{e}^x \cdot \mathrm{e}^{-x} = \mathrm{e}^0 = 1$.

(c) True. See Section 4 Exercise 2.

(d) False. See 2023 Midterm 2 Q1(iii).

(e) False. f is continuous at 0. If $x_n \to 0$ then for each $n \in \mathbb{N}$, $f(x_n) = x_n$ or $f(x_n) = 0$. In both cases $f(x_n) \to 0$.

Exercise 2.

(a) F is clearly C^1. We want to show that $\partial F/\partial y \neq 0$ at $(0, 0.5, 0.5)$.

$$\frac{\partial F(0,0.5,0.5)}{\partial y} = 1 - x_1 x_2 \mathrm{e}^{x_1 x_2 y}\Big|_{(0,0.5,0.5)} = 1 \neq 0$$

We then have

$$\begin{aligned} Dh &= -\left[\frac{\partial F}{\partial y}\right]^{-1} D_x F \\ &= -\frac{1}{1 - x_1 x_2 \mathrm{e}^{x_1 x_2 y}} \begin{bmatrix} 1 - x_2 y \mathrm{e}^{x_1 x_2 y} & 1 - x_1 y \mathrm{e}^{x_1 x_2 y} \end{bmatrix} \end{aligned}$$

(b) Again, F is clearly C^1. We have

$$\begin{aligned} DF_y(2,-1,2,1) &= \begin{bmatrix} -3y_1^2 & 2y_2 \\ -4y_1 & 12y_2^3 \end{bmatrix}\Bigg|_{(2,-1,2,1)} \\ &= \begin{bmatrix} -12 & 2 \\ -8 & 12 \end{bmatrix} \end{aligned}$$

which has determinant $-144 + 16 = 128 \neq 0$.

$$\begin{aligned} Dh &= -[D_y F]^{-1} D_x F \\ &= -\begin{bmatrix} -3y_1^2 & 2y_2 \\ -4y_1 & 12y_2^3 \end{bmatrix}^{-1} \begin{bmatrix} 2x_1 & -2x_2 \\ 2x_2 & 2x_1 + 2x_2 \end{bmatrix} \\ &= -\frac{1}{-36y_1^2 y_2^3 + 8y_1 y_2} \begin{bmatrix} 12y_2^3 & -2y_2 \\ 4y_1 & -3y_1^2 \end{bmatrix} \begin{bmatrix} 2x_1 & -2x_2 \\ 2x_2 & 2x_1 + 2x_2 \end{bmatrix} \end{aligned}$$

Exercise 3. Compare Module 5 “Differentiation” Exercise 18.

Exercise 4.

(a) Continuous differentiability implies continuity, so π is continuous. Because the domain, $[0, 100]$ is compact, we know that π attains a maximum. $\pi'(q) = p(q) + qp'(q) - c'(q) + s$ and so $\pi''(q) = p'(q) + qp''(q) + p'(q) - c''(q)$. We know that $p'(q) < 0$, $q \geqslant 0$, $p''(q) \leqslant 0$, and $c''(q) \geqslant 0$. It follows that $\pi''(q) < 0$, so that profit is strictly concave in output. This implies that the maximum attained is unique. The FOC is

$$0 \equiv \pi'(q) = p(q) + qp'(q) - c'(q) + s$$

We can therefore write

$$F(q, s) := p(q) + qp'(q) - c'(q) + s$$

We know that $\pi''(q) < 0$, so the conditions of the implicit function theorem are met. We then have

$$\begin{aligned} \frac{dq}{ds} &= - \left[\frac{\partial F}{\partial q} \right]^{-1} \frac{\partial F}{\partial s} \\ &= - \left[\pi''(q) \right]^{-1} (1) \\ &> 0 \end{aligned}$$

(b) We still have a continuous objective function on a compact domain, so we know that a solution exists. We cannot say that it is unique. We have

$$\begin{aligned} \frac{\partial^2 \pi}{\partial s \partial q} &= \frac{\partial}{\partial s} \left[p(q) + qp'(q) - c'(q) + s \right] \\ &= 1 \\ &> 0 \end{aligned}$$

so that π has increasing differences in (q, s). Clearly, $\pi(\cdot, s)$ is supermodular in q (because $q \in \mathbb{R}$, which is totally ordered). By the Theorem of Milgrom and Shannon, the solution set $Q^*(s) := \arg\max_{q \in [0,100]} \pi(q, s)$ is monotone increasing in s, in the strong set order.

Exercise 5.

(a)

$$\begin{aligned}\frac{\partial^2 \pi}{\partial p \partial \alpha} &= \frac{\partial}{\partial p}\left[-\alpha p^{-\alpha}\log(p)\cdot(p-c)\right]\\ &= -\alpha\left[-\alpha p^{-\alpha-1}\log(p)\cdot(p-c) + p^{-\alpha}\cdot\frac{1}{p}\cdot(p-c) + p^{-\alpha}\log(p)\right]\\ &= -\alpha p^{-\alpha-1}\left[-\alpha\log(p)\cdot(p-c)+p-c+p\log(p)\right]\end{aligned}$$

The term outside the brackets is negative. For fixed $p<\infty$, as $\alpha\to 0$, the term inside the brackets approaches $p-c+p\log p$. If $p>\max\{c,1\}$, this expression is positive; if $p<\min\{c,1\}$, this expression is negative. Therefore, $\partial^2\pi/\partial p\partial\alpha$ takes both positive and negative values over $\mathbb{R}^2_{++}$.

(b) $(\log\circ\pi)(p,\alpha) = -\alpha\log(p)+\log(p-c)$. This has cross derivative $-1/p<0$.

(c) By the Theorem of Milgrom and Shannon, $-p^*(\alpha)$ is monotone increasing in α. It follows that $p^*(\alpha)$ is monotone decreasing. It follows, in turn, that $D(p^*,\alpha) := (p^*)^{-\alpha}$ is monotone increasing in α.

(d) It suffices to show that $\log D(p,\alpha)$ has increasing differences. We are given that elasticity,

$$\frac{\partial \log D(p,\alpha)}{\partial p}$$

is increasing in α. Then

$$\frac{\partial^2 \log D(p,\alpha)}{\partial\alpha\partial p} \geqslant 0$$

as required.

Exercise 6. *This exercise uses dynamic programming, which we did not cover in 2024.*

Exercise 7.

Note that what Suraj calls the "single-crossing property", we call "single-crossing differences".

(a) From the definitions.

(b) Module 7 "Comparative Statics" Exercise 7.

(c) Any function $f(x,t)$ that does not satisfy increasing differences but such that $g(t) := f(x',t) - f(x,t)$, does not cross or intersect with the t-axis for all $x' > x$ will work. For example, if $f : \mathbb{R}^2_{++} \to \mathbb{R}$ is defined by $f(x,t) = x/t$, then

$$\frac{\partial^2 f}{\partial x \partial t} = -\frac{1}{t^2} < 0$$

but if $x' > x$

$$f(x',t) - f(x,t) = \frac{x' - x}{t} > 0$$

Therefore, $f(x',t') > f(x,t')$ for all t' and all $x' > x$, so single-crossing differences is satisfied.