ECON 6170 Section 10

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Section Exercise 1 (2023 Midterm 2, Q3)**.**

(i) Suppose f and g are continuous on $[a, b] \subseteq \mathbb{R}$ and differentiable on (a, b), and that $g'(x) \neq 0$ for all $x \in (a, b)$ and $g(b) \neq g(a)$. Show that there exists a number $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Hint: Use Rolle's theorem. Rearranging the above equation with 0 on one side might give you an idea of the function to which you should apply Rolle's theorem.

Rearranging,

$$\begin{aligned} \frac{f(b) - f(a)}{g(b) - g(a)} &= \frac{f'(c)}{g'(c)} \\ \frac{f(b) - f(a)}{g(b) - g(a)} - \frac{f'(c)}{g'(c)} &= 0 \\ (f(b) - f(a))g'(c) - (g(b) - g(a))f'(c) &= 0 \end{aligned} \tag{1}$$

Write

$$F(x) := (f(b) - f(a))g(x) - (g(b) - g(a))f(x) = 0$$

so that

$$F(a) = f(b)g(a) - g(b)f(a) = F(b)$$

Then $F(a) - F(b) = 0$ so $F'(c) = 0$ for some $c \in (a, b)$. That is,

$$(f(b) - f(a))g'(c) - (g(b) - g(a))f'(c) = 0$$

as in (1)

(ii) How does this result relate to the mean value theorem?

The mean value theorem is the special case where $g(x) := x$.

(iii) Why can't we apply the mean value theorem on f and g separately to prove the result above?

Because the c obtained when applying the MVT to f need not be the same as that obtained when applying the MVT to g.

Section Exercise 2 (2023 Midterm 3 Q2)**.**

(i) Suppose $X \times Y \subseteq \mathbb{R} \times \mathbb{R}$ is open and consider $f : X \times Y \to \mathbb{R}$. Let $(x_0, y_0) \in X \times Y$ be such that $f(x_0, y_0) = k$ for some $k \in \mathbb{R}$. State the implicit function theorem for this case including all the necessary assumptions and the conclusions.

In addition to the assumptions given, we need that f is C^1 and that $\frac{\partial f(x_0,y_0)}{\partial y} \neq 0$. Then there exists open balls $B_{\varepsilon_x}(x_0)$ and $B_{\varepsilon_y}(y_0)$ such that for all $y \in B_{\varepsilon_y}(y_0)$ there exists a unique $x \in B_{\varepsilon_x}(x_0)$ satisfying $f(x, y) = k$. Therefore, the equation $f(x, y) = k$ implicitly defines a function $g : B_{\varepsilon_x}(x_0) \to B_{\varepsilon_y}(y_0)$ with the property

$$f(x, g(x)) = k$$

for all $x \in B_{\varepsilon_x}(x_0)$. Moreover, g is C^1 and

$$\frac{dg(x)}{dx} = -\left(\frac{\partial f(x, g(x))}{\partial y}\right)^{-1} \frac{\partial f(x, g(x))}{\partial x}$$

(ii) Give an example of a smooth function f and a point (x_0, y_0) where the key necessary condition of the implicit function theorem fails, but the conclusion that the set of (x, y) that solve $f(x, y) = k$ is locally the graph of a (not necessarily differentiable) function of x still holds.

Consider the function $f(x, y) := x - y^3$ and the point $(x_0, y_0) := (0, 0)$. f is C^1 but

$$\frac{\partial f(0,0)}{\partial y} = -3 \cdot 0^2 = 0$$

so the implicit function theorem cannot be applied. But $x - y^3 = 0$ implicitly defines the function $g(x) := x^{1/3}$ for $x \in \mathbb{R}$. Note that $g(x)$ is not differentiable at 0: $g'(x) = x^{-2/3}$, which evaluates to ∞ at $x = 0$.

Section Exercise 3 (Based on 2023 PS 10, Ex 6)**.**

(i) Define $f(x) := Ax$, $A \in \mathbb{R}^{m \times d}$. Show that $Df(x) = A$.

$$\begin{aligned}\frac{\|f(x+h) - f(x) - Ah\|}{\|h\|} &= \frac{\|A(x+h) - Ax - Ah\|}{\|h\|} \\ &= 0\end{aligned}$$

which trivially approaches 0 as $h \to 0$.

(ii) Now define $f(x) := x^\top Ax$ where $A \in \mathbb{R}^{d \times d}$ is symmetric. Show that $Df(x) = 2(Ax)^\top$. *Hint: Use the Cauchy-Schwartz inequality:* $|u^\top v| \leq \|u\| \cdot \|v\|$.

$$\begin{aligned}\frac{|f(x+h) - f(x) - 2(Ax)^\top h|}{\|h\|} &= \frac{|(x+h)^\top A(x+h) - x^\top Ax - 2x^\top Ah|}{\|h\|} \\ &= \frac{|x^\top Ax + x^\top Ah + h^\top Ax + h^\top Ah - x^\top Ax - 2x^\top Ah|}{\|h\|} \\ &= \frac{|x^\top Ah + h^\top Ax + h^\top Ah - 2x^\top Ah|}{\|h\|} \\ &= \frac{|h^\top Ah|}{\|h\|}\end{aligned}$$

where the last inequality follows because $x^\top Ah = h^\top Ax$, because A is symmetric. By the Cauchy-Schwartz inequality,

$$\begin{aligned}\frac{|h^\top Ah|}{\|h\|} &\leq \frac{\|h\| \cdot \|Ah\|}{\|h\|} \\ &= \|Ah\|\end{aligned}$$

which approaches $\|A0\| = \|0\| = 0$ as $h \to 0$.

(iii) Consider the following problem where the objective function is quadratic and the constraints are linear:

$$\max_{x \in \mathbb{R}^d} c^\top x + \frac{1}{2} x^\top Dx \text{ s.t. } Ax = b,$$

where $c \in \mathbb{R}^d$ is a (column) vector, $D \in \mathbb{R}^{d \times d}$ is symmetric, negative definite and $A \in \mathbb{R}^{m \times d}$ has full rank. Set up the Lagrangian to obtain the first-order conditions and solve for the optimal vector x^* as a function of A, b, c and D. *Hint: Because A has full rank, we know that $A^\top A$ is invertible.*

Not solved in class.