Econ 6190 Problem Set 1

Fall 2024

1. [Hong 6.1] Consider an independent and identically distributed random sample $\{X_1, X_2, X_3\}$, where X_i follows from binary distribution with $P\{X_i = 0\} = P\{X_i = 1\} = \frac{1}{2}$ for each $i = 1, 2, 3$. Define the sample mean $\bar{X} = \frac{1}{3}\sum_{i=1}^{3} X_i$. Find

   (a) Find the sampling distribution of $\bar{X}$;

   (b) Mean of $\bar{X}$;

   (c) Variance of $\bar{X}$.

2. [Hong 6.2] A community has five families whose annual incomes are 1, 2, 3, 4, and 5 respectively. Suppose a survey is to be made to two of the five families and the choice of the two is random. Find the sampling distribution of the sample mean of the family income. Give your reasoning clearly.

3. [Mid term, 2022] Let X be a random variable with density $f(x)$. Show that if the density satisfies $f(x) = f(-x)$ for all $x \in \mathbb{R}$, then:

   (a) The distribution function satisfies $F(-x) = 1 - F(x)$ for all $x \in \mathbb{R}$.

   (b) $\mathbb{E}[X] = 0$.

4. [Hansen 6.6] Show that $\mathbb{E}[s] \leq \sigma$. where $s = \sqrt{s^2}$ and s^2 is the sample variance.

5. [Hong 3.31] Show that if X is a continuous random variable, then

$$\min_a \mathbb{E}|X - a| = \mathbb{E}|X - m|,$$

   where m is the median of X.

6. [Mid-term, Fall 2021] Let X be a random variable with conditional density

$$f(x|\theta) = \begin{cases} \frac{1}{\theta} & \text{if } 0 < x < \theta \\ 0 & \text{otherwise} \end{cases}.$$

Usually we treat parameter θ as a constant. Now suppose $\theta > 0$ is treated as a random variable with density

$$g(\theta) = \begin{cases} \theta e^{-\theta} & \text{if } \theta > 0 \\ 0 & \text{if } \theta \leq 0 \end{cases},$$

where we use notation θ as both the random variable and the specific values it can take. Answer this following questions. (This question does not require any prior knowledge on Bayesian statistics, but is a test of your understanding of the key notions introduced in class.)

(a) Find $f(x)$, the marginal density of X.

(b) Find $g(\theta|x)$, the conditional density of θ given $X = x$.

(c) Find $\mathbb{E}[(\theta - a)^2|X = x]$ for some given constant a. (You are NOT required to work out the final integration.)

1. (a). Since $P(X_i = 0) = P(X_i = 1) = 1/2$

and X_1, X_2, X_3 are independent & identically distributed

then $\bar{X} = \frac{1}{3}(X_1 + X_2 + X_3)$ could be $0, 1/3, 2/3, 1$

$P(\bar{X} = 0) = (\frac{1}{2})^3 = 1/8$; $P(\bar{X} = 1/3) = C_3^1 (\frac{1}{2})^3 = 3/8$

$P(\bar{X} = 2/3) = C_3^2 (\frac{1}{2})^3 = 3/8$; $P(\bar{X} = 1) = (\frac{1}{2})^3 = 1/8$

sampling distribution PMF

$$P(\bar{X}) = \begin{cases} 1/8 & \text{if } \bar{x} = 0 \\ 3/8 & \text{if } \bar{x} = 1/3 \\ 3/8 & \text{if } \bar{x} = 2/3 \\ 1/8 & \text{if } \bar{x} = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(b). $E\bar{X} = 1/8 \cdot 0 + 3/8 \cdot 1/3 + 3/8 \cdot 2/3 + 1/8 \cdot 1 = 1/2$

(c). $Var\,\bar{X} = E(\bar{X} - E\bar{X})^2$

$$= \frac{1}{8}(0 - \frac{1}{2})^2 + \frac{3}{8}(\frac{1}{3} - \frac{1}{2})^2 + \frac{3}{8}(\frac{2}{3} - \frac{1}{2})^2 + \frac{1}{8}(1 - \frac{1}{2})^2$$

$$= 1/12$$

2. Income Bundle:

$\{(1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5), (4,5)\}$

Then we know

$P(\bar{X} = 3/2) = 1/10$; $P(\bar{X} = 2) = 1/10$

$P(\bar{X} = 5/2) = 1/5$; $P(\bar{X} = 3) = 1/5$

$P(\bar{X} = 7/2) = 1/5$; $P(\bar{X} = 4) = 1/10$

$P(\bar{X} = 9/2) = 1/10$

p. m. f. $$P(\bar{X}) = \begin{cases} 1/10 & \text{if } \bar{X} = 3/2, 2, 4 \text{ or } 9/2 \\ 1/5 & \text{if } \bar{X} = 5/2, 2, 7/2 \\ 0 & \text{otherwise.} \end{cases}$$

3. For each $x \in \mathbb{R}$,

$$F(-x) = P(X \le -x)$$
$$= 1 - P(X > -x)$$
$$= 1 - \int_{-x}^{\infty} f(t)dt \qquad (1)$$

By change of variable, note that $t = -u$

$$\int_{-x}^{\infty} f(t)\,dt = \int_{-x}^{\infty} f(-u)d(-u)$$
$$= -\int_{-x}^{\infty} f(-u)du$$
$$= \int_{-\infty}^{x} f(-u)du = \int_{-\infty}^{x} f(-t)dt$$

By $f(t) = f(-t)$, $\forall t \in \mathbb{R}$, $(1) = 1 - \int_{-\infty}^{x} f(t)dt = 1 - F(x)$.

(2) $$E[X] = \int x f(x)\,dx = \int_{-\infty}^{0} x f(x)dx + \int_{0}^{\infty} x f(x)dx$$
$$= \int_{-\infty}^{0} x f(-x)dx + \int_{0}^{\infty} x f(x)dx$$
$$= -\int_{-\infty}^{0} x f(-x)d(-x) + \int_{0}^{\infty} x f(x)dx$$
$$= \int_{0}^{\infty} x \underbrace{f(-x)}_{= f(x)} d(-x) + \int_{0}^{\infty} x f(x)dx$$

$$= -\int_0^\infty x f(x)\,dx + \int_0^\infty x f(x)\,dx = 0.$$

4. firstly we know $E\, s^2 = \sigma^2$

WTS. $E\, S \le \sigma$

since we know $\sigma = \underbrace{\sqrt{E\, S^2} \ge E\sqrt{S^2}} = E\, S$

then $E\, S \le \sigma$

we have this, simply because $\sqrt{t}$ is concave

5. $$E|X-a| = \int_{-\infty}^{a} (a-x)\cdot f(x)\,dx + \int_a^{+\infty} (x-a)\cdot f(x)\,dx$$

$$= a\int_{-\infty}^{a} f(x)\,dx - \int_{-\infty}^{a} x f(x)\,dx + \int_a^{+\infty} x f(x)\,dx$$

$$- a\int_a^{+\infty} f(x)\,dx$$

$$\frac{d E|X-a|}{da} = \int_{-\infty}^{a} f(x)\,dx - \int_a^{+\infty} f(x)\,dx$$

let $\frac{dE|X-a|}{da} = 0 \Rightarrow \int_{-\infty}^{a} f(x)dx = \int_a^\infty f(x)dx \Rightarrow a = m$ the median.

To the left of m, $\left.\frac{dE|X-a|}{da}\right|_{a=m-\varepsilon} = \underbrace{\int_{-\infty}^{m-\varepsilon} f(x)dx}_{<1/2} - \underbrace{\int_{m-\varepsilon}^{\infty} f(x)dx}_{>1/2} < 0$

To the right of m, $\left.\frac{dE|X-a|}{da}\right|_{a=m+\varepsilon} = \underbrace{\int_{-\infty}^{m+\varepsilon} f(x)dx}_{>1/2} - \underbrace{\int_{m+\varepsilon}^{\infty} f(x)dx}_{<1/2} > 0$

6.

(a) The joint density of θ and x is

$$h(\theta, x) = g(\theta)f(x|\theta) = \begin{cases} \theta e^{-\theta}\frac{1}{\theta} = e^{-\theta} & \text{if } 0 < x < \theta \\ 0 & \text{otherwise} \end{cases}$$

Hence the marginal density of X is

$$f(x) = \int_{-\infty}^{\infty} h(\theta, x)d\theta = \int_{x}^{\infty} e^{-\theta}d\theta = e^{-x}, x > 0$$

and $f(x) = 0$ for $x \leq 0$.
(b) for each $x > 0$,

$$g(\theta|x) = \frac{h(\theta, x)}{f(x)} = \begin{cases} \frac{e^{-\theta}}{e^{-x}} = e^{x-\theta} & \theta > x \\ \frac{0}{e^{-x}} = 0 & \theta \leq x \end{cases}.$$

Since $f(x) = 0$ for $x \leq 0$, $g(\theta|x)$ is not defined for $x \leq 0$.
(c) $\mathbb{E}[(\theta - a)^2|X = x] = e^x \int_x^{\infty}(\theta - a)^2 e^{-\theta}d\theta$

$$\begin{aligned} &\mathbb{E}[(\theta - a)^2|X = x] \\ =& \int (\theta - a)^2 g(\theta|x)d\theta \\ =& \int_x^{\infty} (\theta - a)^2 e^{x-\theta}d\theta \\ =& e^x \int_x^{\infty} (\theta - a)^2 e^{-\theta}d\theta \end{aligned}$$