ECON 6190

Problem Set 7

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November 11, 2024

1. Let $\{X_1, \ldots, X_n\}$ be a sequence of i.i.d. random variables with mean μ and variance σ^2. Let $\hat{\mu} = \frac{1}{n}\sum_{i=1}^n X_i$ and $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \hat{\mu})^2$.

   (a) Recall that $\mathrm{Var}(X_i) = \sigma^2 = \mathbb{E}[X_i^2] - (\mathbb{E}[X_i])^2$. Also recall that $\hat{\mu} \xrightarrow{p} \mu = \mathbb{E}[X_i]$, by the Weak Law of Large Numbers. Thus, we have that

   $$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \mu + \mu - \hat{\mu})^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \mu)^2 + 2(\mu - \hat{\mu})\frac{1}{n}\sum_{i=1}^n (X_i - \mu) + (\mu - \hat{\mu})^2$$

   Thus, since the second and third terms approach 0 probabilistically by the Weak Law of Large Numbers, we have that

   $$\hat{\sigma}^t = \frac{n-1}{n}s_n^2 \xrightarrow{p} \sigma^2$$

   since the first term is the plug-in estimator for variance, which is unbiased in large samples.

   (b) Note that

   $$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \hat{\mu})^2 = \frac{1}{n}\sum_{i=1}^n X_i^2 - \left(\frac{1}{n}\sum_{i=1}^n X_i\right)^2$$

   Define a function g such that $g(a,b) = b - a^2$. We have that $g(\hat{\mu}, \tilde{\mu}) = \hat{\sigma}^2$, where $\tilde{\mu} = \frac{1}{n}\sum_{i=1}^n X_i^2$. From part (a), we have that $\hat{\sigma}^2$ is consistent. Thus, we can use Delta method. Note that

   $$\sqrt{n}\begin{pmatrix} \hat{\mu} - \mathbb{E}[X_i] \\ \tilde{\mu} - \mathbb{E}[X_i^2] \end{pmatrix} = \sqrt{n}\begin{pmatrix} \frac{1}{n}\sum_{i=1}^n (X_i - \mathbb{E}[X_i]) \\ \frac{1}{n}\sum_{i=1}^n (X_i^2 - \mathbb{E}[X_i^2]) \end{pmatrix} = \sqrt{n}\frac{1}{n}\sum_{i=1}^n Y_i$$

   where $Y_i = \begin{pmatrix} X_i - \mathbb{E}[X_i] \\ X_i^2 - \mathbb{E}[X_i^2] \end{pmatrix}$. Assuming that $\mathbb{E}[X_i^2] < \infty$, applying the vector-valued central limit theorem gets us

   $$\sqrt{n}\frac{1}{n}\sum_{i=1}^n Y_i \xrightarrow{d} \mathcal{N}(0, V)$$

   where

   $$V = \mathrm{Var}(Y_i) = \mathbb{E}[Y_i Y_i'] = \mathbb{E}\left[\begin{pmatrix} X_i - \mathbb{E}[X_i] \\ X_i^2 - \mathbb{E}[X_i^2] \end{pmatrix}\begin{pmatrix} X_i - \mathbb{E}[X_i] \\ X_i^2 - \mathbb{E}[X_i^2] \end{pmatrix}'\right]$$

   Additionally, taking the first order Taylor expansion of g, we get that

   $$g(\hat{\mu}, \tilde{\mu}) = g(\mathbb{E}[X_i], \mathbb{E}[X_i^2]) + \begin{pmatrix} \frac{\partial g(a,b)}{\partial a} \mid_{a,b=\mu_a',\mu_b'} \\ \frac{\partial g(a,b)}{\partial b} \mid_{a,b=\mu_a',\mu_b'} \end{pmatrix}' \begin{pmatrix} \hat{\mu} - \mathbb{E}[X_i] \\ \tilde{\mu} - \mathbb{E}[X_i^2] \end{pmatrix}$$

   Thus, combining them, we have that

   $$\sqrt{n}(g(\hat{\mu}, \tilde{\mu}) - g(\mathbb{E}[X_i], \mathbb{E}[X_i^2])) \xrightarrow{d} \begin{pmatrix} \frac{\partial g(a,b)}{\partial a} \mid_{a,b=\mu_a',\mu_b'} \\ \frac{\partial g(a,b)}{\partial b} \mid_{a,b=\mu_a',\mu_b'} \end{pmatrix}' \mathcal{N}(0, V)$$

and we have that since $\hat{\mu} \xrightarrow{p} \mathbb{E}[X_i]$ and $\tilde{\mu} \xrightarrow{p} \mathbb{E}[X_i^2]$,

$$\begin{pmatrix} \frac{\partial g(a,b)}{\partial a} \mid_{a,b=\mu'_a,\mu'_b} \\ \frac{\partial g(a,b)}{\partial b} \mid_{a,b=\mu'_a,\mu'_b} \end{pmatrix}' \mathcal{N}(0,V) = \mathcal{N}\left(0, \begin{pmatrix} -2\hat{\mu} \\ 1 \end{pmatrix}' V \begin{pmatrix} -2\hat{\mu} \\ 1 \end{pmatrix}\right)$$

Recalling that

$$V = \mathbb{E}\left[\begin{pmatrix} X_i - \mathbb{E}[X_i] \\ X_i^2 - \mathbb{E}[X_i^2] \end{pmatrix} \begin{pmatrix} X_i - \mathbb{E}[X_i] \\ X_i^2 - \mathbb{E}[X_i^2] \end{pmatrix}'\right] = \begin{pmatrix} \mathrm{Var}(X_i) & \mathrm{cov}(X_i, X_i^2) \\ \mathrm{cov}(X_i, X_i^2) & \mathrm{Var}(X_i^2) \end{pmatrix}$$

we finally get that

$$\sqrt{n}(\hat{\sigma}^2 - \sigma) \xrightarrow{d} \mathcal{N}\left(0, \begin{pmatrix} -2\hat{\mu} \\ 1 \end{pmatrix}' \begin{pmatrix} \mathrm{Var}(X_i) & \mathrm{cov}(X_i, X_i^2) \\ \mathrm{cov}(X_i, X_i^2) & \mathrm{Var}(X_i^2) \end{pmatrix} \begin{pmatrix} -2\hat{\mu} \\ 1 \end{pmatrix}\right)$$

2. Let $X \sim U[0,b]$ and $M_n = \max_{i \le n} X_i$, where X_i is a random sample from X. Derive the asymptotic distribution using the following steps.

   (a) We have that

$$F(x) = \mathbb{P}\{X \le x\} = \begin{cases} 1 & x > b \\ \frac{x}{b} & 0 \le x \le b \\ 0 & x < 0 \end{cases}$$

   so

$$F(x) = \frac{\min\{x, b\}}{b} \cdot \mathbb{1}_{x \ge 0}$$

   Illustrated graphically, because I'm a visual person, it looks like:

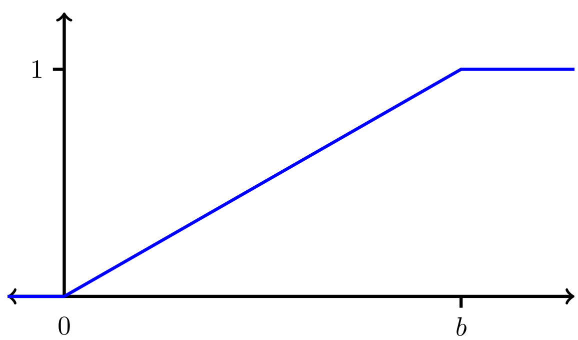

Figure 1: $F(x)$

   (b) We have that, by the fact $n \in \mathbb{N}$ and $b \in \mathbb{R}_+$ are constant,

$$Z_n = n\left(\max_{i \le n} X_i - b\right) = n \max_{i \le n}(X_i - b) = \max_{i \le n} n(X_i - b)$$

   (c) We have that

$$G_n(x) = \mathbb{P}\{Z_n \le x\} = \mathbb{P}\left\{\max_{i \le n} n(X_i - b) \le x\right\}$$

   which becomes

$$\mathbb{P}\left\{n \max_{i \le n} X_i - nb \le x\right\} = \mathbb{P}\left\{\max_{i \le n} X_n \le \left(b + \frac{x}{n}\right)\right\} = \left(F\left(b + \frac{x}{n}\right)\right)^n$$

(d) We have that

$$G_n(x) = \left(F\left(b + \frac{x}{n}\right)\right)^n = \left(\frac{b + \frac{x}{n}}{b}\right)^n = \left(1 + \frac{\frac{x}{b}}{n}\right)^n$$

So,

$$\lim_{n\to\infty} G_n(x) = \lim_{n\to\infty} \left(1 + \frac{\frac{x}{b}}{n}\right)^n = e^{\frac{x}{b}}$$

(e) We have that, since $b + \frac{x}{n} \geq b$, that $F\left(b + \frac{x}{n}\right) = 1$, and so $G_n(x) = \left(F\left(b + \frac{x}{n}\right)\right)^n = 1$.

(f) We have that

$$Z_n \xrightarrow{d} G(x) = \begin{cases} e^{\frac{x}{b}} & x \leq 0 \\ 1 & x > 0 \end{cases}$$

or

$$G(x) = \exp\left(\min\left\{\frac{x}{b}, 0\right\}\right)$$