Solutions to Ungraded Problems

1. An individual with initial wealth $w_0 > 0$ has EU preferences over gambles with payoff function $u(x) = \sqrt{x}$. She has a lottery ticket that pays off \$0 with probability $1/2$ and \$12 with probability $1/2$. For how much would she be willing to sell the ticket?

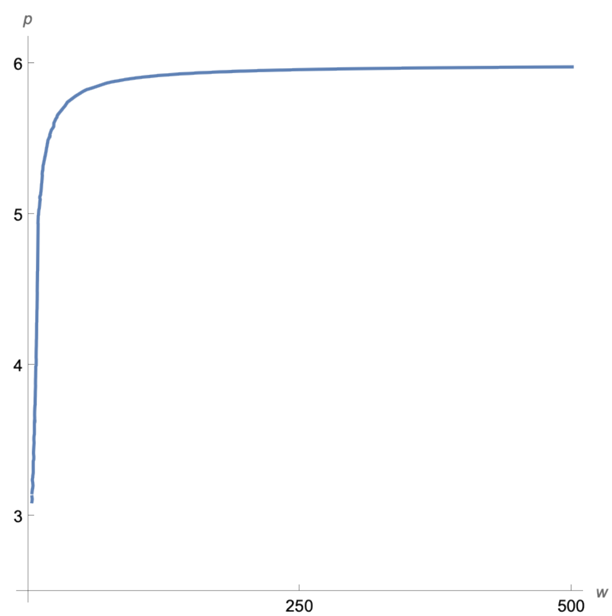
   Solution: The expected utility of the gamble if the individual pays p is

   $$\frac{1}{2}\sqrt{w_0 + 12 - p} + \frac{1}{2}\sqrt{w_0 - p}$$

   and the utility of not accepting the gamble is $\sqrt{w_0}$.

   Obviously you are not going to solve this analytically. And this is the point of the problem. What do you do when you need an answer and your model isn't going to conveniently give you one.

   A good trick is to approximate the utility function around w_0 to get a rough answer (which is quite good when w_0 is large). $\sqrt{x+h} \approx (1/2\sqrt{x})h$. Replacing the radicals with their first-order approximation around w_0 gives the *ex post* obvious answer that $p = 6$.



2. Suppose your friend is a big NBA fan, and a game between the Celtics and the Knicks is coming up. It's another Boston–New York thing, and the friend has intense feelings about the outcome. Boston can either Win, Tie, or Lose. Possible distributions over outcomes are $p = (p_w, p_t, p_l)$ where the three coefficients are non-negative and sum to 1. Your friend's preferences are that $p \succ q$ if either $p_w > q_w$ or $p_w = q_w$ and $p_t > q_t$. Obviously, you think, your friends preferences do not have an EU representation because they do not have any utility representation (why?). Nonetheless, it's interesting to ask (for some of us, at least) which of the vNM axioms are violated by this order?

   Solution: Preferences are clearly transitive and complete. What about independence? Suppose $p \succeq q$ and $0 < \alpha \leq 1$. Choose r and consider two cases:

   i) Suppose that $p_w > q_w$. Then $\alpha p_w + (1-\alpha)r_w > \alpha p_w + (1-\alpha)r_w$ so $\alpha p + (1 - \alpha r) \succ \alpha q + (1 - \alpha r)$. Clearly the converse holds as well.

ii) Suppose $p_w = q_w$ and $p_t > q_t$. Then $\alpha p_w + (1-\alpha)r_w = \alpha p_w + (1-\alpha)r_w$ and $\alpha p_t + (1-\alpha)r_t > \alpha p_t + (1-\alpha)r_t$. Again, $\alpha p + (1-\alpha r) \succ \alpha q + (1-\alpha r)$. Clearly the converse holds as well.

3. Show that the three vNM axioms for preferences with a finite set of outcomes are independent, in the sense that no two implies a third.

Solution: The solution requires constructing examples; of an order that satisfies each pair of axioms but not the third.

(a) To show that completeness is independent, consider the partial order described by two continuous utility functions, $u(x)$ and $v(x)$, which are not positive affine transforms of each other, such that $p \succeq q$ iff $E_p u \geq E_q u$ and $E_p v > E_q v$. This partial order satisfies independence and the Archimedean axiom.

(b) About transitivity, I lied. After thinking about it, I realized that independence implies transitivity. To see why, suppose $\succeq$ is an intransitive order which satisfies independence. Recall that the independence axiom forces indifference surfaces to be hyperplanes (intersected with the set of prob dists). So take some set of indifference hyperplanes which are ordered cyclicly. Draw a line transverse to them all, and choose some finite set of points $p_1, \ldots, p_n$ on the line such that $p_1 \succ p_2 \succ \cdots \succ p_n \succ p_1$. Since $p_n \succ p_1$, $\alpha p_n + (1-\alpha)p_n \succ p_1 + (1-\alpha)p_n$ by independence. Choose α so that $\alpha p_n + (1-\alpha)p_1 = p_{n-1}$. Then $p_n \succ p_{n-1}$, but by construction $p_{n-1} \succ p_n$. A contradiction.

(c) Here is an order which satisfies independence, completeness, and transitivity, but not Archimedes. Again consider two continuous utility functions u and v, and define $p \succ q$ is $E_p u > E_q u$ or $E_p u = E_q u$ and $E_p v > E_q v$.

(d) Obviously one can describe a utility function on the set of probability distributions which is continuous and not linear in probabilities.

4. Suppose that X is a subset of the set of real numbers. A probability distribution μ *first-order stochastic dominates (FOSD)* a probability distribution ν If their CDFs are ordered $F_\mu(x) \leq F_\nu(x)$ for all x, with strict inequality for some real x. Intuitively, ν puts more weight on all small numbers than does μ. A payoff function $u : X \to \mathbf{R}$ *respects first-order stochastic dominance* if for all μ and ν such that μ FOSD ν, $E_\mu u > E_\nu u$; that is, μ is better than ν. Show that expected utility preferences with payoff function u respect first-order stochastic dominance if and only if u is increasing almost surely with respect to μ and ν.

Solution: Actually, once again ...FOSD implies u is increasing, but the converse is not.

Suppose $x_1 < x_2 \in X$, and $u(x_1) > u(x_2)$. Consider two lotteries $p = (p_1, p_2)$ and $q = (q_1, q_2)$ that put all their mass on x_1 and x_2. If $p_1 > q_1$ then q stochastically dominates p. But clearly $E_p u > E_q u$, so if $\succeq$ respects FOSD, then u cannot represent it.

An equivalent definition of FOSD is that μ stochastically dominates ν iff $E_\mu f > E_\nu f$ for every strictly increasing $f : X \to \mathbf{R}$. Another kind of stochastic dominance is *second-order stochastic dominance*: μ second-order dominates ν if $E_\mu f > E_\nu f$ for every strictly increasing *and concave* $f : X \to \mathbf{R}$.

5. A problem only for those who have (or want to have) theory chops. MWG prove the EU representation theorem for finite sets. In this case we can think of combining lotteries by taking convex combinations. Lotteries that have certain outcomes are called *simple lotteries*. The lottery $\alpha p + (1 - \alpha)q$ can be thought of as a lottery whose outcomes are simple lotteries: p with probability α and q with probability $(1 - \alpha)$. Such lotteries are called *compound lotteries*. We implicitly assumed that compound lotteries can be reduced to simple lotteries by multiplying through by the scalars and adding. So it suffices to think of mixing lotteries as taking convex combinations. This doesn't really work well for more complex settings. Instead we talk about *mixture spaces*.

   A *mixture space* is a set P and a family of operations $\circ_a : P \times P \to P$ for $0 \le a \le 1$ such that

   a. $p \otimes_1 q = p$,
   b. $p \otimes_a q = q \otimes_{1-a} p$,
   c. $(p \otimes_b q) \otimes_a q = p \otimes_{ab} q$.

   Show that these axioms imply that

   d. $p \otimes_a p = p$,
   e. $(p \otimes_b q) \otimes_a (p \otimes_c q) = p \otimes_{ab+(1-a)c} q$.

   Solution: For part d),

$$
\begin{aligned}
p \otimes_a p &= (p \otimes_1 q) \otimes_a p && \text{by a} \\
&= (q \otimes_0 p) \otimes_a p && \text{by b} \\
&= q \otimes_{0a} p = q \otimes_0 p && \text{by c} \\
&= p \otimes_1 q && \text{by b} \\
&= p && \text{by a}
\end{aligned}
$$

For part e)

$$
\begin{aligned}
(p \otimes_b q) \otimes_a (p \otimes_c q) &= \big((p \otimes_c q) \otimes_{b/c} q\big) \otimes_a (p \otimes_c q) && \text{by c} \\
&= \big(q \otimes_{1-b/c} (p \otimes_c q)\big) \otimes_a (p \otimes_c q) && \text{by b} \\
&= q \otimes_{a-ab/c} (p \otimes_c q) && \text{by c} \\
&= (p \otimes_c q) \otimes_{1-a+ab/c} q && \text{by b} \\
&= p \otimes_{ab+(1-a)c} q && \text{by c}
\end{aligned}
$$

It seems that mixtures are just a tedious way of rewriting convex sets. But this is not true. The deeply interested among you might see if you can construct an example of a non-convex mixture space.