ECON 6190

Problem Set 6

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1. We have that $\mathbb{E}[Z] = 0$ and $\text{Var}(Z) = 1$. Using Chebyshev's Inequality, we have that

$$\mathbb{P}\{|Z| > \delta\} \leq \frac{\text{Var}(Z)}{\delta^2}$$

so when $\delta = \sqrt{20} \approx 4.47$, $\mathbb{P}\{|Z| > \delta\} \leq 0.05$. In constrast, when $Z \sim \mathcal{N}(0,1)$, we have that $\mathbb{P}\{|Z| > \delta\} = 0.05$ when $\delta = 1.96$. This number is lower because we have a bound on the tail probabilities in a normal distribution – we know that they decay exponentially. We don't know that with an arbitrary distribution.

2. We have $X \sim \mathcal{N}(\mu, \sigma^2)$, draw a random sample and construct a sample mean statistic $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$.

   (a) From Markov's Inequality, we have that $\mathbb{P}\{|Z| > \delta\} \leq \frac{\mathbb{E}[|Z|^r]}{\delta^r}$. From the properties of normal distributions, we have that $\bar{X}_n \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$, meaning that $\bar{X}_n - \mu \sim \mathcal{N}\left(0, \frac{\sigma^2}{n}\right)$. Thus, taking $r = 2$, we get that

$$\mathbb{P}\left\{|\bar{X}_n - \mu| > \delta\right\} \leq \frac{\mathbb{E}[|\bar{X}_n - \mu|^2]}{\delta^2} = \frac{\sigma^2}{n\delta^2}$$

   (b) Recall that the fourth moment of a normal distribution, the kurtosis, is $3\sigma^4$. Thus, taking $r = 4$, we get that

$$\mathbb{P}\left\{|\bar{X}_n - \mu| > \delta\right\} \leq \frac{\mathbb{E}[|\bar{X}_n - \mu|^4]}{\delta^4} = \frac{3\sigma^4}{n^2\delta^4}$$

   (c) Assuming that $\delta = \sigma$ and $n > 2$, we get that

$$\frac{\sigma^2}{n\delta^2} = \frac{1}{n} \qquad \text{and} \qquad \frac{3\sigma^4}{n^2\delta^4} = \frac{3}{n^2}$$

   Where we have that $\frac{1}{n} \geq \frac{3}{n^2}$ for all $n \geq 3$. Thus, Markov's Inequality with $r = 4$ provides a tighter bound.

   (d) Again, we have that $\bar{X}_n - \mu \sim \mathcal{N}\left(0, \frac{\sigma^2}{n}\right)$. This means that

$$\mathbb{P}\left\{|\bar{X}_n - \mu| > \delta\right\} = \mathbb{P}\left\{\bar{X}_n - \mu > \delta\right\} + \mathbb{P}\left\{\bar{X}_n - \mu < -\delta\right\} = 2\mathbb{P}\left\{\bar{X}_n - \mu > \delta\right\}$$

   So we have that

$$\mathbb{P}\left\{|\bar{X}_n - \mu| > \delta\right\} = 2\left(1 - \Phi\left(\frac{\sqrt{n}\delta}{\sigma}\right)\right) = 2\Phi\left(-\frac{\sqrt{n}\delta}{\sigma}\right)$$

   (e) We have that for $Z \sim \mathcal{N}(\mu, \sigma^2)$,

$$\mathbb{P}\{|Z - \mu| > \delta\} \leq 2\exp\left(-\frac{\delta^2}{2\sigma^2}\right)$$

   and recalling that

$$\mathbb{P}\{|\bar{X} - \mu| \leq c\} > 0.95 \Longleftrightarrow \mathbb{P}\{|\bar{X} - \mu| > c\} \leq 0.05$$

we have that

$$\mathbb{P}\{|\bar{X} - \mu| > c\} \leq 2\exp\left(-\frac{nc^2}{2\sigma^2}\right) \leq 0.05$$

So for any c_1 where $2\exp\left(-\frac{nc^2}{2\sigma^2}\right) \leq 0.05$, $\mathbb{P}\{|\bar{X} - \mu| \leq c\} > 0.95$. So

$$c = \sqrt{-\frac{2\sigma^2}{n}\log\left(\frac{1}{40}\right)} = \frac{\sigma}{\sqrt{n}}\sqrt{2\log 40}$$

Using Chebyshev's Inequality, we get that

$$\mathbb{P}\{|\bar{X} - \mu| > c\} \leq \frac{\mathrm{Var}(\bar{X})}{c^2} = \frac{\sigma^2}{nc^2} < 0.05$$

so we get that $c_2 = \sqrt{\frac{20\sigma^2}{n}} = \frac{\sigma}{\sqrt{n}}\sqrt{20}$.

(f) These are equal when

$$\begin{aligned} \frac{\sigma}{\sqrt{n_1}}\sqrt{2\log 40} &= \frac{\sigma}{\sqrt{n_2}}\sqrt{20} \\ \frac{2\sigma^2}{n_1}\log 40 &= \frac{20\sigma^2}{n_2} \\ n_2 &= \frac{10}{\log 40}n_1 \end{aligned}$$

So $n_2 \approx 2.7n_1$. We need to collect approximately 1.71 times more data.

3. Consider a sample X_i, where $X_i = \mu + \sigma_i e_i$ for some constants $\{\sigma_i\}$ and μ, and e_i i.i.d. with mean 0 and variance 1.

(a) We have that $\hat{\mu}_1$ is consistent if $\hat{\mu}_1 \xrightarrow{p} \mu$. This is true if, for all $\delta > 0$,

$$\mathbb{P}\{|\hat{\mu}_1 - \mu| > \delta\} \to 0 \text{ as } n \to \infty$$

This becomes

$$\mathbb{P}\{|\hat{\mu}_1 - \mu| > \delta\} = \mathbb{P}\left\{\left|\frac{1}{n}\sum_{i=1}^{n}\sigma_i e_i\right| > \delta\right\} \leq \mathbb{P}\left\{|\max_i \sigma_i e_i| > \delta\right\} = \mathbb{P}\left\{|\max_i e_i| > \frac{\delta}{\max_i \sigma_i}\right\} \to 0$$

Note that as $n \to \infty$, $\max_i e_i \xrightarrow{p} \mu_e = 0$ by the weak law of large numbers. Thus, this holds as long as $\max_i \sigma_i < \infty$ as $n \to \infty$ and $\max_i \sigma_i > 0$ for all n.

First, note that $\hat{\mu}_1$ is unbiased, since $\mathbb{E}[\hat{\mu}_1] = \frac{1}{n}\left(n\mu + \sum_{i=1}^{n}\sigma_i\,\mathbb{E}[e_i]\right) = \mu$.

We have that $\hat{\mu}_1 - \mu = O_p\left(\sqrt{\mathrm{MSE}(\hat{\mu}_1)}\right)$, and we have that since $\hat{\mu}_1$ is unbiased

$$\mathrm{MSE}(\hat{\mu}_1) = \mathrm{Var}(\hat{\mu}_1) = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}(X_i) = \frac{1}{n}\sum_{i=1}^{n}\sigma_i^2$$

Thus, for $\hat{\mu}_1 - \mu$ to equal $O_p\left(\frac{1}{\sqrt{n}}\right)$, it must be the case that $\sum_{i=1}^{n}\sigma_i^2 = 1$.

(b) Note first that this is a continuous function of $\hat{\mu}_1$, because considering $\sigma^2 \in \mathbb{R}^n$ (*i.e.*, considering σ^2 to be a vector), we have that $\hat{\mu}_2 = \sigma^2 \cdot \hat{\mu}_1 / \|\sigma^2\|$, so $\hat{\mu}_2 \xrightarrow{p} \sigma^2 \cdot \mu / \|\sigma^2\| = \mu$. Thus, $\hat{\mu}_2$ is consistent.

Note also that $\hat{\mu}_2$ is unbiased, since $\mathbb{E}[\hat{\mu}_2] = \mu$. To see:

$$\mathbb{E}\left[\frac{\sum_{i=1}^n \frac{X_i}{\sigma_i^2}}{\sum_{i=1}^n \frac{1}{\sigma_i^2}}\right] = \frac{\sum_{i=1}^n \frac{\mathbb{E}[X_i]}{\sigma_i^2}}{\sum_{i=1}^n \frac{1}{\sigma_i^2}} = \frac{\sum_{i=1}^n \frac{\mu}{\sigma_i^2}}{\sum_{i=1}^n \frac{1}{\sigma_i^2}} = \mu \frac{\sum_{i=1}^n \frac{1}{\sigma_i^2}}{\sum_{i=1}^n \frac{1}{\sigma_i^2}} = \mu$$

Then we have that $\hat{\mu}_2 - \mu = O_p\left(\sqrt{\text{MSE}(\hat{\mu}_2)}\right)$. Since $\hat{\mu}_2$ is unbiased, we have that

$$\text{MSE}(\hat{\mu}_2) = \text{Var}(\hat{\mu}_2)) = \frac{\sum_{i=1}^n \frac{\text{Var}(X_i)}{\sigma_i^4}}{\left(\sum_{i=1}^n \frac{1}{\sigma_i^2}\right)^2} = \frac{\sum_{i=1}^n \frac{1}{\sigma_i^2}}{\left(\sum_{i=1}^n \frac{1}{\sigma_i^2}\right)^2} = \frac{1}{\sum_{i=1}^n \frac{1}{\sigma_i^2}}$$

Thus, for it to be true that $\hat{\mu}_2 - \mu = O_p\left(\frac{1}{\sqrt{n}}\right)$, we need that $\sum_{i=1}^n \frac{1}{\sigma_i^2} = n$.

(c) We have that

$$\text{MSE}(\hat{\mu}_1) = \frac{1}{n}\sum_{i=1}^n \sigma_i^2 \quad \text{and} \quad \text{MSE}(\hat{\mu}_2) = \frac{1}{\sum_{i=1}^n \frac{1}{\sigma_i^2}}$$

Thus,

$$\frac{\text{MSE}(\hat{\mu}_1)}{\text{MSE}(\hat{\mu}_2)} = \frac{\frac{1}{n}\sum_{i=1}^n \sigma_i^2}{\frac{1}{\sum_{i=1}^n \frac{1}{\sigma_i^2}}} = \frac{1}{n}\sum_{i=1}^n \sigma_i^2 \frac{1}{\sum_{i=1}^n \frac{1}{\sigma_i^2}} \geq \frac{1}{n}\sum_{i=1}^n \sigma_i^2 \frac{1}{\sigma^2} = 1$$

Thus, $\text{MSE}(\hat{\mu}_1) \geq \text{MSE}(\hat{\mu}_2)$, and $\hat{\mu}_2$ is more efficient.

4. We have that, from the definition of derivatives,

$$\frac{f(Y_n) - f(0)}{Y_n} \xrightarrow{p} f'(0)$$

Thus, we have that

$$X_n(f(Y_n) - f(0)) = X_n Y_n \frac{f(Y_n) - f(0)}{Y_n} \xrightarrow{p} X_n Y_n f'(0)$$

and since $f'(0)$ is a constant, we have that

$$X_n(f(Y_n) - f(0)) \xrightarrow{d} X_n Y_n f'(0) \xrightarrow{d} f'(0)Y$$

Thus, $X_n(f(Y_n) - f(0)) \xrightarrow{d} f'(0)Y$.

5. Assume that X_i are iid with mean μ and variance σ^2. Let $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$

(a) Note that the continuous mapping theorem is not directly applicable here, as $(\bar{X})^2$ is a function of $\bar{X}$, not $\sqrt{n}(\bar{X} - \mu)$. Instead, we will use the delta method. Define $h(x) = x^2$. Since we have that $\sqrt{n}(\bar{X} - \mu) \xrightarrow{d} \mathcal{N}(0, 1)$ from the Lindéberg-Levy Central Limit Theorem, we have from the Delta Theorem that since $h(\cdot)$ is continuously differentiable in a neighborhood around μ, that

$$\sqrt{n}((\bar{X})^2 - \mu^2) \xrightarrow{d} \mathcal{N}(0, (2\mu\sigma)^2)$$

(b) If we were to use the Delta method, we would get that this converges to $\mathcal{N}(0, 0)$, which is a degenerate distribution (and gives us no information about the asymptotic distribution). We know by the Central Limit Theorem that

$$\sqrt{n}(\bar{X} - \mu) \xrightarrow{d} \mathcal{N}(0, \sigma^2) \implies \frac{\sqrt{n}\bar{X}}{\sigma} \xrightarrow{d} \mathcal{N}(0, 1)$$

Thus, from the Continuous Mapping Theorem, we have that

$$h\left(\frac{\sqrt{n}\bar{X}}{\sigma}\right) \xrightarrow{d} \chi^2_n$$