Econ 6190 Practice Final Exam

Instructions

This 2.5 hour practice exam consists of four questions, not of equal length or difficulty. Answer all questions. Remember to always explain your answer.

1. **[30 pts]** Suppose that random variables $Y_1, \dots Y_n$ satisfy

$$Y_i = \beta X_i + \varepsilon_i,$$

for $i = 1 \dots n$ where $X_1 \dots X_n$ are fixed constants and $\{\varepsilon_i\}_{i=1}^n$ follows iid $\mathrm{N}(0, \sigma^2)$, where β is **unknown** but σ^2 is **known**.

   (a) Find the MLE for β and check if it is unbiased. Give your reasoning.

   (b) Find the variance of $\hat{\beta}$.

   (c) Find the finite sample distribution of $\hat{\beta}$.

   (d) Consider another estimator $\tilde{\beta} = \frac{\sum_{i=1}^n Y_i}{\sum_{i=1}^n X_i}$. Is it unbiased for β? Give your reasoning.

   (e) Is $\tilde{\beta}$ more efficient than your MLE in terms of the MSE criterion? Give your reasoning.

   (f) Find the likelihood ratio statistic LR_n for testing $\mathbb{H}_0 : \beta = b$ vs $\mathbb{H}_0 : \beta \neq b$ for some b. Show $LR_n = T^2$, where $T = \frac{\hat{\beta} - b}{\sqrt{\mathrm{var}(\hat{\beta})}}$.

2. **[25 pts]** Suppose there are k independent random samples from a normal $\mathrm{N}(\mu, \sigma^2)$ distribution, but each sample has a different sample size. Let S_i^2 be the sample variance of the i-th sample variance of the i-th random sample whose sample size is n_i.

   (a) Assuming μ is known, find the Cramer Rao Lower Bound for estimating σ^2.

   (b) Define a class of estimators for σ^2 as follows: $\bar{S}^2 = \sum_{i=1}^k c_i S_i^2$. Find the best unbiased estimator of σ^2 from this class. Denote this best unbiased estimator as $\bar{S}^{*2}$. Give your reasoning.

   (c) Define a pooled sample variance S^2 by pooling all k random samples together. That is, S^2 is the sample variance from the pooled random sample that combines all k independent random samples (so that the total sample size $n = \sum_{i=1}^k n_i$.) Which estimator, $\bar{S}^{*2}$ or S^2, is better? Give your reasoning.

(d) Derive an **exact** 95% confidence interval for σ^2 based on S^2.

3. **[25 pts]** Consider a random sample $\{X_i\}_{i=1}^n$ from random variable $X \in \mathbb{R}$. Our object of interest is the probability $\theta_0 = P\{X \le c\}$, for a given constant c.

   (a) Define the method of moments estimator (say, $\hat{\theta}$) for θ_0 and derive its asymptotic distribution. Carefully state the assumptions you need to derive the asymptotic distribution.

   (b) Suppose we know that X follows a parametric distribution with pdf $f(x, \gamma_0)$ and cdf $F(x, \gamma_0)$, where γ_0 is an unknown scalar parameter, and the functional form of f and F are known. Propose an estimator of θ_0 (say, $\tilde{\theta}$) by using an estimator of γ_0. Then derive the asymptotic distribution of $\tilde{\theta}$. Carefully state the assumptions you need to derive the results.

   (c) Suppose we want to test hypothesis $\mathbb{H}_0 : \theta_0 = c$ for some $0 < c < 1$. Propose a test with asymptotic size α based on the estimator in (b). Impose suitable assumptions as you need.

4. **[20 pts]** Answer the following questions.

   (a) Let the true parameter of interest be $\theta \in \mathbb{R}$ and $\hat{\theta}_n$ be an estimator of θ. The distribution of $\hat{\theta}_n$ is as follows:
   $$P\{\hat{\theta}_n = \theta\} = 1 - \frac{1}{n}, \quad P\{\hat{\theta}_n = n^{\alpha}\} = \frac{1}{n}$$
   for some $\alpha > 0$.

      i. Show $\hat{\theta}_n$ is consistent.

      ii. Derive the bias of $\hat{\theta}_n$. Is there any situation when $\hat{\theta}_n$ is asymptotically unbiased? Is there any situation when the bias of $\hat{\theta}_n$ converges to ∞?

   (b) Let $\{X_1 \dots X_n\}$ be a sequence of iid random variables with mean μ and variance $\sigma^2 > 0$. Let $\hat{\mu} = \frac{1}{n}\sum_{i=1}^n X_i$, and $s^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \hat{\mu})^2$. Impose suitable assumptions to find the stochastic orders of $s^2 - \sigma^2$.