Econ 6190 Problem Set 3

Fall 2024

1. [Hong 6.8] Establish the following recursion relations for sample means and sample variances. Let $\bar{X}_n$ and s_n^2 be the sample mean and sample variances based on random sample $\{X_1, X_2 \ldots X_n\}$. Then suppose another observation, X_{n+1}, becomes available. Show:

   (a) $\bar{X}_{n+1} = \frac{X_{n+1} + n\bar{X}_n}{n+1}$.

   (b) $ns_{n+1}^2 = (n-1)s_n^2 + \frac{n}{n+1}(X_{n+1} - \bar{X}_n)^2$.

2. [Hong 6.6] Suppose $\mathbf{X}^n = (X_1, \ldots, X_n)$ is an iid $\mathrm{N}(\mu, \sigma^2)$ random sample, $\mathbf{Y}^n = (Y_1, \ldots, Y_n)$ is an iid $\mathrm{N}(\mu, \sigma^2)$ random sample, and the two random samples are mutually independent. Let $\bar{X}_n$ and $\bar{Y}_n$ be the sample means of the first and second random samples, respectively, and let s_X^2 and s_Y^2 be the sample variances of the first and second random samples respectively. Find:

   (a) the distribution of $(\bar{X}_n - \bar{Y}_n)/\sqrt{2\sigma^2/n}$;

   (b) the distribution of $(\bar{X}_n - \bar{Y}_n)/\sqrt{2s_X^2/n}$;

   (c) the distribution of $(\bar{X}_n - \bar{Y}_n)/\sqrt{2s_Y^2/n}$;

   (d) the distribution of $(\bar{X}_n - \bar{Y}_n)/\sqrt{(s_X^2 + s_Y^2)/n}$;

   (e) the distribution of $(\bar{X}_n - \bar{Y}_n)/\sqrt{s_n^2/n}$, where s_n^2 is the sample variance of the difference sample $\mathbf{Z}^n = (Z_1, Z_2 \ldots Z_n)$, where $Z_i = X_i - Y_i$, $i = 1, 2 \ldots n$.

3. [Hong 6.9] Let $X_i, i = 1, 2, 3$ be independent with $\mathrm{N}(i, i^2)$ distributions. For each of the following situations, use X_1, X_2, X_3 to construct a statistic with the indicated distribution:

   (a) Chi-square distribution of 3 degrees of freedom;

   (b) t distribution with 2 degrees of freedom;

4. [Final exam, 2022] Let $\{X_1, \ldots X_n\}$ be i.i.d with pdf $f(x \mid \theta) = e^{-(x-\theta)}\mathbf{1}\{x \geq \theta\}$. Show $Y = \min\{X_1, \ldots X_n\}$ is a sufficient statistic for θ **without** using the Factorization Theorem.

5. Let $\{X_1, \ldots, X_n\}$ be a random sample with the pdf for each X_i

$$f(x|\theta) = \begin{cases} e^{i\theta - x}, & x \geq i\theta \\ 0 & x < i\theta \end{cases}.$$

Show $\min_i \left(\frac{X_i}{i}\right)$ is a sufficient statistic for θ.

6. Show that the following claim is true: any one-to-one function of a sufficient statistic is a also sufficient statistic.

7. Let X be one observation from $\mathrm{N}(0, \sigma^2)$. Is $|X|$ a sufficient statistic for σ^2? Give your reasoning clearly.