ECON 6090-Microeconomic Theory. TA Section 6

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In Section notes

Cost Minimization Problem (CMP)

$$\text{cost function } c(w,q) = \min_{z} w \cdot z \text{ subject to } f(z) = q$$

Conditional input demand: $z(w,q)$

Properties of the cost function:

1. c(w,q) is HoD1 in w.
2. c(w,q) is nondecreasing in q if we assume free disposal.
3. c(w,q) is concave in w.
4. If f(z) is HoD k, then c(w,q) is HoD $\frac{1}{k}$ in q.

Profit Maximization Problem (PMP)

$$\pi(p,w) = \max_{x} pf(x) - wx$$

Where $x(p,w)$ is the input demand and $y(p,w)$ is the output supply.

Properties:

1. Nondecreasing in p and nonincreasing in w_i for all i.
2. HoD 1 in (p,w).
3. Convex in (p,w).
4. Continuous on $\mathbb{R}^1_{++}$ x $\mathbb{R}^n_{++}$

Derivatives

1. $\frac{\partial \pi(p,w)}{\partial p} = f(x(p,w)) = y(p,w)$
2. $\frac{\partial \pi(p,w)}{\partial w_i} = -x_i(p,w)$
3. $\frac{\partial c(w,q)}{\partial w_i} = z_i(w,q)$
4. $\frac{\partial c(w,q)}{\partial q} = MC(\text{Marginal Cost})$

Exercises

Solving problems with continuum of inputs

The problem is,

$$\pi(p,w) = \max_{z(j)} pf(z) - \int_0^1 w(j)z(j)dj$$

Given the production function $f(z) = \int_0^1 z(j)^\alpha \, dj$, the profit maximization problem becomes

$$\pi(p,w) = \max_{z(j)} \left[p \int_0^1 z(j)^\alpha \, dj - \int_0^1 w(j)z(j) \, dj \right].$$

Simplifying,

$$\pi(p,w) = \max_{z(j)} \int_0^1 \left[pz(j)^\alpha - w(j)z(j) \right] \, dj.$$

The problem is separable[1], so the maximization for each $z(j)$ can be solved independently,

$$\max_{z(j)} \left[pz(j)^\alpha - w(j)z(j) \right].$$

Taking the derivative with respect to $z(j)$ and setting it to zero (FOC),

$$\frac{\partial}{\partial z(j)} \left[pz(j)^\alpha - w(j)z(j) \right] = 0,$$

$$p\alpha z(j)^{\alpha-1} - w(j) = 0.$$

Solving for $z(j)$:

$$z(j) = \left(\frac{w(j)}{p\alpha} \right)^{\frac{1}{\alpha-1}} = x(j,p,w).$$

To confirm a maximum, check the second derivative:

$$\frac{\partial^2}{\partial z(j)^2} \left[pz(j)^\alpha - w(j)z(j) \right] = p\alpha(\alpha-1)z(j)^{\alpha-2}.$$

Since $\alpha \in (0,1)$, the term $(\alpha - 1) < 0$, so the second derivative is negative, confirming a maximum.

Extra (not required): The optimal allocation is:

$$z^*(j) = \left(\frac{w(j)}{p\alpha} \right)^{\frac{1}{\alpha-1}}.$$

Substitute $z^*(j)$ into the profit function:

$$\pi(p,w) = \int_0^1 \left[p \left(z^*(j) \right)^\alpha - w(j)z^*(j) \right] \, dj.$$

Substitute $z^*(j)$ explicitly. First, compute $(z^*(j))^\alpha$:

$$(z^*(j))^\alpha = \left(\frac{w(j)}{p\alpha} \right)^{\frac{\alpha}{\alpha-1}}.$$

And:

[1]For an optimization problem, separability means that the objective function can be expressed as a sum (or integral) of terms, each depending only on a single variable (or a small subset of variables). This property allows the optimization over all variables to be broken down into independent subproblems that can be solved separately for each variable.
The objective function is separable because: The term $pz(j)^\alpha - w(j)z(j)$ depends only on $z(j)$ for a fixed j. There is no interaction between $z(j)$ and $z(k)$ for $j \neq k$.

$$w(j)z^*(j) = w(j)\left(\frac{w(j)}{p\alpha}\right)^{\frac{1}{\alpha-1}}.$$

Thus:

$$\pi(p,w) = \int_0^1 \left[p\left(\frac{w(j)}{p\alpha}\right)^{\frac{\alpha}{\alpha-1}} - w(j)\left(\frac{w(j)}{p\alpha}\right)^{\frac{1}{\alpha-1}}\right] dj.$$

Simplify further to compute the explicit profit if $w(j)$ is specified.

A question from a past Q exam

(a) The problem is

$$\begin{aligned} E_p[\pi(p,w)] &= \max_x E_p[px_1^\alpha x_2^\beta - w_1x_1 - w_2x_2] \\ &= \max_x E_p[p]x_1^\alpha x_2^\beta - w_1x_1 - w_2x_2 \end{aligned}$$

Where $E[p] = \delta p_1 + (1-\delta)p_2$.

The FOCs are,

$$\begin{aligned} x_1 &: \ \alpha E(p)x_1^{\alpha-1}x_2^\beta = w_1 \\ x_2 &: \ \beta E(p)x_1^\alpha x_2^{\beta-1} = w_2 \end{aligned}$$

Then,

$$\begin{aligned} x_1^* &= E(p)^{\frac{1}{1-\alpha-\beta}}\alpha^{\frac{1-\beta}{1-\alpha-\beta}}\beta^{\frac{\beta}{1-\alpha-\beta}}w_1^{\frac{\beta-1}{1-\alpha-\beta}}w_2^{\frac{-\beta}{1-\alpha-\beta}} \\ x_2^* &= E(p)^{\frac{1}{1-\alpha-\beta}}\alpha^{\frac{\alpha}{1-\alpha-\beta}}\beta^{\frac{1-\alpha}{1-\alpha-\beta}}w_1^{\frac{-\alpha}{1-\alpha-\beta}}w_2^{\frac{\alpha-1}{1-\alpha-\beta}} \end{aligned}$$

And the optimal output is,

$$q(E(p),w) = f(x^*) = E(p)^{\frac{\alpha+\beta}{1-\alpha-\beta}}\alpha^{\frac{\alpha}{1-\alpha-\beta}}\beta^{\frac{\beta}{1-\alpha-\beta}}w_1^{\frac{-\alpha}{1-\alpha-\beta}}w_2^{\frac{\beta}{1-\alpha-\beta}}$$

(b) In this case we replace $E(p) = p_1$ and $E(p) = p_2$ respectively, and get,

$$q(p_1,w) = f(x^*) = p_1^{\frac{\alpha+\beta}{1-\alpha-\beta}}\alpha^{\frac{\alpha}{1-\alpha-\beta}}\beta^{\frac{\beta}{1-\alpha-\beta}}w_1^{\frac{-\alpha}{1-\alpha-\beta}}w_2^{\frac{\beta}{1-\alpha-\beta}}$$

$$q(p_2,w) = f(x^*) = p_2^{\frac{\alpha+\beta}{1-\alpha-\beta}}\alpha^{\frac{\alpha}{1-\alpha-\beta}}\beta^{\frac{\beta}{1-\alpha-\beta}}w_1^{\frac{-\alpha}{1-\alpha-\beta}}w_2^{\frac{\beta}{1-\alpha-\beta}}$$

(c) Let $g(w) = \alpha^{\frac{\alpha}{1-\alpha-\beta}}\beta^{\frac{\beta}{1-\alpha-\beta}}w_1^{\frac{-\alpha}{1-\alpha-\beta}}w_2^{\frac{\beta}{1-\alpha-\beta}}$, and $\alpha+\beta = \frac{1}{2}$, then,

$$\implies \frac{\alpha+\beta}{1-\alpha-\beta} = 1$$

$$\begin{aligned} \implies q(E(p),w) &= E(p)\alpha^{\frac{\alpha}{1-\alpha-\beta}}\beta^{\frac{\beta}{1-\alpha-\beta}}w_1^{\frac{-\alpha}{1-\alpha-\beta}}w_2^{\frac{\beta}{1-\alpha-\beta}} \\ &= (\delta p_1 + (1-\delta)p_2)\alpha^{\frac{\alpha}{1-\alpha-\beta}}\beta^{\frac{\beta}{1-\alpha-\beta}}w_1^{\frac{-\alpha}{1-\alpha-\beta}}w_2^{\frac{\beta}{1-\alpha-\beta}} \\ &= \delta q(p_1,w) + (1-\delta)q(p_2,w) \end{aligned}$$

Therefore, the expectation of the outputs in part (b) equals the output in part (a).

(d) By a) we have $E_p[\pi(p,w)] = \pi(\delta p_1 + (1-\delta)p_2, w)$.
By expected b) we have $\delta q(p_1,w) + (1-\delta)q(p_2,w)$
Since the profit function is convex in (p,w), it is convex in p, and we can apply Jensen's inequality to conclude,

$$(a) \leq \text{expected } (b)$$

The Cost Function

We are given $C(w,q)|_{q=0} = C(w,0) = 0$ and Marginal Cost $= \frac{\partial C(w,q)}{\partial q} = k$.
Now we take integral back on q,

$$\implies C(w,q) = kq + T(w)$$

Given our initial condition $C(w,0) = T(w) = 0$,

$$\implies C(w,q)kq$$

Which means that it is HoD1 in q. Therefore, the production function is HoD1 in (p,w).
Some examples are:

1. Cobb-Douglas

$$f(x_1,...,x_n) = \Pi_{i=1}^{n} x_i^{\beta_i} \text{ where } \sum_{i=1}^{n} \beta_i = 1$$

2. Leontief

$$f(x_1,...,x_n) = \min\{\beta_1 x_1, ..., \beta_n x_n\}$$

3. Constant Elasticity of Substitution (CES)

$$f(x_1,...,x_n) = (\sum_{i=1}^{n} \beta_i x_i^r)^{\frac{1}{r}}$$