ECON 6170

Problem Set 5

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Exercise 3. Let $S \subseteq \mathbb{R}^d$. Prove that $co(S)$ is the collection of all finite convex combinations of elements in S.

Proof. We have that $co(S) = \bigcap\{T \subseteq \mathbb{R}^d : S \subseteq T, T \text{ convex}\}$. We wish to show that

$$\left\{x \in \mathbb{R}^d : x = \sum_{i=1}^n \alpha_i y_i, \sum_{i=1}^n \alpha_i = 1, \alpha_i \geq 0 \;\forall\, i, y_i \in S \;\forall\, y_i\right\} = \bigcap\{T \subseteq \mathbb{R}^d : S \subseteq T, T \text{ convex}\}$$

We will use set containment.

($\subseteq$): Take some $x \in \mathbb{R}^d$, such that $\exists\ \{y_1, \dots, y_n\}, \{\alpha_1, \dots, \alpha_n\}$ s.t. $y_i \in S \;\forall\; i \in \{1, \dots, n\}$, $\sum_{i=1}^n \alpha_i = 1$, $\alpha_i \in [0,1] \;\forall\; i \in \{1, \dots, n\}$ where $x = \sum_{i=1}^n \alpha_i y_i$. Since $y_i \in S \;\forall\; i \in \{1, \dots, n\}$, $y_i \in T \;\forall\; T$ since $S \subseteq T$. Since each T is convex, by Proposition 1 in the Convexity notes, $\sum_{i=1}^t \alpha_i y_i \in T \;\forall\; T$ where $S \subseteq T$ and T convex. Thus, since $x = \sum_{i=1}^t \alpha_i y_i$, $x \in T \;\forall\; T$, meaning that $x \in \bigcap\{T \subseteq \mathbb{R}^d : S \subseteq T, T \text{ convex}\}$.

($\supseteq$): Take some $x \in \bigcap\{T \subseteq \mathbb{R}^d : S \subseteq T, T \text{ convex}\}$. Consider two cases. First, if $x \in S$, then choosing $\alpha_1 = 1$, $\alpha_2 = 0$, and some $y \in S$ where $y \neq x$, we have that $x = \alpha_1 x + \alpha_2 y$, so $x \in \left\{x \in \mathbb{R}^d : x = \sum_{i=1}^n \alpha_i y_i, \sum_{i=1}^n \alpha_i = 1, \alpha_i \geq 0 \;\forall\, i, y_i \in S \;\forall\, y_i\right\}$.

Next, assume that $x \notin S$. The fact that $x \in co(S)$ implies that $\exists\ n \in \mathbb{N}, \{y_i, \alpha_i\}_{i=1}^n$ s.t. $x = \sum_{i=1}^n y_i \alpha_i$ for $y_i \in T \;\forall\, i, T$. We also have that for at least one j, $y_j \notin S$. If y_j can be written as a finite convex combination of elements of S, then writing it as such creates a finite convex combination of elements of S that equal x. If y_j cannot by written as a finite convex combination of elements of S, then there exists T_j such that $S \subseteq T_j$ and $x, y_j \notin T_j$, where T_j convex. Thus, if x cannot be written as a finite convex combination of elements of S, $x \notin co(S)$. By contrapositive, $x \in \left\{x \in \mathbb{R}^d : x = \sum_{i=1}^n \alpha_i y_i, \sum_{i=1}^n \alpha_i = 1, \alpha_i \geq 0 \;\forall\, i, y_i \in S \;\forall\, y_i\right\}$. □

Exercise 7. Prove that $\overline{co}(S) = cl(co(S))$.

Proof. We wish to show that

$$\bigcap\left\{T \subseteq \mathbb{R}^d : S \subseteq T, T \text{ is convex and closed}\right\} = \bigcap\left\{T \subseteq \mathbb{R}^d : co(S) \subseteq T, T \text{ closed}\right\}$$

($\subseteq$): If $x \in \overline{co}(S)$, then $x \in T \;\forall\; T$ convex and closed, where $S \subseteq T$. Since $S \subseteq T$ and T convex, $co(S) \subseteq T$. Since T is also closed, and these hold for all T, $x \in cl(co(S))$.

($\supseteq$): If $x \in cl(co(S))$, then $x \in T$ for all $co(S) \subseteq T$ where T is closed. Since $S \subseteq co(S)$, $S \subseteq T$. Since not all $T \ni x$ are necessarily convex, the set of T that are convex, closed, and contain S is a subset of the set of T that x are in. Thus, $x \in \overline{co}(S)$. □

Exercise 10. Prove that a function is concave (convex) if and only if its subgraph (epigraph) is convex.

Proof.

($\Rightarrow$): Assume that a function f is concave. Take some $(x,y),(x',y') \in sub(f)$, so we have that $f(x) \geq y$ and $f(x') \geq y'$. Fix $\alpha \in (0,1)$. It suffices to show that $(\alpha x + (1-\alpha)x', \alpha y + (1-\alpha)y') \in sub(f)$. Since f is concave, we have that

$$f(\alpha x + (1-\alpha)x') \geq \alpha f(x) + (1-\alpha)f(x') \geq \alpha y + (1-\alpha)y'$$

Where the second inequality follows from the assumption that $(x,y),(x',y') \in sub(f)$. Thus, $sub(f)$ is convex.

($\Leftarrow$): We have that $sub(f)$ is convex. FSOC, assume that f is not concave, meaning that there exist $x, x', \alpha \in (0,1)$ such that

$$f(\alpha x + (1-\alpha)x') < \alpha f(x) + (1-\alpha)f(x')$$

This implies that there is positive distance between the two quantities, so there exist $y, y' \in \mathbb{R}$ such that $f(\alpha x + (1-\alpha)x') < \alpha y + (1-\alpha)y' \leq \alpha f(x) + (1-\alpha)f(x')$. However, that would imply that $(x,y),(x',y') \in sub(f)$, but $(\alpha x + (1-\alpha)x', \alpha y + (1-\alpha)y') \notin sub(f)$, which contradicts the assumption that $sub(f)$ is convex. Thus, f is concave.

The same proof applies for f being convex if and only if its epigraph is convex, flipping the respective inequalities. $\square$

Example 11. Prove that an affine function is both convex and concave.

Proof. We have that $f : X \to \mathbb{R}$ is affine, meaning that $f(x) = ax + b$ for some $a, b \in \mathbb{R}^d, \mathbb{R}$. Consider $x, x' \in X$. We have that

$$f(\alpha x + (1-\alpha)x') = a(\alpha x + (1-\alpha)x') + b = \alpha(ax+b) + (1-\alpha)(ax'+b) = \alpha f(x) + (1-\alpha)f(x')$$

Thus, $f(\alpha x + (1-\alpha)x') \geq \alpha f(x) + (1-\alpha)f(x')$ meaning that f is concave, and $f(\alpha x + (1-\alpha)x') \leq \alpha f(x) + (1-\alpha)f(x')$, meaning that f is convex. $\square$

Exercise 12. Prove that a function is quasiconcave (resp. quasiconvex) if and only if the upper (resp. lower) contour sets are convex.

Proof. ($\Rightarrow$): We have that $f : X \to \mathbb{R}$ is quasiconcave. Take some x, y in the upper contour set r of f. That means that for some r, $f(x) \geq r$ and $f(y) \geq r$. Then, for some $\alpha \in (0,1)$,

$$f(\alpha x + (1-\alpha)y) \geq \min\{f(x), f(y)\} \geq r$$

where the first inequality follows from quasiconcavity of f. Thus, $\alpha x + (1-\alpha)y$ is in the upper contour set r of f, and the upper contour sets of f are convex.

($\Leftarrow$): We have that the upper contour sets of f are convex. Consider some $x, y \in X$. Take $r = \min\{f(x), f(y)\}$. Since $f(x) \geq r$ and $f(y) \geq r$ by construction, x and y are in the upper contour set r of f. Since the upper contour sets are convex, $\alpha x + (1-\alpha)y$ for some $\alpha \in (0,1)$ is also in the upper contour set r of f, which means that $f(\alpha x + (1-\alpha)y) \geq r = \min\{f(x), f(y)\}$. Thus, f is quasiconcave.

The same proof follows, reversing the inequalities, for quasiconvex and the lower contour sets. $\square$

Exercise 13. True or false:If f is a (quasi)concave function and $h : \mathbb{R} \to \mathbb{R}$ is a nondecreasing function, then $h \circ f$ is (quasi)concave.

(i) True!

Proof. Consider some x, y. Take $r = \min\{f(x), f(y)\}$. Then x, y are each in the upper contour set r of f, which is convex because f is quasiconcave. This means that $f(\alpha x + (1-\alpha)y) \geq \min\{f(x), f(y)\}$. WLOG, assume that $f(x) \leq f(y)$. Since h is nondecreasing, we have that

$$f(\alpha x + (1-\alpha)y) \geq f(x) = \min\{f(x), f(y)\} \implies (h \circ f)(\alpha x + (1-\alpha)y) \geq (h \circ f)(x)$$

Thus, $h \circ f$ is quasiconcave. $\square$

(ii) False! Consider the example of $f(x) = 1$ and be any strictly increasing and strictly convex function. $f(x)$ is affine and thus concave, but $h \circ f = h$ which is strictly convex (and thus not concave).

Exercise 1. Let $X \subseteq \mathbb{R}^d$ be convex. Prove or give a counterexample:

(i) True!

Proof. We have that f and g are convex. Consider for some $x, y \in X, \alpha \in (0,1)$:

$$\begin{aligned}(f+g)(\alpha x + (1-\alpha)y) &= f(\alpha x + (1-\alpha)y) + g(\alpha x + (1-\alpha)y) \\ &\leq \alpha f(x) + (1-\alpha)f(y) + \alpha g(x) + (1-\alpha)g(y) \\ &= \alpha(f+g)(x) + (1-\alpha)(f+g)(y)\end{aligned}$$

Where the inequality follows from the assumption that f and g are convex. $\square$

(ii) False! Consider $f(x) = -x$ and $g(x) = x - \frac{|x|}{2}$. f is convex and thus quasiconvex, and g is monotonically increasing and thus quasiconvex, but their sum is the function $(f+g)(x) = -\frac{|x|}{2}$ which is not quasiconvex because the lower contour set for, *e.g.*, $r = -1$ is the disjoint intervals $(-\infty, -2] \cup [2, \infty)$ which is not convex by inspection.

(iii) True!

Proof. f is concave implies that for arbitrary $x, y \in X$, $\alpha \in (0,1)$,

$$f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha)f(y) \geq \min\{f(x), f(y)\}$$

where the second inequality follows from a direct property of minima. Thus, f is quasiconcave. $\square$

(iv) True!

Proof. f is concave implies that for arbitrary distinct $x, y \in X$, $\alpha \in (0,1)$,

$$f(\alpha x + (1-\alpha)y) > \alpha f(x) + (1-\alpha)f(y) \geq \min\{f(x), f(y)\}$$

where the second inequality follows from a direct property of minima. Thus, f is strictly quasiconcave. $\square$