ECON 6090

Problem Set 4

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1. We have that $u(w) = -\exp(-r_a w)$, for $r_a > 0$. First, note that the decision maker is risk-averse, as this Bernoulli utility function is concave in w. Furthermore, her coefficient of absolute risk aversion is

$$A(w) = -\frac{u''(w)}{u'(w)} = \frac{r_a^2 \exp(-r_a w)}{r_a \exp(-r_a w)} = r_a$$

which is constant, meaning that the decision maker has constant absolute risk aversion, so we may feel free to ignore wealth effects. Saying that the agent invests x in the risky asset, which has (random) gross return $\varepsilon \sim \mathcal{N}(\mu, \sigma)$, and $w_0 - x$ in the risk-free asset, where the risk-free asset has a gross return of r_f, her wealth is

$$w = x\varepsilon + (w_0 - x)r_f = x\mu + x(R - \mu) + (w_0 - x)r_f$$

with first and second moments

$$\mathbb{E}[w] = x\mu + (w_0 - x)r_f \quad \text{and} \quad \text{Var}(w) = x^2\sigma^2$$

Using the moment generating function for $X \sim \mathcal{N}(\mu, \sigma^2)$, we get that

$$\mathbb{E}[\exp(tX)] = \exp\left(t\mu + \frac{t^2\sigma^2}{2}\right)$$

So her expected utility under CARA utility is

$$\mathbb{E}[u(w)] = -\exp\left(-r_a\,\mathbb{E}[w] + \frac{r_a^2}{2}\,\text{Var}(w)\right) = -\exp\left(-r_a x\mu - r_a r_f(w_0 - x) + \frac{r_a^2 x^2 \sigma^2}{2}\right)$$

Maximizing this function is equivalent to maximizing the exponent. The first order condition with respect to x gives

$$-r_a\mu + r_a r_f + r_a^2 x\sigma^2 = 0$$

Thus, we have that

$$x^\star = \frac{\mu - r_f}{r_a\sigma^2}$$

Taking into account corners, we get that the optimal level of investment is

$$x^\star = \begin{cases} 0 & r_f \geq \mu \\ \max\left\{\frac{\mu - r_f}{r_a\sigma^2}, w_0\right\} & \text{otherwise} \end{cases}$$

(note that this is in real dollar values – to get the share of wealth, simply divide everything by w_0)

2. Suppose that $\succeq$ satisfies the Savage axioms with state space S and outcome space X, and suppose that it has an SEU representation with payoff function u and belief distribution μ. Prove that for every non-null event A the preference order σ_A has an SEU representation. What is it?

Proof. We will define the preference order σ_A as follows:

$$f \succeq_A g \text{ if and only if } f\mid_A \succeq g\mid_A$$

(intuitively, f is weakly preferred to g conditional on A if and only if the restriction of f to A is preferred to the restriction of g to A under the global preference relation)

Since $\succeq$ has an SEU representation, the expected utility of f is

$$\mathop{\mathbb{E}}_{\mu}[u \circ f] = \int_{s \in S} u(f(s)) d\mu(s)$$

To construct the SEU representation of σ_A, we need a conditional utility function and a conditional belief distribution. The conditional utility function is over outcomes, and will coincide with u. Define the conditional belief distribution $\mu(\cdot \mid A)$ as follows, using the definition of conditional probabilities:

$$\mu(B \mid A) = \frac{\mu(B \cap A)}{\mu(A)}$$

Thus, we can show that σ_A has an SEU representation as follows. Consider two acts $f, g \in F$. From above, we have that

$$f \succeq_A g \Longleftrightarrow \mathop{\mathbb{E}}_{\mu}[u \circ f \mid A] \geq \mathop{\mathbb{E}}_{\mu}[u \circ g \mid A]$$

Expanding, we get that

$$f \succeq_A g \Longleftrightarrow \int_{s \in A} u(f(s)) d\mu(s \mid A) \geq \int_{s \in A} u(g(s)) d\mu(s \mid A)$$

The SEU representation for σ_A is

$$\mathop{\mathbb{E}}_{\mu}[u \circ f \mid A] = \int_{s \in A} u(f(s)) d\mu(s \mid A)$$

$\square$

3. Let M denote the right triangle in the plane with vertices $x = (0, 1)$, $y = (0, 0)$, and $z = (1, 0)$. Each $m \in M$ can be written uniquely as $\alpha_m x + (1 - \alpha_m)(\beta_m y + (1 - \beta_m) z)$. Define the mixture operators

$$m \otimes_\lambda n = \begin{cases} z & \text{if } m = n = z; m = z \ \&\ \lambda = 1; \text{ or } n = z \ \&\ \lambda = 0 \\ (\lambda \alpha_m + (1 - \lambda)\alpha_n) x + \\ (1 - (\lambda \alpha_m + (1 - \lambda)\alpha_n)) y & \text{otherwise} \end{cases}$$

   (a) This is not a mixture space. Consider the following counterexample, showing that it violates the first axiom of mixture spaces:

   Counterexample. This is not a mixture space. Consider $m = (0.5, 0.5)$, which admits the unique coordinates $\alpha_m = 0.5$, $\beta_m = 0$. For arbitrary n, we have that $m \otimes_1 n = \alpha_m x + (1 - \alpha_m) y = (0, 0.5) \neq m$.

   (b) It doesn't. It admits no indifference curves.

4. We have that X has density $f(x) = x^{-6/5}/5$ and Y has density $g(x) = x^{-3/2}/2$.

   (a) Note first that neither of the functions are densities over the domains $(-\infty, \infty)$ or $(0, \infty)$, as they are (respectively) not well-defined over the negative real numbers and diverge on $(0, 1)$. However, if we consider the domain $[1, \infty)$, we have that

$$\int_1^\infty f(x) dx = \int_1^\infty g(x) dx = 1$$

   Thus, we will restrict them each to the domain $[1, \infty)$.

Recall that a distribution X first order stochastically dominates Y if their CDFs are ordered $F_X(x) \leq F_Y(x)$ for all x, with strict inequality holding for at least one x. We construct the CDFs by integrating the densities. Formally, we have that

$$F(x) = \int_1^x f(t)dt = \left(-\frac{1}{t^{1/5}}\right|_1^x = 1 - \frac{1}{x^{1/5}}$$

and

$$G(x) = \int_1^x g(t)dt = \left(-\frac{1}{t^{1/2}}\right|_1^x = 1 - \frac{1}{x^{1/2}}$$

Since $x \in [1, \infty)$, we can say that for any x, $F(x) \leq G(x)$. Additionally, taking $x = 2$, we have that $F(x) \approx 0.13 < 0.29 \approx G(x)$. Thus, X first-order stochastically dominates Y.

(b) We have that $u(x) = \sqrt{x}$. Since this function is strictly increasing, the decision maker will always prefer a lottery that first-order stochastically dominates, so they will always prefer X. To see why concretely, consider that the decision maker will prefer X to Y if

$$\int_1^\infty u(x)f(x)dx > \int_1^\infty u(x)g(x)dx \implies \int_1^\infty u(x)d(F(x) - G(x)) > 0$$

Note that, integrating by parts, we have that for some CDF F,

$$\int_1^\infty u(x)dF(x) = u(x)F(x)|_{x=1}^{x=\infty} - \int_1^\infty u(x)F(x)dx$$

Thus, since $F(1) = G(1) = 0$ and $F(\infty) = G(\infty) = 1$, we have that

$$\int_1^\infty u(x)d(F(x) - G(x)) = -\int_1^\infty u(x)(F(x) - G(x))dx = \int_1^\infty u(x)(G(x) - F(x))dx > 0$$

since $G(x) \geq F(x) \; \forall \; x$.

5. We have that

	s_1	s_2
a_1	0	-8
a_2	-10	0
a_3	-4	-3

(a) If the decision maker believes that $p_1 = 1/4$ and $p_1 = 3/4$ with equal probability, her expectation is that

$$p_1 = \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{3}{4} = \frac{1}{2}$$

(b) Given that $\mathbb{E}[p_1] = \frac{1}{2}$, we have that $\mathbb{E}[a_1] = -4$, $\mathbb{E}[a_2] = -5$, and $\mathbb{E}[a_3] = -3.5$. She will choose a_3.

(c) Define p' as the decision maker's posterior belief over the probability that the probability of state 1 is 3/4. Her prior belief is that $p' = 1/2$. Having been told that the previous draw was of s_1, we have that by Bayes' Rule

$$p' = \mathbb{P}\left\{p_1 = \frac{3}{4} \middle| s_{-1} = s_1\right\} = \frac{\mathbb{P}\{s_{-1} = s_1 \mid p_1 = 3/4\}}{\mathbb{P}\{s_{-1} = s_1 \mid p_1 = 3/4\} + \mathbb{P}\{s_{-1} = s_1 \mid p_1 = 1/4\}} = \frac{3/4}{3/4 + 1/4} = \frac{3}{4}$$

Thus, her expectation is that

$$\mathbb{E}[p_1] = p'\frac{3}{4} + (1 - p')\frac{1}{4} = \frac{9}{16} + \frac{1}{16} = \frac{5}{8}$$

Her expected utilities from each choice are:

$$\mathbb{E}[a_1] = \frac{5}{8} \cdot 0 + \frac{3}{8} \cdot -8 = -3$$
$$\mathbb{E}[a_2] = \frac{5}{8} \cdot -10 + \frac{3}{8} \cdot 0 = -6.25$$
$$\mathbb{E}[a_3] = \frac{5}{8} \cdot -4 + \frac{3}{8} \cdot -3 = -3.625$$

Thus, she will choose a_1

(d) Again define p' as the posterior that the probability of state 1 is 3/4. Again by Bayes' rule, we have that

$$p' = \mathbb{P}\left\{p_1 = \frac{3}{4}\middle| s_{-1} = s_2\right\} = \frac{\mathbb{P}\{s_{-1} = s_2 \mid p_1 = 3/4\}}{\mathbb{P}\{s_{-1} = s_2 \mid p_1 = 3/4\} + \mathbb{P}\{s_{-1} = s_2 \mid p_1 = 1/4\}} = \frac{1/4}{1/4 + 3/4} = \frac{1}{4}$$

Thus, her expectation is that

$$\mathbb{E}[p_1] = p'\frac{3}{4} + (1 - p')\frac{1}{4} = \frac{3}{16} + \frac{3}{16} = \frac{3}{8}$$

Her expected utilities from each choice are

$$\mathbb{E}[a_1] = \frac{3}{8} \cdot 0 + \frac{5}{8} \cdot -8 = -5$$
$$\mathbb{E}[a_2] = \frac{3}{8} \cdot -10 + \frac{5}{8} \cdot 0 = -3.75$$
$$\mathbb{E}[a_3] = \frac{3}{8} \cdot -4 + \frac{5}{8} \cdot -3 = -3.375$$

Thus, she will choose a_3

(e) From part (b), we know that the decision maker's expected utility when she has no information is -3.5. From part (c), we know that her expected utility when she is told s_1 is -3 and from part (d), her expected utility when she is told s_2 is -3.375. She has prior expectation that the probability of s_1 is $\frac{1}{2}$, so we have that her expected expected utility is

$$\frac{1}{2} \cdot -3 + \frac{1}{2} \cdot -3.375 = -3.1875$$

so she gains, in expectation, $-3.1875 - (-3.5) = 0.3125$ from knowing the value of the state in the previous period.

6. In the three-color Ellsberg paradox, we have that $R = 30$ and $B + G = 60$. We also have that, under the generally accepted results,

$$R \succ B \quad \text{and} \quad B + G \succ R + G$$

Note first that we *do* have complete preferences, over the acts that we have been given, despite the fact that we do not know how they rank, for example, G and B. Since we have that the act f (pay \$100 if Red, nothing if Green or Blue) is preferred to g (pay \$100 if Blue, nothing if Red or Green). Define h as "pay nothing if Green" and k as "pay \$100 if Green". Then we have that $f \mid_A h \succ g \mid_A h$, where $A = \{\text{Red or Blue}\}$, but $f \mid_A k \prec g \mid_A k$. Thus, the second Savage axiom is violated. The Savage axioms three through five concern outcomes. None of them are violated, as long as we make the (reasonable) assumption that people prefer \$100 to \$0.

So the three-color Ellsberg paradox violates Savage P2.