ECON 6090 - Microeconomics I
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Contents

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0 Conventions

$x := y$	x is defined as equal to y
$\mathbb{N}$	$\{1, 2, 3, \dots\}$
$\mathbb{R}_+$	$\{x \in \mathbb{R} \mid x \geq 0\}$
$\mathbb{R}_{++}$	$\{x \in \mathbb{R} \mid x > 0\}$
$\mathbb{R}^L_{++}$	$\mathbb{R}_{++} \times \cdots \times \mathbb{R}_{++}$
$x \geq y$	$x_i \geq y_i$ for $i = 1, \dots, L$
$x \gg y$	$x_i > y_i$ for $i = 1, \dots, L$
$S \subseteq T$	S is a subset of T.
$S \subset T$	S is a *proper* subset of T, *i.e.*, $S \subseteq T$ and $S \neq T$
S^{C}	Complement of the set S
A^{T}	Transpose of the matrix A
$x \cdot y$	$x^{\mathsf{T}} y$
	Vectors are column vectors, unless otherwise specified
$D_x f(x)$	Derivative matrix of f, with entries $\partial f_i(x)/\partial x_j$
MWG	Mas-Colell, Whinston, Green – *Microeconomic Theory*
Kreps	Kreps – *Microeconomic Foundations I*

Note: Where possible, all stated results are proven. Proofs that were not given in the lectures are marked with *. In addition, all results and proofs in sections titled *Appendix* were not included in the lectures.

1 Choice Theory

1.1 Preference Theory

Let X be a finite set of objects.

Example 1.1. $X := \{x, y, z\}$

Definition 1.2. Define a preference relation $\succsim$ on X – "x is **at least as good as** y".

Definition 1.3.

(i) $x \succ y$: x is **strictly preferred** to y if $x \succsim y$ and not $y \succsim x$.

(ii) $x \sim y$: x is **indifferent** to y if $x \succsim y$ and $y \succsim x$.

Definition 1.4. The preference relation $\succsim$ is **complete**, if for all $x, y \in X$ either $x \succsim y$, $y \succsim x$ or both.

Remark 1.5. Completeness implies **reflexivity**: For all $x \in X$, $x \succsim x$.

Definition 1.6. The preference relation $\succsim$ is **transitive** if for all $x, y, z \in X$, if $x \succsim y$ and $y \succsim z$ then $x \succsim z$.

Example 1.7. Suppose $\succ$ is intransitive. In particular, let $x \succ y$, $y \succ z$ and $z \succ x$. Making a choice for the agent becomes impossible.

Definition 1.8. Preference relation $\succsim$ is **rational** if it is complete and transitive.

Definition 1.9. For any nonempty subset B of X define

$$C^*(B, \succsim) := \{x \in B \mid x \succsim y,\ \forall y \in B\}$$

Remark 1.10.

(i) Suppose $x \in C^*(B, \succsim)$ and $y \in C^*(B, \succsim)$. Then $x \sim y$.

(ii) Suppose $x \in B$, $x \notin C^*(B, \succsim)$ and $C^*(B, \succsim) \neq \varnothing$. Then there exists $y \in B$ such that $y \succ x$.

Definition 1.11. Define the **power set** of X as $\mathcal{P}(X) := \{B \subseteq X \mid B \neq \varnothing\}$.

Proposition 1.12. *If $\succsim$ is a rational preference relation on X then*

$$C^* \colon \mathcal{P}(X) \to \mathcal{P}(X)$$

In words, C^* maps nonempty subsets of X to nonempty subsets of X.

Proof. (By induction on $\#B$) Let $B \subseteq \mathcal{P}(X)$ be nonempty. Suppose B contains exactly one element, x. Then, by completeness, $x \succsim x$ and $C^*(B) = \{x\} \subseteq \mathcal{P}(X)$. Suppose alternatively that any set containing $n \in \mathbb{N}$ elements is mapped by C^* to another nonempty subset of X. Let B contain $n+1$ elements and let x be an element of B. Let $B' := B \setminus \{x\}$ and let x' be an element of $C^*(B', \succsim)$. Note that the latter is nonempty by the induction hypothesis. By completeness, $x \succsim x'$ or $x' \succsim x$, implying $C^*(B, \succsim) \in \{C^*(B', \succsim), \{x\}, C^*(B', \succsim) \cup \{x\}\} \subseteq \mathcal{P}(X)$ using transitivity. □

To see how choice and rationality relate, we need some further definitions.

Definition 1.13. C^* satisfies **Sen's α** if $x \in A \subseteq B$ and $x \in C^*(B, \succsim)$ implies $x \in C^*(A, \succsim)$.

Remark 1.14. Sen's α is also known as independence of irrelevant alternatives.

Proposition 1.15. *If $\succsim$ is a rational preference relation then C^* satisfies Sen's α.*

Proof. The result is trivially true if $A = B$. Suppose then that $A \subset B$. Let $x \in C^*(B, \succsim)$. Then $x \succsim y$ for all $y \in B$. In particular, if $y \in A \subseteq B$, then $x \succsim y$. Thus, $x \in C^*(A, \succsim)$. □

Definition 1.16. C^* satisfies **Sen's β** if $x, y \in C^*(A, \succsim)$, $A \subseteq B$ and $y \in C^*(B, \succsim)$ implies $x \in C^*(B, \succsim)$.

Remark 1.17. Sen's β is also known as expansion consistency.

Proposition 1.18. *If $\succsim$ is a rational preference relation then C^* satisfies Sen's β.*

Proof. Let $x, y \in C^*(A, \succsim)$, $A \subseteq B$ and $y \in C^*(B, \succsim)$. Since $x \in C^*(A, \succsim)$ we have $x \succsim y$ as $y \in A$. Since $y \in C^*(B, \succsim)$ we have $y \succsim z$ for all $z \in B$. By transitivity, $x \succsim y$ and $y \succsim z$ implies $x \succsim z$. So we have $x \succsim z$ for all $z \in B$ and thus $x \in C^*(B, \succsim)$. □

Definition 1.19. C^* satisfies **Houthaker's weak axiom of revealed preference** if for all $A, B \in \mathcal{P}(X)$, if $x, y \in A \cap B$, $x \in C^*(A, \succsim)$ and $y \in C^*(B, \succsim)$, then $x \in C^*(B, \succsim)$ and $y \in C^*(A, \succsim)$.

Proposition 1.20. *$C^* \colon \mathcal{P}(X) \to \mathcal{P}(X)$ satisfies Sen's α and β if and only if it satisfies Houthaker's weak axiom of revealed preference (HWARP).*

Proof.

(i) ($\alpha + \beta \implies$ HWARP) Suppose $x, y \in A \cap B \subseteq \mathcal{P}(X)$, $x \in C^*(A, \succsim)$ and $y \in C^*(B, \succsim)$. By Sen's α, both x and y are in $C^*(A \cap B, \succsim)$. Then by Sen's β, $x \in C^*(B, \succsim)$ and $y \in C^*(A, \succsim)$.

(ii) (HWARP $\implies \beta$) Say $x, y \in C^*(A, \succsim)$, $A \subseteq B$ and $y \in C^*(B, \succsim)$. Because $A = A \cap B$, $x, y \in C^*(A \cap B, \succsim)$. Applying HWARP, we get $x \in C^*(B, \succsim)$.

(iii) (HWARP $\implies \alpha$) Say $x \in A \subseteq B$ and $x \in C^*(B, \succsim)$. Suppose $x \notin C^*(A, \succsim)$. Then by Proposition 1.12, there exists $y \in C^*(A, \succsim)$. Note that $x, y \in A = A \cap B$, $x \in C^*(B, \succsim)$ and $y \in C^*(A, \succsim)$. By HWARP, $x \in C^*(A, \succsim)$, which is a contradiction. □

Proposition 1.21. *The following are equivalent for $C^* : \mathcal{P}(X) \to \mathcal{P}(X)$*

(i) $\succsim$ is rational

(ii) C^* satisfies Sen's α and Sen's β

(iii) C^* satisfies HWARP

Proof. The equivalence of (ii) and (iii) is given by Proposition 1.21. The implication (i) $\implies$ (ii) is given by Propositions 1.15 and 1.18. Finally, (iii) $\implies$ (i) is given later, in the proof of Proposition 1.29. □

1.2 Observed Choice

Definition 1.22. For $B \in \mathcal{P}(X)$, let $C(B) := \{x \in B \mid x \text{ is chosen}\}$

More generally,

Definition 1.23. For $\mathcal{B}$ a collection of nonempty subsets of X, $(\mathcal{B}, C)$, is called a **choice structure** if $C(B) \subseteq B$ and $C(B) \neq \varnothing$ for all $B \in \mathcal{B}$.

Example 1.24. Let $X = \{x, y, z\}$ and $\mathcal{B} = \{\{x, y\}, \{x, y, z\}\}$. Then a choice structure could be $C(\{x, y\}) = \{x\}$, $C(\{x, y, z\}) = \{y\}$. Note that in this example, the choice structure is not consistent with rational choice.

Definition 1.25. The choice structure $(\mathcal{B}, C)$ satisfies the **weak axiom of revealed preference** (WARP) if for all $A, B \in \mathcal{B}$, x and y are in both A and

B, and $x \in C(A)$ and $y \in C(B)$, then $x \in C(B)$ and $y \in C(A)$.[1]

Example 1.26. Let $X = \{x, y, z\}$ and $\mathcal{B} = \mathcal{P}(X)$. Suppose $C(\{x, y\}) = \{x, y\}$, $C(\{y, z\}) = \{y, z\}$ and $C(\{x, z\}) = \{z\}$. If we look at $C(\{x, y, z\})$ we will find a contradiction with WARP.

Remark 1.27. WARP and HWARP are defined in different contexts in our notes but are essentially the same theorem with a small caveat – HWARP is WARP with $\mathcal{B} = \mathcal{P}(X)$.

Definition 1.28. Given a choice structure $(\mathcal{B}, C)$, the **revealed preference relation** $\succsim^*$ is defined by $x \succsim^* y$ if there exists $B \in \mathcal{B}$ such that $x, y \in B$ and $x \in C(B)$.

Proposition 1.29. *Suppose that X is finite and $\mathcal{B} = \mathcal{P}(X)$. If $(\mathcal{B}, C)$ satisfies WARP then the revealed preference relation that it induces, $\succsim^*$, is rational and $C(B) = C^*(B, \succsim^*)$ for all $B \in \mathcal{B}$.*

**Proof.* If $\mathcal{B} = \mathcal{P}(X)$ and $(\mathcal{B}, C)$ is a choice structure, then $C(X)$ is defined and nonempty for every $\{x, y\} \subseteq X$. This implies $x \succsim^* y$ or $y \succsim^* x$ for all $x, y \in X$ and so $\succsim^*$ is complete.

Suppose $x \succsim^* y$ and $y \succsim^* z$. Then there exists $A \subseteq X$ containing x and y such that $x \in C(A)$, and $B \subseteq X$ containing y and z such that $y \in C(B)$. Moreover, $\{x, y, z\} \in \mathcal{B}$ and $C(x, y, z)$ is nonempty. Suppose $y \in C(x, y, z)$. Then by WARP, $x \in C(x, y, z)$. Suppose $z \in C(x, y, z)$. Then again by WARP, $y \in C(x, y, z)$ and thus $x \in C(x, y, z)$. In that case, $x \in C(x, y, z)$ reveals that $x \succsim^* z$ and so $\succsim^*$ is transitive.

Let x be an element of $C^*(B, \succsim^*)$. Then $x \succsim^* y$ for all $y \in B$. Since $C(B)$ is nonempty, we have $z \in C(B)$ for some z. By $x \succsim^* z$, there exists $A \in \mathcal{B}$ such that $x, z \in A$ and $x \in C(A)$. Therefore by $(\mathcal{B}, C)$ satisfying WARP, $x \in C(B)$. Conversely, suppose $x \in C(B)$. Then $x \succsim^* y$ for all $y \in B$, and so $x \in C^*(B, \succsim^*)$. $\square$

Remark 1.30. Note that the above proof holds more generally for any $\mathcal{B}$ that contains all subsets of X of cardinality 3 or less.

Example 1.31. A counterexample for Remark 1.30: Suppose $X := \{x, y, z, w\}$ and $\mathcal{B} := \{\{x, y\}, \{y, z\}, \{z, w\}, \{w, x\}\}$. Let C be defined by:

[1] Note the difference in phrasing from the HWARP definition – this is because $A \cap B$ is not necessarily in $\mathcal{B}$.

$$\begin{aligned}
\{x, y\} &\mapsto \{x, y\} \\
\{y, z\} &\mapsto \{y, z\} \\
\{z, w\} &\mapsto \{z, w\} \\
\{w, x\} &\mapsto \{x\}
\end{aligned}$$

Because no pair of elements of X are both in two elements of $\mathcal{B}$, WARP is vacuously satisfied. But neither $x \succsim^* z$ nor $z \succsim^* x$, so $\succsim^*$ is incomplete. It can also be shown that $\succsim^*$ is intransitive. Moreover, if we extend C^* to the family of all doubletons in X, such that every doubleton apart from $\{w, x\}$ is mapped to itself, $\succsim^*$ becomes complete but remains intransitive.

1.3 Incomplete preferences

Definition 1.32. $\succ$ is a **strict partial order**[2] if

(i) For any $x, y \in X$ if $x \succ y$ then $\neg(y \succ x)$.

(ii) $\succ$ is transitive.

Remark 1.33. Note that we do not want to define $\sim$ by $x \sim y$ if $\neg(x \succ y)$ and $\neg(y \succ x)$. It could be the case that x and y are not comparable.

Proposition 1.34. *Define choice by*

$$C^*(A, \succ) := \{x \in A \mid \forall y \in A, \neg(y \succ x)\}$$

where $\succ$ is a strict partial order. Then C^* satisfies Sen's α but not Sen's β.

**Proof.*

(i) Suppose $x \in A \subseteq B$ and $x \in C^*(B, \succ)$. Then there does not exist $y \in B$ such that $y \succ x$. It follows that no such y exists in $A \subseteq B$ either, so $x \in C^*(A, \succ)$.

(ii) By way of counterexample, suppose $x, y \in C^*(A, \succ)$, $A \subseteq B$, $y \in C^*(B, \succ)$ and there is some $z \succ x$ in B such that y and z are incomparable. Then the hypotheses of Sen's β are satisfied, but $x \notin C^*(B, \succ)$.

$\square$

[2] As opposed to the $\succ$ induced by complete preferences, which is a strict *total* order. In the language of order theory, the complete weak preference relation, $\succsim$, is a total *pre*order. If the consumer was never indifferent between distinct consumption bundles, $\succsim$ would be a total order.

2 Consumer Choice

2.1 WARP and the Slutsky Matrix

Assumptions 2.1.

(i) L commodities, $x := (x_1, \ldots, x_L) \in \mathbb{R}^L_+$

(ii) Prices, $p := (p_1, \ldots, p_L) \in \mathbb{R}^L_{++}$. That is, $p \gg 0$ or $p_i > 0$ for all i.

(iii) Wealth, $w > 0$

(iv) Budget set, $B_{p,w} := \{x \in \mathbb{R}^L_+ \mid p \cdot x \leq w\}$

Let $x(p, w)$ be the consumer choice at (p, w).

Definition 2.2. The **Walrasian demand function**[3] is given by

$$x\colon \mathbb{R}^L_{++} \times \mathbb{R}_{++} \to \mathbb{R}^L_+$$

Assumptions 2.3.

(i) $x(p, w)$ is homogeneous of degree 0:

$$x(\alpha p, \alpha w) = x(p, w)$$

for all $(p, w) \in \mathbb{R}^L_{++} \times \mathbb{R}_{++}$ and $\alpha > 0$.

(ii) $x(p, w)$ satisfies Walras' Law:

$$p \cdot x(p, w) = w$$

for all $(p, w) \in \mathbb{R}^L_{++} \times \mathbb{R}_{++}$.

Proposition 2.4. *Let $\mathcal{B}^W := \{B_{p,w} \mid (p, w) \in \mathbb{R}^L_{++} \times \mathbb{R}_{++}\}$ and $C_x(B_{p,w}) := \{x(p, w)\}$ and let x be homogeneous of degree* 0 *and satisfy Walras' Law. Then $\left(\mathcal{B}^W, C_x\right)$ is a choice structure.*

**Proof.* We want to show that $C_x(B_{p,w})$ is a uniquely-defined nonempty subset of $B_{p,w}$ for all $B_{p,w} \in \mathcal{B}^W$. That $C_x(B_{p,w})$ is nonempty follows from the definition of x as a function. Homogeneity of degree 0 implies that for $B_{p,w} = B_{\alpha p, \alpha w}$, $C_x(B_{p,w}) = C_x(B_{\alpha p, \alpha w})$. Walras' Law implies that $C_x(B_{p,w}) \subseteq B_{p,w}$. $\square$

[3] If (p, w) does not uniquely specify a value, x, then we instead have the **Walrasian demand correspondence**, $X\colon \mathbb{R}^L_{++} \times \mathbb{R}_{++} \rightrightarrows \mathbb{R}^L_+$.

Definition 2.5. In the context of consumer choice, $x(p, w)$ satisfies the **weak axiom of revealed preference** if the following holds:
If (p, w), $(p', w') \in \mathbb{R}^L_{++} \times \mathbb{R}_{++}$ are such that $p' \cdot x(p, w) \leq w'$ and $x(p', w') \neq x(p, w)$, then $p \cdot x(p', w') > w$.

Remark 2.6. "In words, the weak axiom says that if x' is ever chosen when x is available, then there can be no budget set containing both alternatives for which x is chosen and x' is not." – MWG

Definition 2.7. A **Slutsky compensated price change** is a price change from p to p' accompanied by a change in wealth from w to w' that makes the old bundle just affordable. That is, such that $p' \cdot x(p, w) = w'$.

Remark 2.8. Recall that good i is normal at (p, w) if $\partial x_i / \partial w \geq 0$ and inferior at (p, w) if $\partial x_i / \partial w < 0$.

Proposition 2.9 (**Law of compensated demand**). *Suppose that consumer demand $x(p, w)$ is homogeneous of degree 0 and satisfies Walras' Law. Then $x(p, w)$ satisfies WARP if and only if for any compensated price change from (p, w) to $(p', w') := (p', p' \cdot x(p, w))$ we have*

$$(p' - p) \cdot (x(p', w') - x(p, w)) \leq 0$$

with strict inequality if $x(p', w') \neq x(p, w)$.

Proof. The wealth change can be defined as

$$\begin{aligned} \Delta w := w' - w &= p' \cdot x(p, w) - p \cdot x(p, w) \\ &= (p' - p) \cdot x(p, w) \\ &= \Delta p \cdot x(p, w) \end{aligned}$$

where $\Delta p := p' - p$. By WARP, $p \cdot x(p', w') \geq p \cdot x(p, w) = w$ with strict inequality if $x(p', w') \neq x(p, w)$. Note that $p' \cdot x(p', w') = p' \cdot x(p, w) = w'$ by Walras' Law and the definition of a compensated price change. Subtracting the two gives us:

$$\begin{aligned} & (p - p') \cdot x(p', w') \geq (p - p') \cdot x(p, w) \\ \implies & (p' - p) \cdot x(p', w') \leq (p' - p) \cdot x(p, w) \\ \implies & (p' - p) \cdot (x(p', w') - x(p, w)) \leq 0 \end{aligned}$$

Conversely, say

$$(p' - p) \cdot (x(p', w') - x(p, w < 0$$

Then

$$\begin{aligned}
&p' \cdot x(p', w') - p' \cdot x(p, w) - p \cdot (x(p', w') + p \cdot x(p, w)) < 0 \\
\implies &p \cdot (x(p', w') - x(p, w)) > 0 \\
\implies &p \cdot x(p', w') > w
\end{aligned}$$

The case of strict inequality is analogous. □

Proposition 2.10. *Let $x \colon \mathbb{R}^L_+ \times \mathbb{R}_+ \to \mathbb{R}^L_+$ be continuously differentiable. Then*

$$\frac{\partial x_j(p, w)}{\partial p_j} + x_j(p, w) \frac{\partial x_j(p, w)}{\partial w} \leq 0$$

Proof. Let p change solely in p_j, by Δp_j, and let Δw be the compensating changing in wealth, as above. Let $\Delta x := x(p', w') - x(p, w)$. Then, by the law of compensated demand,

$$\Delta p_j \Delta x_j = \Delta p \cdot \Delta x \leq 0$$

and

$$\begin{aligned}
&\Delta p_j (x_j(p', w') - x_j(p, w)) \leq 0 \\
\implies &\frac{x_j(p', w') - x_j(p, w)}{\Delta p_j} \leq 0
\end{aligned}$$

Write $x_j(p', w') - x_j(p, w) = x_j(p', w) - x_j(p, w) + x_j(p', w') - x_j(p', w)$. So

$$\frac{x_j(p', w) - x_j(p, w)}{\Delta p_j} + \frac{x_j(p', w') - x_j(p', w)}{\Delta p_j} \leq 0$$

Using $\Delta w = \Delta p_j x_j(p, w)$,

$$\frac{x_j(p', w) - x_j(p, w)}{\Delta p_j} + x_j(p, w) \frac{x_j(p', w') - x_j(p', w)}{\Delta w} \leq 0$$

Taking the limit as $\Delta p_j \to 0$ (implying $\Delta w \to 0$ and $p' \to p$) and using the continuity of partials of x_j, we have

$$\frac{\partial x_j(p, w)}{\partial p_j} + x_j(p, w) \frac{\partial x_j(p, w)}{\partial w} \leq 0$$

provided that x_j is continuously differentiable. □

Definition 2.11. The **Slutsky matrix**,

$$\begin{aligned}
S(p,w) &:= D_p x(p,w) + D_w x(p,w) x(p,w)^\mathsf{T} \\
&= \begin{bmatrix} \frac{\partial x_1}{\partial p_1} & \cdots & \frac{\partial x_1}{\partial p_L} \\ \vdots & & \vdots \\ \frac{\partial x_L}{\partial p_1} & \cdots & \frac{\partial x_L}{\partial p_L} \end{bmatrix} + \begin{bmatrix} \frac{\partial x_1}{\partial w} \\ \vdots \\ \frac{\partial x_L}{\partial w} \end{bmatrix} \begin{bmatrix} x_1(p,w) & \cdots & x_L(p,w) \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial x_1}{\partial p_1} & \cdots & \frac{\partial x_1}{\partial p_L} \\ \vdots & & \vdots \\ \frac{\partial x_L}{\partial p_1} & \cdots & \frac{\partial x_L}{\partial p_L} \end{bmatrix} + \begin{bmatrix} x_1 \frac{\partial x_1}{\partial w} & \cdots & x_L \frac{\partial x_1}{\partial w} \\ \vdots & & \vdots \\ x_1 \frac{\partial x_L}{\partial w} & \cdots & x_L \frac{\partial x_L}{\partial w} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial x_1}{\partial p_1} + x_1 \frac{\partial x_1}{\partial w} & \cdots & \frac{\partial x_1}{\partial p_L} + x_L \frac{\partial x_1}{\partial w} \\ \vdots & & \vdots \\ \frac{\partial x_L}{\partial p_1} + x_1 \frac{\partial x_L}{\partial w} & \cdots & \frac{\partial x_L}{\partial p_L} + x_L \frac{\partial x_L}{\partial w} \end{bmatrix}
\end{aligned}$$

Proposition 2.12. *$S(p,w)$ is negative semidefinite.*

Proof. Let $dp := (dp_1, \ldots, dp_L)$ be an arbitrary element of $\mathbb{R}^L$. Then for all i,

$$dx_i = \frac{\partial x_i}{\partial p_1} dp_1 + \cdots + \frac{\partial x_i}{\partial p_L} dp_L + \frac{\partial x_i}{\partial w} x_1(p,w) dp_1 + \cdots + \frac{\partial x_i}{\partial w} x_L(p,w) dp_L$$

$$\begin{aligned}
\implies dx &= D_p x(p,w) dp + D_w x(p,w) \left(x(p,w) \cdot dp \right) \\
&= \left(D_p x(p,w) + D_w x(p,w) x(p,w)^\mathsf{T} \right) dp
\end{aligned}$$

By WARP, $dp \cdot dx \leq 0$, hence

$$dp^\mathsf{T} \left(D_p x(p,w) + D_w x(p,w) x(p,w)^\mathsf{T} \right) dp \leq 0$$

Since dp is arbitrary, this implies $S(p,w)$ is negative semidefinite. □

2.1.1 Appendix: More on the Slutsky Matrix

The following propositions are from Sections 2.E and 2.F of MWG.

*Proposition A*2.13. If x is homogeneous of degree 0, then

$$D_p x(p,w) p + D_w x(p,w) w = 0$$

for all p and w.

Proof. Homogeneity of degree 0 implies

$$x(\alpha p, \alpha w) - x(p, w) = 0$$

for all $\alpha > 0$. Differentiating with respect to α and applying the chain rule, we obtain

$$D_{\alpha p} x(\alpha p, \alpha w) p + D_{\alpha w} x(\alpha p, \alpha w) w = 0$$

Since this is true for all $\alpha > 0$, it is true in particular for $\alpha = 1$. The result follows. $\square$

*Proposition A*2.14. If x satisfies Walras' law, then

$$\text{(i)} \;\; p^{\mathsf{T}} D_p x(p, w) + x(p, w)^{\mathsf{T}} = 0^{\mathsf{T}}$$

and

$$\text{(ii)} \;\; p^{\mathsf{T}} D_w x(p, w) = 1$$

Proof. Using matrix notation, Walras' law is

$$p^{\mathsf{T}} x = w$$

Differentiating with respect to p and applying the multiplication rule yields

$$p^{\mathsf{T}} D_p x + x^{\mathsf{T}} I = 0^{\mathsf{T}}$$

while differentiating with respect to w gives (ii). $\square$

*Proposition A*2.15. If x is homogeneous of degree 0 and satisfies Walras' law, then

$$p^{\mathsf{T}} S(p, w) = S(p, w) p = 0$$

Proof.

$$\begin{aligned} p^{\mathsf{T}} S(p, w) &= p^{\mathsf{T}} D_p x(p, w) + p^{\mathsf{T}} D_w x(p, w) x(p, w)^{\mathsf{T}} \\ &= -x(p, w)^{\mathsf{T}} + x(p, w)^{\mathsf{T}} \\ &= 0 \end{aligned}$$

where the second equality follows from the previous proposition. By first applying Walras' law and then Proposition A2.13, we also get

$$\begin{aligned} S(p, w) p &= D_p x(p, w) p + D_w x(p, w) x(p, w)^{\mathsf{T}} p \\ &= D_p x(p, w) p + D_w x(p, w) w \\ &= 0 \end{aligned}$$

$\square$

2.2 Consumer Choice from $\succsim$

As before, let $X := \mathbb{R}^L_+$.

Definition 2.16. A **utility function** representing $\succsim$ on X is a function $u\colon X \to \mathbb{R}$ such that for all $x, y \in X$:

$$u(x) \geq u(y) \text{ if and only if } x \succsim y$$

Proposition 2.17. *If $u\colon X \to R$ represents $\succsim$ on X and $f\colon \mathbb{R} \to \mathbb{R}$ is strictly increasing, then $f \circ u$ represents $\succsim$.*

**Proof.*

$$x \succsim y \iff u(x) \geq u(y) \iff (f \circ u)(x) \geq (f \circ u)(y)$$

□

Example 2.18. Lexicographic preferences, defined by

$$(x_1, x_2) \succsim (y_1, y_2) \text{ if and only if } x_1 > y_1 \text{ or } x_1 = y_1 \text{ and } x_2 \geq y_2$$

are rational but cannot be represented by a utility function.

Definition 2.19.

(i) The **upper contour set**, $R(x) := \{y \in X \mid y \succsim x\}$, is the set of all bundles that are at least as good as x. Denote its complement by $P^{-1}(x)$.

(ii) The **lower contour set**, $R^{-1}(x) = \{y \in X \mid x \succsim y\}$, is the set of all bundles that x is at least as good as. Denote its complement by $P(x)$.[4]

Definition 2.20. The preference relation $\succsim$ on X is **continuous** if $R(x)$ and $R^{-1}(x)$ are closed subsets of X for all $x \in X$.

Remark 2.21. Lexicographic preferences are not continuous. Let $x := (0, 2)$ and consider the sequence $((\frac{1}{n}, 1))_{n=1}^{\infty}$ in the upper contour set of x. This sequence converges to a point outside the upper contour set, $(0, 1) \prec (0, 2)$.

Proposition 2.22 (Debreu's theorem). *Suppose the rational preference relation $\succsim$ on X is continuous. Then there is a continuous utility function representing $\succsim$.*

[4] $R(x), R^{-1}(x), P(x)$ and $P^{-1}(x)$ are sometimes called the no-worse-than, no-better-than, better-than and worse-than sets, respectively.

Proof. We prove the proposition with the additional assumption of strict monotonicity (see Definition 2.23).[5]

Choose any $x \in \mathbb{R}^L_+$. Let $e = (1, 1, \ldots, 1)$. Define the ray from the origin as $Z := \{\alpha e \in \mathbb{R}^L_+ \mid \alpha \in \mathbb{R}_+\}$, which is closed. By continuity of $\succsim$, $R(x)$ and $R^{-1}(x)$ are also closed. If follows that $R(x) \cap Z$ and $R^{-1}(x) \cap Z$ are closed. Call them $S(x)$ and $S^{-1}(x)$. By monotonicity, $x \succsim 0$ and there exists $\bar{\alpha}$ such that $\bar{\alpha} e \succsim x$. Thus, $S(x)$ and $S^{-1}(x)$ are both nonempty. Furthermore, completeness also implies that $S(x) \cup S^{-1}(x) = Z$. But the ray Z is connected – that is, Z is not the disjoint union of two nonempty, closed subsets. Given that $S(x)$ and $S^{-1}(x)$ are nonempty and closed, they must not be disjoint. In other words, $S(x) \cap S^{-1}(x) = \{\alpha \in \mathbb{R}^+ \mid \alpha e \sim x\}$ is nonempty.

Define $u\colon \mathbb{R}^L_+ \to \mathbb{R}$ by $u(x) = \alpha$, where α is defined as above. Say $u(x) \geq u(y)$. Then $x \sim u(x)e \succsim u(y)e \sim y$ by monotonicity. Conversely, say $x \succsim y$. Then $u(x)e \succsim u(y)e$ and thus $u(x) \geq u(y)$ by monotonicity. Therefore, u is a utility function representing the preference relation $\succsim$.

To show that u is continuous, note that by continuity of $\succsim$, $P^{-1}(x)$ and $P(x)$ are open for any $x \in \mathbb{R}^L_+$. Fix x and y such that $x \succ y$. Our choice of y implies that y is in $P^{-1}(x)$ and openness of the latter implies so too is any y' sufficiently close to y. This in turn implies that $x \succ y'$ and so $x \in P(y')$, which is open. Thus for all x' sufficiently close to x, $x' \in P(y')$, or $x' \succ y'$. Now fix $\varepsilon > 0$ and note that $(u(x) + \varepsilon)e \succ x \succ (u(x) - \varepsilon)e$. From what we just found, we know that for x' sufficiently close to x, $(u(x) + \varepsilon)e \succ x' \succ (u(x) - \varepsilon)e$. By strict monotonicity, this implies $u(x) + \varepsilon > u(x') > u(x) - \varepsilon$. Therefore, u is continuous. □

Definition 2.23. The preference relation $\succsim$ is **monotone** if for all $x, y \in X$, $x \geq y$ implies $x \succsim y$.[6] It is **strictly monotone** if $x \geq y$ and $x \neq y$ implies $x \succ y$.

Remark 2.24. Strict monotonicity implies monotonicity.

Definition 2.25. The preference relation $\succsim$ is **locally non-satiated** if for every $x \in X$ and $\delta > 0$ there is a $y \in X$ such that $\|x - y\| \leq \delta$ and $y \succ x$.

[5]Note that this is a complete proof, based primarily on the proof of Proposition 3.C.1 in MWG and secondarily on sections 1.5 and 2.3 in Kreps. In the lecture, only a sketch of this proof was given.

[6]This is the definition used in Kreps; the definition in MWG is slightly different. MWG defines weak monotonicity as $x \gg y$ implying $x \succ y$, which Kreps calls "strict monotonicity for strict increases in the bundle."

Remark 2.26. Strict monotonicity implies local non-satiation.

Definition 2.27. The preference relation $\succsim$ on X is **convex** if for all $x, y, z \in X$ and all $\alpha \in [0,1]$,

$$y \succsim x \text{ and } z \succsim x \implies \alpha y + (1-\alpha)z \succsim x$$

It is **strictly convex** if for all $x, y, z \in X$ and all $\alpha \in (0,1)$,

$$y \succsim x \text{ and } z \succsim x \text{ and } y \neq z \implies \alpha y + (1-\alpha)z \succ x$$

Remark 2.28. Preferences are convex if and only if for every $x \in X$, $R(x)$ is a convex set.

Definition 2.29. The function $u\colon \mathbb{R}^L_+ \to \mathbb{R}$ is **quasiconcave** if for all $x, y \in \mathbb{R}^L_+$ and any $\alpha \in [0,1]$

$$u(\alpha x + (1-\alpha)y) \geq \min\{u(x), u(y)\}$$

Definition 2.30. The function $u\colon \mathbb{R}^L_+ \to \mathbb{R}$ is **concave** if for all $x, y \in \mathbb{R}^L_+$ and any $\alpha \in [0,1]$

$$u(\alpha x + (1-\alpha)y) \geq \alpha u(x) + (1-\alpha)u(y)$$

Remark 2.31. Strict quasiconcavity and strict concavity are defined by restricting α to be in $(0,1)$, requiring that $x \neq y$, and changing to a strict inequality in the above definitions.

Proposition 2.32. *u representing $\succsim$ is quasiconcave if and only if $\succsim$ is convex.*

**Proof.* Assuming quasiconcavity, $u(y), u(z) \geq u(x)$ implies $u(\alpha y + (1-\alpha)z) \geq \min\{u(y), u(z)\} \geq u(x)$. Conversely, suppose without loss of generality that $y \succsim z$. Note also that $z \succsim z$. Thus by convexity of preferences, $\alpha y + (1-\alpha)z \succsim z$. So $u(\alpha y + (1-\alpha)z) \geq u(z) = \min\{u(y), u(z)\}$. □

Remark 2.33. An analogous result holds if we make both properties strict.

2.2.1 Appendix: More on Utility Representations

*Proposition A*2.34. Suppose $\succsim$ is a rational preference relation on a set X (not necessarily $\mathbb{R}^L_+$). Any of the following properties of X are sufficient to guarantee the existence of a utility function representing $\succsim$:[7]

[7] Note that each of these properties generalizes the previous, so strictly speaking the sufficiency of (iii) implies the sufficiency of the other two.

(i) X is finite.

(ii) X is countable.

(iii) X has a countable subset, X^*, such that if $x, y \in X$ and $x \succ y$ then there exists $x^* \in X^*$ with $x \succsim x^* \succ y$.

Proof.

(i) Define $u(x) := \#R^{-1}(x)$ and note that $x \succsim y$ if and only if $R^{-1}(y) \subseteq R^{-1}(x)$.

(ii) Enumerate X as $\{x_1, x_2, \dots\}$ and define $d \colon X \to \mathbb{R}$ by $d(x_i) = \left(\frac{1}{2}\right)^i$. Then define $u(x) := \sum_{y \in R^{-1}(x)} d(y)$.

(iii) Enumerate X^* as $\{x_1^*, x_2^*, \dots\}$ and define $d \colon X^* \to \mathbb{R}$ by $d(x_i^*) = \left(\frac{1}{2}\right)^i$ and $u \colon X \to \mathbb{R}$ by $u(x) = \sum_{x^* \in X^* \cap R^{-1}(x)} d(x^*)$. □

*Proposition A*2.35. If $\succsim$ has a utility representation then it is rational.

Proof. For all $x, y \in X$, either $u(x) \geq u(y)$ or $u(y) \geq u(x)$, so $\succsim$ is complete. For all $x, y, z \in X$, $u(x) \geq u(y)$ and $u(y) \geq u(z)$ implies $u(x) \geq u(z)$. Thus $\succsim$ is transitive. □

*Proposition A*2.36. That $\succsim$ has a utility representation does not imply that $\succsim$ is continuous.

Proof. Let $X := \mathbb{R}_+$ and define the preference relation $\succsim$ by

$$x \succsim y \iff (x > 0 \text{ or } y = 0)$$

$R(1) = \mathbb{R}_{++}$, which is not closed. Thus $\succsim$ is not continuous. However, it does have a utility representation given by

$$u(x) := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

□

*Proposition A*2.37. If $\succsim$ can be represented by a continuous utility function u, then $\succsim$ is continuous.

Proof. If u is continuous, then $R(x) = u^{-1}[u(x), \infty)$ and $R^{-1}(x) = u^{-1}(-\infty, u(x)]$ are both closed. $\square$

Proposition A2.38. $\succsim$ is continuous and can be represented by u does not imply u is continuous.

Proof. This follows from Proposition 2.17. $\square$

Proposition A2.39. Lexicographic preferences cannot be represented by a utility function.

Proof. Suppose that they can. First note that any utility function would have to be injective, as a consumer with lexicographic preferences is never indifferent between distinct bundles. Note also that for all $x_1 \in \mathbb{R}$, $u(x_1, 0) < u(x_1, 1)$. Because the rational numbers are dense in the real line, we know that there is a rational number between $u(x_1, 0)$ and $u(x_1, 1)$. Call this rational number $q(x_1)$, and define an associated function, $q\colon \mathbb{R} \to \mathbb{Q}$. Clearly if $x_1 \neq x_2$ then $q(x_1) \neq q(x_2)$, so q is injective. This implies that the cardinality of $\mathbb{R}$ is less than or equal to that of $\mathbb{Q}$, a contradiction. $\square$

2.3 Consumer Optimization

Definition 2.40. The **consumer's problem** is the optimization problem

$$\begin{aligned} &\max_{x \in \mathbb{R}^L_+} u(x) \\ &\text{st } p \cdot x \leq w \end{aligned} \tag{2.1}$$

Proposition 2.41 (Properties of Walrasian demand correspondence)**.** *Let u be a continuous utility function representing $\succsim$ on $\mathbb{R}^L_+$.*

(i) If $p \in \mathbb{R}^L_{++}$ and $w \in \mathbb{R}_{++}$, then there exists an $x^* \in \mathbb{R}^L_+$ that solves the consumer's problem.

(ii) If $\lambda > 0$, then this x^* also solves the consumer's problem for λp and λw (homogeneity of degree 0).

(iii) If in addition, $\succsim$ satisfies local non-satiation, then $p \cdot x^* = w$ for any solution (Walras' Law).

(iv) If in addition, $\succsim$ is strictly convex then x^* is unique and the Walrasian demand function

$$x\colon \mathbb{R}^L_{++} \times \mathbb{R}_{++} \to \mathbb{R}^L_+$$

is well-defined and continuous.

**Proof.*

(i) $B_{p,w}$ is nonempty and compact and u is continuous so, by the extreme value theorem, u attains a maximum on $B_{p,w}$.

(ii) Observe that $p \cdot x \leq w \iff \lambda p \cdot x \leq \lambda w$, so the constraint set is the same in both problems.

(iii) Suppose not: suppose $p \cdot x^* < w$. Choose $\varepsilon > 0$ such that $\|x^* - y\| < \varepsilon$ implies $p \cdot y < w$. By local non-satiation, there exists y within ε distance of x^* such that $y \succ x$, which is a contradiction.

(iv) Let $\hat{x}$ be a distinct solution. Then by strict convexity of preferences, for all $\alpha \in (0,1)$, $\alpha x^* + (1-\alpha)\hat{x} \succ x^*$. Moreover, convexity of the budget set implies that $\alpha x^* + (1-\alpha)\hat{x}$ is affordable, so this is a contradiction. Continuity of x is proven in Kreps (Proposition 3.3). □

Proposition 2.42 (Necessary conditions)**.** *Suppose*

(i) The consumer's preferences on $\mathbb{R}^L_+$ can be represented by a twice continuously differentiable utility function u.

(ii) The preferences are strictly monotone.

(iii) $p \gg 0$ and $w > 0$

If x^* is an interior solution of (2.1)*, that is, a solution satisfying $x^* \gg 0$, then*

$$\mathrm{MRS}_{ij}(x^*) := \frac{\frac{\partial u(x^*)}{\partial x_i}}{\frac{\partial u(x^*)}{\partial x_j}} = \frac{p_i}{p_j}$$

Proof. Strict monotonicity implies $p \cdot x^* = w$ and $\frac{\partial u(x^*)}{\partial x_j} > 0$. We know x^* solves (2.1) and the constraint qualification holds. By the Karush-Kuhn-Tucker conditions, there exists $\lambda > 0$ such that $\nabla u(x^*) = \lambda p$. The result follows. □

Proposition 2.43 (Sufficient conditions)**.** *Suppose, in addition to hypotheses (i) to (iii) of the previous proposition, we have* *(iv) $\succsim$ are strictly convex.*

If x^* satisfies $x^* \gg 0$ and $p \cdot x^* = w$, and there exists a $\lambda > 0$ such that

$$\nabla u(x^*) = \lambda p$$

then x^* is the unique solution to the consumer's problem.

Proposition 2.44. *Suppose that $f\colon \mathbb{R}^n \to \mathbb{R}$ and $h\colon \mathbb{R}^n \to \mathbb{R}$ are twice continuously differentiable functions, and consider the problem*[8]

$$\begin{aligned} &\max_x f(x) \\ &st\ h(x) = 0 \end{aligned} \tag{2.2}$$

Suppose $x^* \in \mathbb{R}^n$ and $\lambda^* > 0$ are such that

$$\frac{\partial \mathcal{L}(x^*, \lambda^*)}{\partial x_i} = 0$$

for $i = 1, \ldots, n$, and

$$\frac{\partial \mathcal{L}(x^*, \lambda^*)}{\partial \lambda} = 0$$

where $\mathcal{L}(x, \lambda) := f(x) - \lambda h(x)$ is the Lagrangian associated with (2.2). Suppose further that the leading principal minors of the Hessian of $\mathcal{L}$ satisfy

$$\begin{vmatrix} 0 & \mathcal{L}_{\lambda x_1} & \mathcal{L}_{\lambda x_2} \\ \mathcal{L}_{x_1 \lambda} & \mathcal{L}_{x_1 x_1} & \mathcal{L}_{x_1 x_2} \\ \mathcal{L}_{x_2 \lambda} & \mathcal{L}_{x_2 x_1} & \mathcal{L}_{x_2 x_2} \end{vmatrix} > 0, \quad \begin{vmatrix} 0 & \mathcal{L}_{\lambda x_1} & \mathcal{L}_{\lambda x_2} & \mathcal{L}_{\lambda x_3} \\ \mathcal{L}_{x_1 \lambda} & \mathcal{L}_{x_1 x_1} & \mathcal{L}_{x_1 x_2} & \mathcal{L}_{x_1 x_3} \\ \mathcal{L}_{x_2 \lambda} & \mathcal{L}_{x_2 x_1} & \mathcal{L}_{x_2 x_2} & \mathcal{L}_{x_2 x_3} \\ \mathcal{L}_{x_3 \lambda} & \mathcal{L}_{x_3 x_1} & \mathcal{L}_{x_3 x_2} & \mathcal{L}_{x_3 x_3} \end{vmatrix} < 0, \ldots$$

at (x^*, λ^*). That is, for $r \geq 3$, the r-th order leading principal minor is positive if r is odd and negative if r is even. Then x^* is a strict local maximizer of $f(x)$ subject to the constraint $h(x) = 0$.

Proof. Omitted. □

Now we prove Proposition 2.43 using Proposition 2.44.

Proof of Proposition 2.43.

Set up the Lagrangian:

$$\mathcal{L}(\lambda, x) = u(x) + \lambda(w - p \cdot x)$$

[8] This is Theorem 5 in the optimization handout.

By hypothesis, x^* and λ^* satisfy the first-order conditions:

$$p \cdot x^* = w$$
$$u_i(x^*) := \frac{\partial u(x^*)}{\partial x_i} = \lambda^* p_i$$

for $i = 1, \ldots, L$, and the Hessian satisfies:

$$\begin{aligned}
\bar{H} &= \begin{vmatrix} 0 & \mathcal{L}_{\lambda x_1} & \ldots & \mathcal{L}_{\lambda x_L} \\ \mathcal{L}_{x_1 \lambda} & \mathcal{L}_{x_1 x_1} & \ldots & \mathcal{L}_{x_1 x_L} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}_{x_L \lambda} & \mathcal{L}_{x_L x_1} & \ldots & \mathcal{L}_{x_L x_L} \end{vmatrix} \\
&= \begin{vmatrix} 0 & -p_1 & \ldots & -p_L \\ -p_1 & u_{11} & \ldots & u_{1L} \\ \vdots & \vdots & \ddots & \vdots \\ -p_L & u_{L1} & \ldots & u_{LL} \end{vmatrix} \\
&= \begin{vmatrix} 0 & \frac{-u_1}{\lambda^*} & \ldots & \frac{-u_L}{\lambda^*} \\ \frac{-u_1}{\lambda^*} & u_{11} & \ldots & u_{1L} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-u_L}{\lambda^*} & u_{L1} & \ldots & u_{LL} \end{vmatrix} \\
&= \left(-\frac{1}{\lambda^*}\right)^2 \begin{vmatrix} 0 & u_1 & \ldots & u_L \\ u_1 & u_{11} & \ldots & u_{1L} \\ \vdots & \vdots & \ddots & \vdots \\ u_L & u_{L1} & \ldots & u_{LL} \end{vmatrix}
\end{aligned}$$

where the second-to-last inequality follows from the first-order conditions. Finally, by the strict quasiconcavity of u (or equivalently, the strict convexity of preferences) the determinant in the last line is positive if L is even and negative if L is odd, and its leading principal minors alternate sign. The conditions of Proposition 2.44 are met and thus x^* is a strict local maximizer.

Now we want to show that x^* is a strict *global* maximizer – in other words, that x^* is the unique solution to the consumer's problem. Suppose there is a distinct global maximizer $\bar{x}$. Then $\bar{x} \succsim x^*$. Let $\alpha \in (0, 1)$ and consider $x_\alpha = \alpha x^* + (1-\alpha)\bar{x}$. Since the budget set is convex, x_α is affordable. Moreover, $x_\alpha \succ x^*$ by the strict convexity of $\succsim$. This is true for α arbitrarily close to 1, so this contradicts x^* being a strict local maximizer. By the first part of Proposition 2.41, we know that there exists some global maximizer. We have shown that this global maximizer is unique and must be x^*. $\square$

2.4 Indirect Utility Function

Definition 2.45. The **indirect utility function**, $V\colon \mathbb{R}^L_{++} \times \mathbb{R}_{++} \to \mathbb{R}$, is defined by

$$V(p,w) := \max_{x \in \mathbb{R}^L_+} u(x)$$
$$\text{st } p \cdot x \le w$$

Remark 2.46. If $x(p,w)$ is a solution then

$$V(p,w) = u(x(p,w))$$

Assumptions 2.47.

(i) $\succsim$ on $\mathbb{R}^L_+$ are locally non-satiated

(ii) u is continuous

(iii) $p \gg 0$ and $w > 0$

Proposition 2.48 (Properties of V)**.**

(i) Continuous.

(ii) Nonincreasing in p_i for $i = 1, \dots, L$.

(iii) Strictly increasing in w.

(iv) Quasiconvex: that is, $\{(p,w) \mid V(p,w) \le k\}$ is a convex set, for all k.

(v) Homogeneous of degree 0.

**Proof.*

(i) In the case where the solution, x, is unique, $V = u \circ x$. Continuity of u is assumed, while continuity of x is proven in Proposition 2.41 (given continuity of u). The general case is proven in Kreps Proposition 3.3.

(ii) Fix i and suppose $p'_i \ge p_i$. Then $B_{p',w} \subseteq B_{p,w}$ and so $V(p',w) \le V(p,w)$.

(iii) Let $x := x(p,w)$ and suppose $w' > w$. Then $p \cdot x < w'$ and by local non-satiation, there exists $x' \succ x$ such that $p \cdot x' \le w'$. This implies that $V(p,w') \ge u(x') > u(x) = V(p,w)$.

(iv) Suppose

$$x \in B(\alpha p + (1-\alpha)p', \alpha w + (1-\alpha)w')$$

Then

$$\begin{aligned}
&\alpha p \cdot x + (1-\alpha)p' \cdot x \leq \alpha w + (1-\alpha)w' \\
\implies &\alpha(p \cdot x - w) + (1-\alpha)(p' \cdot x - w') \leq 0 \\
\implies &p \cdot x \leq w \text{ or } p' \cdot x \leq w' \\
\implies &x \in B_{p,w} \cup B_{p',w'} \\
\implies &B(\alpha p + (1-\alpha)p', \alpha w + (1-\alpha)w') \subseteq B_{p,w} \cup B_{p',w'} \\
\implies &V(\alpha p + (1-\alpha)p', \alpha w + (1-\alpha)w') \leq \max\{V(p,w), V(p',w')\} \\
\implies &V \text{ is quasiconvex}
\end{aligned}$$

(v) This follows from the Walrasian correspondence, X, being homogeneous of degree 0 (Proposition 2.41(ii)). □

Proposition 2.49. *If u and x are continuously differentiable, then so too is V and*

$$\frac{\partial V}{\partial w} = \lambda$$

where λ is the Lagrange multiplier in $\mathcal{L}(\lambda, x) = u(x) + \lambda(w - p \cdot x)$.

Proof. This can be seen immediately as an application of the envelope theorem

$$\frac{\partial V}{\partial w} = \frac{\partial u}{\partial w} + \lambda$$

Note that u is not a function of w, so

$$\frac{\partial u}{\partial w} = 0$$

The result follows. We can also obtain the result by applying the chain rule

$$\frac{\partial V}{\partial w} = \sum_{i=1}^{L} \frac{\partial u}{\partial x_i}\frac{\partial x_i}{\partial w} = \lambda \sum_{i=1}^{L} p_i \frac{\partial x_i}{\partial w} = \lambda$$

where the last equality uses differentiation of both sides of Walras' law with respect to w. □

Remark 2.50. This proposition gives economic meaning to the Lagrange multiplier: it is the marginal utility obtained from relaxing the budget constraint by one unit of income.

2.5 Expenditure Minimization

Definition 2.51. The **expenditure minimization problem** is the optimization problem

$$\min_{x \in \mathbb{R}^L_+} p \cdot x$$
$$\text{st } u(x) \geq \bar{u}$$

Definition 2.52. The associated value function, the **expenditure function**, is defined by

$$e(p, \bar{u}) := \min_{x \in \mathbb{R}^L_+} p \cdot x$$
$$\text{st } u(x) \geq \bar{u}$$

Definition 2.53. The **Hicksian demand correspondence**, $H\colon \mathbb{R}_{++} \times \mathbb{R} \rightrightarrows \mathbb{R}_+$ gives the solutions to the expenditure minimization problem:

$$H(p, \bar{u}) := \arg\min_{x \in \mathbb{R}^L_+} p \cdot x$$
$$\text{st } u(x) \geq \bar{u}$$

If $H(p, \bar{u})$ is singleton-valued for all p and $\bar{u}$, then we have the **Hicksian demand function**, $h\colon \mathbb{R}_{++} \times \mathbb{R} \to \mathbb{R}_+$.

Proposition 2.54 (Properties of Hicksian demand correspondence)**.** *Let preferences be continuous.*

(i) If $u(0) \leq \bar{u} \leq \sup_{x \in \mathbb{R}^L_+} u(x)$ where the right-hand side is possibly infinite, then there exists an $h^* \in \mathbb{R}^L_+$ that solves the EMP.

(ii) If $\lambda > 0$, then this h^* also solves the consumer's problem for λp and λw (homogeneity of degree 0).

(iii) If h^* solves the EMP, then $u(h^*) = \bar{u}$.

(iv) If in addition, $\succsim$ is strictly convex then h^* is unique and the Hicksian demand function

$$h\colon \mathbb{R}^L_{++} \times \mathbb{R} \to \mathbb{R}^L_+$$

is well-defined and continuous.

**Proof.*

(i) By the continuity of u and the intermediate value theorem, there exists $x^0 \in \mathbb{R}^L_+$ such that $u(x^0) = \bar{u}$. We can then restrict the constraint set to $\{x \in R^L_+ \mid u(x) \geq \bar{u} \text{ and } p \cdot x \leq p \cdot x^0\}$ without changing the solution. This set is nonempty and compact, so we can apply the extreme value theorem.

(ii) This follows from $p \cdot h^* \geq p \cdot x \iff \lambda p \cdot h^* \geq \lambda p \cdot x$.

(iii) Suppose $u(h^*) > \bar{u}$. Then, by continuity of u, there exists $x \neq h^*$ such that $x \leq h^*$ and $\bar{u} < u(x) < u(h^*)$. This implies that x is in the constraint set and $p \cdot x < p \cdot h^*$, contradicting our choice of h^*.

(iv) The existence proof is identical to that of Proposition 2.41(iv), noting that the constraint set is an interval in $\mathbb{R}$ and therefore convex. Continuity of h is proven in Kreps (Proposition 10.3). □

Proposition 2.55 (Properties of e)**.**

(i) Continuous.

(ii) Nondecreasing in p_i for $i = 1, \ldots, L$.

(iii) Strictly increasing in $\bar{u}$.

(iv) Homogeneous of degree 1 in p.

(v) Concave in p.

**Proof.*

(i) See Kreps Proposition 10.3b.

(ii) Let $p' \geq p$ and $h' \in H(p', \bar{u})$. Then

$$e(p', \bar{u}) = p' \cdot h' \geq p \cdot h' \geq e(p, \bar{u})$$

(iii) Suppose not. Then there exists $\bar{u}, \bar{u}'$ such that $\bar{u}' > \bar{u}$, and such that for $x \in H(p, \bar{u})$, $h' \in H(p, \bar{u}')$, we have $p \cdot x \geq p \cdot h'$. By continuity of u, for sufficiently large $\alpha \in (0, 1)$, $u(\alpha h') > u(x)$. But $p \cdot x \geq p \cdot h' > p \cdot \alpha h'$, contradicting the definition of x.

(iv) This follows from H being homogeneous of degree 0.

(v) Let $p'' := \alpha p + (1 - \alpha)p'$ and $h'' \in H(p'', \bar{u})$. Then

$$e(p'', \bar{u}) = p'' \cdot h'' = \alpha p \cdot h'' + (1 - \alpha)p' \cdot h'' \geq \alpha e(p, \bar{u}) + (1 - \alpha)e(p', \bar{u}) \quad \square$$

Henceforth, we assume 2.47.

Proposition 2.56. *Assume $\succsim$ is continuous and locally non-satiated. Then:*

(i) $H(p, V(p, w)) = X(p, w)$

(ii) $X(p, e(p, \bar{u})) = H(p, \bar{u})$

(iii) $e(p, V(p, w)) = w$

(iv) $V(p, e(p, \bar{u})) = \bar{u}$

**Proof.*

(i) Fix p and w. Let x^* and h^* be arbitrary elements of $X(p, w)$ and $H(p, V(p, w))$, respectively. Since $u(x^*) = V(p, w)$ and h^* minimizes expenditure over consumption bundles satisfying $u(x) \geq V(p, w)$, we must have $p \cdot h^* \leq p \cdot x^* \leq w$. By Proposition 2.54(iii), $u(h^*) = V(p, w)$. Thus, h^* is a utility-maximizing element of $B_{p,w}$. In other words, $h^* \in X(p, w)$. By Walras' law, $p \cdot h^* = w = p \cdot x^*$, so x^* is an expenditure-minimizing consumption bundle satisfying $u(x^*) \geq V(p, w)$. In other words, $x^* \in H(p, V(p, w))$. Since x^* and h^* are arbitrary, $H(p, V(p, w)) = X(p, w)$.

(ii) Fix p and $\bar{u} \in [u(0), \sup u(x)]$. Let x^* and h^* be arbitrary elements of $X(p, e(p, \bar{u}))$ and $H(p, \bar{u})$, respectively. Since $p \cdot h^* = e(p, \bar{u})$ and x^* maximizes utility over $B(p, e(p, \bar{u}))$, $u(x^*) \geq u(h^*) = \bar{u}$. Moreover, by definition $p \cdot x^* \leq e(p, \bar{u}) = p \cdot h^*$. Thus, x^* also minimizes expenditure over the constraint set. That is, $x^* \in H(p, \bar{u})$. Then $u(x^*) = \bar{u} = u(h^*)$. Clearly, $h^* \in B(p, e(p, \bar{u}))$ so $h^* \in X(p, e(p, \bar{u}))$.

(iii) Let $h^* \in H(p, V(p, w)) = X(p, w)$. Then $e(p, V(p, w)) = p \cdot h^* = w$.

(iv) Let $x^* \in X(p, e(p, \bar{u}))$. Then $V(p, e(p, \bar{u})) = u(x^*) = \bar{u}$. □

Corollary 2.57. *If x and h are well-defined:*

(i) $h(p, V(p, w)) = x(p, w)$

(ii) $x(p, e(p, \bar{u})) = h(p, \bar{u})$

Proposition 2.58 (Shephard's lemma)**.** *In addition to Assumptions 2.47, suppose that $\succsim$ are strictly convex and that $e \in C^1$. Then for $p \gg 0$*

$$h_i(p, \bar{u}) = \frac{\partial e(p, \bar{u})}{\partial p_i}$$

for $i = 1, \ldots, L$.

Proof. Fix some u^* and some $p^* \gg 0$. Let $x^* = h(p^*, u^*)$ and define $g(p) = e(p, u^*) - p \cdot x^*$. Note that $g(p) \leq 0$ for all p and $g(p^*) = 0$. So, $g(p)$ is maximized at p^*. Hence,

$$\frac{\partial g(p^*)}{\partial p_i} = \frac{\partial e(p^*, u^*)}{\partial p_i} - x_i^* = 0$$

for $i = 1, \ldots, L$. This implies that

$$\frac{\partial e(p^*, u^*)}{\partial p_i} = h_i(p^*, u^*) \qquad \square$$

Remark 2.59. Note that for a given $\bar{u} \in \mathbb{R}$, $e(p, \bar{u})$ is not invariant to positive monotonic transformations of the function u. For example, replacing $u(x)$ with $v(x) := 2u(x)$ will change the value of $e(p, \bar{u})$.

Proposition 2.60 (**Roy's identity**)**.** *In addition to Assumptions 2.47, suppose that $\succsim$ are strictly convex and that $e, V \in C^1$. Then for $p \gg 0$*

$$x_i(p, w) = -\frac{\frac{\partial V(p,w)}{\partial p_i}}{\frac{\partial V(p,w)}{\partial w}}$$

for $i = 1, \ldots, L$.

Proof. Let $p^* \gg 0$ and $u^* := u(x(p^*, w^*)) = V(p^*, w^*)$. Then by Proposition 2.56, $u^* = V(p^*, e(p^*, u^*))$. Taking the derivative with respect to p_i and evaluating at p_i^*, we obtain

$$0 = \frac{\partial V(p^*, w^*)}{\partial p_i} + \frac{\partial V(p^*, w^*)}{\partial w} \frac{\partial e(p^*, u^*)}{\partial p_i}$$

So

$$x_i(p^*, w^*) = h_i(p^*, u^*) = \frac{\partial e(p^*, u^*)}{\partial p_i} = -\frac{\frac{\partial V(p^*,w^*)}{\partial p_i}}{\frac{\partial V(p^*,w^*)}{\partial w}} \qquad \square$$

2.6 The Slutsky Equation

Proposition 2.61 (**The Slutsky equation**)**.** *Suppose e and V are both twice continuously differentiable. Fix p and w, and let $u^* := V(p, w)$. Then*

$$\frac{\partial x_i(p, w)}{\partial p_j} = \underbrace{\frac{\partial h_i(p, u^*)}{\partial p_j}}_{\textit{substitution effect}} \underbrace{- x_j(p, w) \frac{\partial x_i(p, w)}{\partial w}}_{\textit{income effect}} \tag{2.3}$$

Proof. From Proposition 2.56, we have $h_i(p,u^*) = x_i(p, e(p,u^*))$. Differentiating both sides with respect to p_j, we obtain

$$\frac{\partial h_i(p,u^*)}{\partial p_j} = \frac{\partial x_i(p,w)}{\partial p_j} + \frac{\partial x_i(p,w)}{\partial w}\frac{\partial e(p,u^*)}{\partial p_j}$$

Now use $\frac{\partial e(p,u^*)}{\partial p_j} = h_j(p,u^*) = x_j(p,w)$. □

Remark 2.62. In matrix form, the Slutsky equation is

$$\begin{bmatrix} \frac{\partial x_1(p,w)}{\partial p_1} & \cdots & \frac{\partial x_1(p,w)}{\partial p_L} \\ \vdots & & \vdots \\ \frac{\partial x_L(p,w)}{\partial p_1} & \cdots & \frac{\partial x_L(p,w)}{\partial p_L} \end{bmatrix} = \begin{bmatrix} \frac{\partial h_1(p,u^*)}{\partial p_1} & \cdots & \frac{\partial h_1(p,u^*)}{\partial p_L} \\ \vdots & & \vdots \\ \frac{\partial h_L(p,u^*)}{\partial p_1} & \cdots & \frac{\partial h_L(p,u^*)}{\partial p_L} \end{bmatrix} - \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial w} \\ \vdots \\ \frac{\partial x_L(p,w)}{\partial w} \end{bmatrix} \begin{bmatrix} x_1(p,w) & \cdots & x_L(p,w) \end{bmatrix}$$

or equivalently

$$D_p x(p,w) = D_p h(p,u^*) - D_w x(p,w) x(p,w)^\mathsf{T}$$

Comparing with Definition 2.11, we can see that

$$D_p h(p, V(p,w)) = S(p,w)$$

Proposition 2.63. *$S(p,w)$ is negative semidefinite.*[9] *If e is twice continuously differentiable, then $S(p,w)$ is symmetric.*

Proof. For symmetry, we have

$$\begin{aligned} S(p,w) = D_p h(p,u^*) = \left[\frac{\partial h_i(p,u^*)}{\partial p_j}\right]_{ij} &= \left[\frac{\partial^2 e(p,u^*)}{\partial p_j \partial p_i}\right]_{ij} = \left[\frac{\partial^2 e(p,u^*)}{\partial p_i \partial p_j}\right]_{ij} \\ &= \left[\frac{\partial h_j(p,u^*)}{\partial p_i}\right]_{ij} = D_p h(p,u^*)^\mathsf{T} = S(p,w)^\mathsf{T} \end{aligned}$$

where the fourth equality follows from Young's theorem. The above sequence of equalities also shows that $S(p,w)$ is the Hessian of e. Therefore, negative semidefiniteness of $S(p,w)$ follows from the concavity of e. □

[9] This part of the proof is just an alternative way of proving Proposition 2.12.

2.7 Envelope Theorem

Theorem 2.64 (**Envelope theorem**). *Let*

$$V(a) := \max_{x \in \mathbb{R}^n_+} f(x, a)$$
$$st\ g(x, a) = 0$$

Then

$$\frac{\partial V(a)}{\partial a} = \frac{\partial f(x^*(a), a)}{\partial a} + \lambda^*(a) \frac{\partial g(x^*(a), a)}{\partial a}$$

where (x^*, λ^*) *is the solution to the maximization problem.*

Proof. Let $x(a) := \arg\max_{x \in \mathbb{R}^n_+} f(x, a)$ st $g(x, a) = 0$. Then $V(a) = f(x(a), a)$ and

$$\begin{aligned}
\frac{\partial V(a)}{\partial a} &= \sum_{i=1}^{n} \left(\frac{\partial f(x(a), a)}{\partial x_i} \frac{\partial x_i(a)}{\partial a} \right) + \frac{\partial f(x(a), a)}{\partial a} \\
&= \lambda(a) \sum_{i=1}^{n} \left(\frac{\partial g(x(a), a)}{\partial x_i} \frac{\partial x_i(a)}{\partial a} \right) + \frac{\partial f(x(a), a)}{\partial a} \\
&= \lambda(a) \frac{\partial g(x(a), a)}{\partial a} + \frac{\partial f(x(a), a)}{\partial a}
\end{aligned}$$

where the second equality uses the first-order conditions and the third uses the derivative of the constraint equation with respect to a. □

2.8 Integrability

Remark 2.65. Suppose $x \colon \mathbb{R}^L_{++} \times \mathbb{R}_{++} \to \mathbb{R}^L_+$ is a continuously differentiable Marshallian demand function generated by rational, continuous and locally non-satiated preferences. We know that x must satisfy:

(i) Walras' law: $p \cdot x(p, w) = w$

(ii) Homogeneity of degree 0

(iii) Symmetric, negative semidefinite Slutsky matrix,

$$\left[\frac{\partial x_i(p, w)}{\partial p_j} + x_j(p, w) \frac{\partial x_i(p, w)}{\partial w} \right]_{ij}$$

We want to show that these necessary conditions are also sufficient for the existence of rational generating preferences.

Proposition 2.66 (Recovering the expenditure function from demand)**.** *Suppose $x\colon \mathbb{R}^L_{++} \times \mathbb{R}_{++} \to \mathbb{R}^L_+$ is a continuously differentiable function satisfying (i) to (iii) above. Then we can recover an expenditure function satisfying:*

$$\frac{\partial e(p, \bar{u})}{\partial p_i} = x_i(p, e(p, \bar{u}))$$

for $i = 1, 2, \ldots, L$ and for all p and all w.

Proof. Fix some $(p^0, w^0) \in \mathbb{R}^L_{++} \times \mathbb{R}_{++}$. Let $x^0 = x(p^0, w^0)$. Assign utility $\bar{u}^0$ to x^0 (we can do this because preferences are invariant under a positive monotonic transformation of the utility function). Using Propositions 2.56 and 2.58, we obtain the following system of partial differential equations:

$$\frac{\partial e(p, \bar{u}^0)}{\partial p_i} = x_i(p, e(p, \bar{u}^0))$$

for $i = 1, \ldots, L$, with initial condition:

$$e(p^0, \bar{u}^0) = w^0$$

A result from the theory of partial differential equations, Frobenius' theorem, tells us that this system has a solution $e(p, \bar{u}^0)$ if assumption (iii), symmetry of the Slutsky matrix, holds. Moreover, the solution $e(p, \bar{u}^0)$ is concave in p by negative semidefiniteness of the Slutsky matrix. □

Remark 2.67. Once we have obtained the expenditure function, we can use Proposition 2.56 to obtain the indirect utility function. Namely, from the identity $e(p, V(p, w)) = w$.

Proposition 2.68 (Recovering preferences from the indirect utility function)**.** *For some (p^0, w^0), let $x^0 := x(p^0, w^0)$ be such that $x^0 \gg 0$. Then*

$$\begin{aligned} u(x^0) &= \min_p V(p, w^0) \\ st\ p \cdot x^0 &= w^0 \end{aligned}$$

Proof. Consider any p such that $p \cdot x^0 = w^0$. As x^0 is feasible at p,

$$x(p, w^0) \succsim x^0$$

and

$$V(p, w^0) \geq V(p^0, w^0) = u(x^0)$$

So p^0 minimizes $V(p, w^0)$ subject to $p \cdot x^0 = w^0$ and $u(x^0)$ equals the minimized value. □

Example 2.69. Suppose we are given a demand function,

$$x_i(p, w) := \frac{\alpha_i w}{p_i}$$

where $\sum_{i=1}^{L} \alpha_i = 1$ and $\alpha_i > 0$ for $i = 1, \ldots, L$. First we recover the expenditure function from x.

$$\frac{\partial e(p, \bar{u})}{\partial p_i} = h_i(p, \bar{u}) = x_i(p, e(p, \bar{u})) = \frac{\alpha_i e(p, \bar{u})}{p_i}$$

This gives

$$\frac{\partial \log e(p, \bar{u})}{\partial p_i} = \frac{1}{e(p, \bar{u})} \frac{\partial e(p, \bar{u})}{\partial p_i} = \frac{\alpha_i}{p_i}$$

Integrating with respect to p_i, we obtain

$$\log e(p, \bar{u}) = \alpha_i \log p_i + C_i(p_{-i}, \bar{u})$$

Following the same steps we could also obtain

$$\log e(p, \bar{u}) = \alpha_j \log p_j + C_j(p_{-j}, \bar{u})$$

Observe that the "constant" of integration C_i includes the term $\alpha_j \log p_j$ for all $j \neq i$ as well as some "constant" in $\bar{u}$. Hence,

$$\log e(p, \bar{u}) = \sum_{i=1}^{L} \alpha_i \log p_i + C(\bar{u})$$

Let $p^0 := (1, \ldots, 1)$. Then,

$$\log e(p^0, \bar{u}) = C(\bar{u})$$

Note that e is strictly increasing in u, so without loss of generality we can transform the utility function $u(x)$ to $e(p^0, u(x))$. Thus we can rewrite

$$\log e(p, \bar{u}) = \sum_{i=1}^{L} \alpha_i \log p_i + \log \bar{u}$$

implying

$$e(p, \bar{u}) = p_1^{\alpha_1} \ldots p_L^{\alpha_L} \bar{u}$$

Next, we recover the indirect utility function from e. We know that $e(p, V(p, w)) = w$. Thus,

$$\log w = \log e(p, V(p, w)) = \sum_{i=1}^{L} \alpha_i \log p_i + \log V(p, w)$$

and

$$V(p,w) = \frac{w}{p_1^{\alpha_1} \dots p_L^{\alpha_L}}$$

We now apply Proposition 2.68:

$$\begin{aligned} u(x) &= \min_p \frac{w}{p_1^{\alpha_1} \dots p_L^{\alpha_L}} \\ &\text{st } p \cdot x = w \end{aligned}$$

To solve the minimization problem, we set up the Lagrangian:

$$\mathcal{L}(p) = -w \prod_{i=1}^{L} p_i^{-\alpha_i} + \lambda(p \cdot x - w)$$

and obtain the first order conditions

$$w\alpha_i p_i^{-\alpha_i - 1} \prod_{j \neq i} p_j^{-\alpha_j} + \lambda x_i = 0$$

for $i = 1, \dots, L$. These imply

$$\frac{\alpha_i}{\alpha_j} = \frac{x_i}{x_j} \frac{p_i}{p_j}$$

for all i, j. Summing across j, using the fact that the α_j sum to 1 and applying Walras' law, we get

$$p_i = \frac{\alpha_i}{x_i} w$$

Plug back in to get

$$u(x) = \frac{w}{p_1^{\alpha_1} \dots p_L^{\alpha_L}} = \frac{w}{wC} \prod_{i=1}^{L} x_i^{\alpha_i} = \frac{1}{C} \prod_{i=1}^{L} x_i^{\alpha_i}$$

where $C := \prod_{i=1}^{L} \alpha_i^{\alpha_i}$. Because $C > 0$, we can represent the same preferences by

$$u(x) := \prod_{i=1}^{L} x_i^{\alpha_i}$$

2.9 Welfare

Remark 2.70. Consider a change in price and income from (p^0, w^0) to (p^1, w^1). We want to know what effect this has on the consumer's welfare. We might compare $V(p^0, w^0)$ to $V(p^1, w^1)$. However, V isn't uniquely determined by the consumer's preferences: it depends on our choice of u to represent those preferences, which is unique only up to a positive monotonic transformation.

Assumptions 2.71. We assume that the consumer's preferences are rational, continuous, and locally non-satiated.

Remark 2.72. Note that, for fixed $\bar{p}$, $e(\bar{p}, V(p, w))$ is a valid indirect utility function, as it is strictly increasing in V. Moreover, it is invariant under a positive monotonic transformation of u. That is, if V and V' are indirect utility functions derived from utility functions U and U' representing the same preference, then $e(\bar{p}, V(p, w)) = e(\bar{p}, V'(p, w))$.

Definition 2.73. A **money metric indirect utility function** is an indirect utility function of the form $e(\bar{p}, V(p, w))$, for some fixed $\bar{p}$.

What $\bar{p}$ should we choose? Henceforth, we only consider changes in prices – wealth is fixed at w. Let prices change from p^0 to p^1. Let $u^0 := V(p^0, w)$ and $u^1 := V(p^1, w)$.

Definition 2.74. The **compensating variation** is the amount of money, CV, such that the consumer is indifferent between having w at the old prices and $w - CV$ at the new prices.

$$CV(p^0, p^1, w) := e(p^1, u^1) - e(p^1, u^0) = w - e(p^1, u^0)$$

Definition 2.75. The **equivalent variation** is the amount of money, EV, such that the consumer is indifferent between having w at the new prices and $w + EV$ at the old prices.

$$EV(p^0, p^1, w) := e(p^0, u^1) - e(p^0, u^0) = e(p^0, u^1) - w$$

Remark 2.76. Both compensating and equivalent variations are positive when the price changes make the consumer better off and negative when the price changes make the consumer worse off.

Proposition 2.77. *Suppose the price of only one good changes. Without loss of generality, let that good have index 1. Then*

$$EV(p^0, p^1, w) = \int_{p_1^1}^{p_1^0} h_1(p_1, p_{-1}, u^1) dp_1$$

and

$$CV(p^0, p^1, w) = \int_{p_1^1}^{p_1^0} h_1(p_1, p_{-1}, u^0) dp_1$$

Proof. We assume that h_1 is well-defined and integrable with respect to p_1. In fact, this can be proven from the hypotheses of the proposition: see Kreps Proposition 12.10.

$$\begin{aligned}
EV(p^0, p^1, w) &= e(p^0, u^1) - w \\
&= e(p^0, u^1) - e(p^1, u^1) \\
&= \int_{p_1^1}^{p_1^0} h_1(p_1, p_{-1}, u^1) dp_1 \\
CV(p^0, p^1, w) &= w - e(p^1, u^0) \\
&= e(p^0, u^0) - e(p^1, u^0) \\
&= \int_{p_1^1}^{p_1^0} h_1(p_1, p_{-1}, u^0) dp_1
\end{aligned}$$

□

2.9.1 Quasilinear Preferences

Definition 2.78. Let $X := \mathbb{R}_+^2$ and normalize $p_2 := 1$. Let $u\colon \mathbb{R}_+^2 \to \mathbb{R}$ be given by

$$u(x, y) = v(x) + y$$

where v is some function of x. The preferences represented by u are called **quasilinear preferences**.

Remark 2.79. In practice, v will often have a "nice" functional form such as $\log x$ or $\sqrt{x}$.

Proposition 2.80. *Suppose that the consumer has quasilinear preferences with strictly increasing, strictly concave, continuously differentiable v that fixes 0 ($v(0) = 0$) and has the Inada condition: $\lim_{x \downarrow 0} v'(x) = +\infty$. Suppose that the price of good 1 changes from p_1^0 to p_1^1 and that at both prices we have an interior solution. Then the associated compensating and equivalent variations are equal:*

$$CV(p^0, p^1, w) = EV(p^0, p^1, w)$$

Proof. Following Proposition 2.77, it suffices to show that $h_1(p, u^0) = h_1(p, u^1)$ for $p_1^0 \leq p_1 \leq p_1^1$. This is true if $x_1(p, e(p, u^0)) = x_1(p, e(p, u^1))$. It suffices, then, to show that $x_1(p, w)$ does not depend upon w for $p_1^0 \leq p_1 \leq p_1^1$. The consumer's problem is the following:

$$\max_x v(x) + w - p_1 x$$

This has an interior solution if $w > p_1 x$ (the Inada condition prevents a boundary solution at $x = 0$). By assumption, this is true for p_1^0 and p_1^1, and thus must be true for any p_1 between those prices. Moreover, we know that v is differentiable, so $x(p, w)$ can be obtained from the first order condition:

$$v'(x) - p_1 = 0$$

This condition implies that, for p_1 in this interval, $x(p, w)$ does not depend upon w, as required. □

2.10 Aggregation

Assumptions 2.81.

(i) I Consumers $i = 1, \dots, I$.

(ii) Individual preferences, $\succsim_i$, are rational and locally non-satiated.

(iii) Individual wealths $w^i > 0$.

(iv) Aggregate wealth $w := \sum_{i=1}^{I} w^i$

(v) Prices $p \gg 0$

Definition 2.82. We define **aggregate demand** by

$$x(p, w^1, \dots, w^I) = \sum_{i=1}^{I} x^i(p, w^i)$$

Definition 2.83. A **representative consumer** exists if the aggregate demand function, $x(p, w)$, is the Walrasian demand function generated by some rational preference relation $\succsim$ on X, with wealth equal to the aggregate wealth.

Remark 2.84. Walras' law and homogeneity of degree 0 hold true in the aggregate. WARP may not. For example, if

$$x_j^1(p, w^1) := \begin{cases} w^1/p_j & \text{if } p_j < p_k \text{ for } k \neq j \\ 0 & \text{otherwise} \end{cases}$$

and

$$x_j^2(p, w^2) := \frac{w^2}{Lp_j}$$

then aggregate demand for good j is given by

$$x_j(p, w^1, w^2) := \begin{cases} \frac{Lw^1+w^2}{Lp_j} & \text{if } p_j < p_k \text{ for } k \neq j \\ \frac{w^2}{Lp_j} & \text{otherwise} \end{cases}$$

The individual consumer's choices satisfy WARP. However, the candidate representative consumer will make different choices from the same budget set, B_{p,w^1+w^2}, depending upon the relative values of w^1 and w^2, violating WARP.

Definition 2.85. An indirect utility function is said to have the **Gorman form** (in some region) if it is of the form

$$V^i(p, w^i) := a^i(p) + b(p)w^i$$

(in that region) where $b(p) > 0$ for all p and both $a^i(p)$ and $b(p)$ are twice continuously differentiable.

Proposition 2.86. *An indirect utility function having the Gorman form has a Walrasian demand function of the form*

$$x_j^i(p, w^i) := A_j^i(p) + B_j(p)w^i$$

where $A_j^i(p) := -\frac{1}{b(p)}\frac{\partial a^i(p)}{\partial p_j}$ *and* $B_j(p) := -\frac{1}{b(p)}\frac{\partial b(p)}{\partial p_j}$.

Proof. Use Roy's identity. □

Proposition 2.87. *If* $I \geq 2$ *and there exists a representative consumer, then individual demand functions have the Gorman form.*

Proof. First we show that the Walrasian demand functions are affine in wealth:

$$\frac{\partial^2 x_j^i}{\partial (w^i)^2} = \frac{\partial}{\partial w^i}\left(\frac{\partial x_j}{\partial w^i}\right) = \frac{\partial}{\partial w^i}\left(\frac{\partial x_j}{\partial w^h}\right) = \frac{\partial}{\partial w^i}\left(\frac{\partial x_j^h}{\partial w^h}\right) = 0$$

where the last equality follows from the fact that x^i is not a function of w^h, and thus neither are its derivatives. We can therefore write

$$x_j^i(p, w^i) = A_j^i(p) + B_j^i(p)w^i$$

Moreover, if aggregate demand is a function of w and not the relative distribution of individual wealths, then we can write

$$\frac{\partial x_j^i}{\partial w^i} = \frac{\partial x_j}{\partial w^i} = \frac{\partial x_j}{\partial w}\frac{\partial w}{\partial w^i} = \frac{\partial x_j}{\partial w}\frac{\partial w}{\partial w^h} = \frac{\partial x_j}{\partial w^h} = \frac{\partial x_j^h}{\partial w^h}$$

which implies that the coefficient on w^i in $x^i(p, w^i)$ is equal to that on w^h in $x^h(p, w^h)$, or

$$x_j^i(p, w^i) = A_j^i(p) + B_j(p)w^i \qquad \square$$

Definition 2.88. A Walrasian demand function $x(p, w)$ satisfies the **uncompensated law of demand** if for any w and any price change from p to p' we have

$$(p' - p) \cdot (x(p', w) - x(p, w)) \leq 0$$

with inequality if $x(p', w) \neq x(p, w)$.

Remark 2.89. Contrast this with the *compensated* law of demand. Here, there is no compensating change in wealth.

Proposition 2.90. *If every consumer's Walrasian demand function satisfies the uncompensated law of demand, then $x(p, w)$ also satisfies the uncompensated law of demand and WARP.*

**Proof.*

(i) First we prove that if all the individual demands satisfy the ULD, then so to does aggregate demand. Say aggregate demand $x(p, w) \neq x(p', w)$. Then there exists some i such that $x^i(p, w^i) \neq x^i(p', w^i)$. By hypothesis, x^i satisfies the ULD, so

$$(p' - p) \cdot (x^i(p', w^i) - x^i(p, w^i)) < 0$$

for all such i. Summing over i, we get

$$(p' - p) \cdot (x(p', w) - x(p, w)) < 0.$$

(ii) Now we prove that if a Walrasian demand function satisfies the ULD, then it also satisfies WARP. Suppose again that $x(p, w) \neq x(p', w')$ and $x(p', w')$ is feasible at (p, w): $p \cdot x(p', w') \leq w$. Let $p'' := \frac{w}{w'}p'$. By homogeneity of degree 0,

$$x(p'', w) = x\left(\frac{w}{w'}p', w\right) = x(p', w')$$

By ULD,

$$(p'' - p) \cdot (x(p'', w) - x(p, w)) < 0$$

Multiplying out the left-hand side and applying Walras' law and our assumption that $p \cdot x(p', w') \leq w$, this becomes

$$w - p'' \cdot x(p, w) - w + w < 0$$

or equivalently

$$\frac{w}{w'} p' \cdot x(p, w) = p'' \cdot x(p, w) > w$$

which implies

$$p' \cdot x(p, w) > w'$$

Thus WARP is satisfied.

□

Proposition 2.91. *If the indirect utility function of each consumer has the Gorman form globally, then there exists a representative consumer.*

**Proof.* The aggregate demand function

$$x_j\,(p, w) = \sum_{i=1}^{I} A_j^i(p) + B_j(p) w$$

depends only on the sum, and not the relative distribution of, $(w^1, \ldots, w^I)$. Note that if we fix any p, because each x^i is a valid Walrasian demand function,

$$0 = \lim_{w^i \downarrow 0} x_j^i(p, w^i) = A_j^i(p) = -\frac{1}{b(p)} \frac{\partial a^i(p)}{\partial p_j}$$

for all j. Therefore, we must have

$$x_j^i(p, w^i) = B_j(p) w^i$$

and

$$V^i(p, w^i) = a^i + b(p) w^i$$

The latter implies

$$0 \geq \frac{\partial V^i}{\partial p_j} = \frac{\partial b}{\partial p_j} w^i$$

which in turn implies

$$0 \leq B_j(p) = \frac{\partial x_j^i}{\partial w^i}$$

Because x^i is a valid Walrasian demand function, we know it must satisfy WARP and therefore the Hicksian demand for each good is nonincreasing in own price (Proposition 2.10). Recall the Slutsky equation (2.3):[10]

$$\frac{\partial x_j^i(p,w)}{\partial p_j} = \frac{\partial h_j^i(p,u^*)}{\partial p_j} - x_j^i(p,w)\frac{\partial x_j^i(p,w)}{\partial w}$$

From what we've shown, we know that the right-hand side is nonpositive, so $x_j^i(p,w)$ is nonincreasing in p_j for all j. It follows immediately that x^i satisfies the uncompensated law of demand, for each i. By Proposition 2.90, this is sufficient to ensure that $x(p,w)$ satisfies WARP. This in turn implies that the associated Slutsky matrix, $S(p,w)$, is negative semidefinite (Proposition 2.12). Moreover, we know that Walras' law and homogeneity of degree 0 hold in the aggregate. If follows from Section 2.8 (Integrability) that $x(p,w)$ is a Walrasian demand function generated by some rational preferences. $\square$

[10] Specifically, for a change in own price.

3 Production

3.1 Firm

Assumptions 3.1.

(i) L commodities

(ii) Production plan $y \in \mathbb{R}^L$

  (1) Net input: good i such that $y_i < 0$

  (2) Net output: good j such that $y_j > 0$

(iii) Production possibility set, $Y \subseteq \mathbb{R}^L$ of feasible production plans

(iv) Prices, $p \geq 0$, are unaffected by the activity of the firm.

We will also often assume:

Assumptions 3.2.

(i) Y is nonempty, closed and (strictly) convex.[11]

(ii) Free disposal: If $y \in Y$ and $y' \leq y$ then $y' \in Y$.

Definition 3.3. A production plan, $y \in Y$ is **efficient** if there does not exist $y' \in Y$ such that $y' \geq y$ and $y'_i > y_i$ for some i.

In the case of a single output, we partition y into output $q \in \mathbb{R}_+$ and inputs $z \in \mathbb{R}_+^{L-1}$. This allows us to define the following:

Definition 3.4. The **production function** $f \colon \mathbb{R}_+^{L-1} \to \mathbb{R}_+$ is defined by

$$\begin{aligned} f(z) = \max q \\ \text{st } (q, -z) \in Y \end{aligned}$$

Definition 3.5. The **input requirement set**

$$V(q) := \{z \in \mathbb{R}_+^{L-1} \mid (q, -z) \in Y\}$$

gives all the input vectors that can be used to produce the output q.

[11]These properties are required for the general existence and/or uniqueness of the maximizers and minimizers defined in this section. In particular, strict convexity allows us to speak exclusively of demand and supply *functions*, rather than correspondences. An additional more technical property, the recession-cone property, is also required: see Kreps Proposition 9.7.

Definition 3.6. The **isoquant**

$$Q(q) := \{z \in \mathbb{R}_+^{L-1} \mid z \in V(q) \text{ and } z \notin V(q') \text{ for any } q' > q\}$$

gives all the input vectors that can be used to produce at most q units of output.

3.2 Cost minimization

Assumptions 3.7.

(i) $L - 1$ inputs z

(ii) One output $q = f(z)$

(iii) $f \in C^2$

(iv) Input price $w \in \mathbb{R}_+^{L-1}$

Remark 3.8. Inputs with zero prices will not affect the decision-making of the firm and can thus be ignored.

The firm's **cost minimization problem** is

$$\begin{aligned} &\min_{z \in \mathbb{R}_+^{L-1}} w \cdot z \\ &\text{st } f(z) = q \end{aligned}$$

Definition 3.9. The associated value function is called the **cost function**:

$$\begin{aligned} C(w, q) := &\min_{z \in \mathbb{R}_+^{L-1}} w \cdot z \\ &\text{st } f(z) = q \end{aligned}$$

Proposition 3.10 (Properties of the cost function)**.**

(i) C is homogeneous of degree 1 in w.

(ii) C is concave in w.

(iii) If we assume free disposal, then C is nondecreasing in q.

(iv) If f is homogeneous of degree k in z, then C is homogeneous of degree $\frac{1}{k}$ in q.

**Proof.*

(i) Increasing w by a factor of α is a positive monotonic transformation and therefore does not affect the optimal choice of z, but does increase $w \cdot z$ by that factor.

(ii) Let $w, w' \in \mathbb{R}_+^{L-1}$. Suppose $C(w,q) = w \cdot z$ and $C(w',q) = w' \cdot z'$. Let $w'' = \alpha w + (1-\alpha)w'$ for some $\alpha \in [0,1]$. Then, for z'' a cost minimizer at w'',

$$\begin{aligned} C(w'', q) &= w'' \cdot z'' \\ &= (\alpha w + (1-\alpha)w') \cdot z'' \\ &= \alpha w \cdot z'' + (1-\alpha) w' \cdot z'' \end{aligned}$$

We know $w \cdot z'' \geq C(w,q)$ and $w' \cdot z'' \geq C(w',q)$. So $C(w'',q) \geq \alpha C(w,q) + (1-\alpha)C(w',q)$.

(iii) Suppose $q' > q$. By free disposal, q can be produced from the same input vector used to produce q'.

(iv) Homogeneity of degree k of f implies

$$f(z) = q \iff \frac{1}{q} f(z) = 1 \iff f\left(\frac{z}{q^{1/k}}\right) = 1$$

Therefore,

$$\begin{aligned} C(w,q) &= \min_z w \cdot z \text{ st } f(z) = q \\ &= \min_z w \cdot z \text{ st } f\left(\frac{z}{q^{1/k}}\right) = 1 \\ &= q^{1/k} \min_z w \cdot \frac{z}{q^{1/k}} \text{ st } f\left(\frac{z}{q^{1/k}}\right) = 1 \\ &= q^{1/k} C(w,1) \end{aligned}$$

□

3.3 Homogeneous functions

Definition 3.11. $f \colon X \subseteq \mathbb{R}^n \to \mathbb{R}$ is **homogeneous of degree k** if

$$f(\alpha x) = \alpha^k f(x)$$

where k is a nonnegative integer, for all $\alpha > 0, x \in X$

Proposition 3.12. *If f is homogeneous of degree k, then for $i = 1, 2, \ldots, n$, $\frac{\partial f}{\partial x_i}$ is homogeneous of degree $k-1$.*

**Proof.* Let $f_i := \frac{\partial f}{\partial x_i}$.

$$\begin{aligned}
f(\alpha x) &= \alpha^k f(x) && \text{hod k} \\
\alpha f_i(\alpha x) &= \alpha^k f_i(x) && \text{differentiating wrt } x_i \\
f_i(\alpha x) &= \alpha^{k-1} f_i(x) && \text{dividing by } \alpha \\
\implies f_i(\alpha x) &\text{ is homogenous of degree } k-1
\end{aligned}$$

□

Proposition 3.13 (**Euler's formula**)**.** *If f is homogeneous of degree k and differentiable, then at any x*

$$\sum_{i=1}^{n} \frac{\partial f(x)}{\partial x_i} x_i = k f(x)$$

**Proof.*

$$\begin{aligned}
f(\alpha x) &= \alpha^k f(x) && \text{hod k} \\
\sum_{i=1}^{n} f_i(\alpha x) x_i &= k\alpha^{k-1} f_i(x) && \text{differentiating wrt } \alpha \\
\sum_{i=1}^{n} f_i(x) x_i &= k f_i(x) && \text{evaluating at } \alpha = 1
\end{aligned}$$

□

Proposition 3.14. *If the production function f is homogeneous of degree k, then*

$$\mathrm{MRTS}_{ij}(z) := \frac{\frac{\partial f(z)}{\partial z_i}}{\frac{\partial f(z)}{\partial z_j}} = \frac{\frac{\partial f(\alpha z)}{\partial z_i}}{\frac{\partial f(\alpha z)}{\partial z_j}} = \mathrm{MRTS}_{ij}(\alpha z)$$

Proof.

$$\frac{f_i(\alpha z)}{f_j(\alpha z)} = \frac{\alpha^{k-1} f_i(z)}{\alpha^{k-1} f_j(z)} = \frac{f_i(z)}{f_j(z)}$$

□

3.4 Profit maximization

The firm's **profit maximization problem** is

$$\max_y p \cdot y$$
$$\text{st } y \in Y$$

Definition 3.15. The associated value function is called the **profit function**:

$$\pi(p) := \max_y p \cdot y$$
$$\text{st } y \in Y$$

In the single-output case, this becomes

$$\pi(p, w) := \max_{z \in \mathbb{R}_+^{L-1}} pf(z) - w \cdot z$$

Henceforth, we consider only the single-output case.

Proposition 3.16 (Properties of the profit function)**.**

(i) Homogeneous of degree 1.

(ii) Nondecreasing in output price p.

(iii) Nonincreasing in input prices w.

(iv) Convex in (p, w).

(v) Continuous.

**Proof.*

(i) $\max_z \alpha(pf(z) - w \cdot z) = \alpha \max_z pf(z) - w \cdot z$.

(ii) $p' \geq p \implies p'f(z) \geq pf(z)$ for all z.

(iii) $w' \geq w \implies w' \cdot z \geq w \cdot z$.

(iv) Let $(p'', w'') := \alpha(p, w) + (1 - \alpha)(p', w')$ and z, z', z'' be the solution to the profit maximization problem with the corresponding output prices and input price vectors. Then by definition of z and z',

$$\pi(p, w) = pf(z) - w \cdot z \geq pf(z'') - w \cdot z''$$
$$\pi(p', w') = p'f(z') - w' \cdot z' \geq p'f(z'') - w' \cdot z''$$

implying

$$\begin{aligned}\alpha\pi(p,w)+(1-\alpha)\pi(p',w') &\geq \alpha(pf(z'')-w\cdot z'')\\ &\quad +(1-\alpha)(p'f(z'')-w'\cdot z'')\\ &=(\alpha p+(1-\alpha)p')f(z'')\\ &\quad -(\alpha w+(1-\alpha)w')\cdot z''\\ &=\pi(p'',z'')\end{aligned}$$

(v) See Kreps Proposition 9.9. □

Remark 3.17. Note that π being convex in (p,w) implies that π is convex in p and w individually.

Definition 3.18. The **unconditional input demand function**

$$x(p,w) := \arg\max_{z\in\mathbb{R}_+^{L-1}} pf(z)-w\cdot z$$

is the solution to the profit maximization problem. The **output supply function**

$$q(p,w) := f(x(p,w))$$

is the output level when the profit is maximized.

Proposition 3.19 (**Hotelling's lemma**)**.** *If π is differentiable,*[12] *then for $(p,w)\in\mathbb{R}_{++}^L$,*

$$\begin{aligned}q(p,w)&=\frac{\partial\pi(p,w)}{\partial p}\\ x_j(p,w)&=-\frac{\partial\pi(p,w)}{\partial w_j}\end{aligned}$$

Proof. Apply the Envelope Theorem and note that $x(p,w)$ is the profit maximizer and $q(p,w)$ is the production function evaluated at the maximizer. □

[12] In fact, if the output supply and unconditional demand functions are well-defined – or equivalently, the associated correspondences are singleton-valued – then π is necessarily differentiable. See Kreps Proposition 9.22. An analogous result holds for the cost function: see Kreps Proposition 9.24j.

Definition 3.20. The **conditional input demand function**

$$z(w,q) := \arg\min_{z\in\mathbb{R}^{L-1}_+} w\cdot z$$
$$\text{st } f(z) = q$$

is the solution to the cost minimization problem.

Proposition 3.21 (**Shepard's lemma**)**.** *If C is differentiable, then for $w \in \mathbb{R}^{L-1}_{++}$,*

$$z_i(w,q) = \frac{\partial C(w,q)}{\partial w_i}$$

Proof. Similarly, apply the Envelope Theorem to the cost function (the value function of the cost minimization problem). Note that the Envelope Theorem also holds true for minimization problems. Equivalently, we can rewrite

$$-C(w,q) := \max_{z\in\mathbb{R}^{L-1}_+} -w\cdot z \text{ st } f(z) = q.$$

and apply the regular Envelope Theorem. $\square$

Proposition 3.22. *Suppose that the profit function is twice continuously differentiable. Then*

(i)
$$\frac{\partial q(p,w)}{\partial p} \geq 0$$

(ii)
$$\frac{\partial x_j(p,w)}{\partial w_j} \leq 0$$

(iii)
$$\frac{\partial x_j(p,w)}{\partial w_i} = \frac{\partial x_i(p,w)}{\partial w_j}$$

Proof. By applying Hotelling's lemma, note that

$$D^2\pi(p,w) = \begin{bmatrix} \frac{\partial q(p,w)}{\partial p} & \frac{\partial q(p,w)}{\partial w_1} & \cdots & \frac{\partial q(p,w)}{\partial w_n} \\ -\frac{\partial x_1(p,w)}{\partial p} & -\frac{\partial x_1(p,w)}{\partial w_1} & \cdots & -\frac{\partial x_1(p,w)}{\partial w_n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\partial x_n(p,w)}{\partial p} & -\frac{\partial x_n(p,w)}{\partial w_1} & \cdots & -\frac{\partial x_n(p,w)}{\partial w_n} \end{bmatrix}$$

is symmetric and positive semidefinite because the profit function π is twice continuously differentiable and convex. Then, (i) and (ii) follows from the fact that a positive semidefinite matrix has nonnegative diagonal entries, and (iii) follows from symmetry. $\square$

Proposition 3.23. *Suppose that the cost function is twice continuously differentiable. Then*

(i)

$$\frac{\partial z_i(w,q)}{\partial w_i} \leq 0$$

(ii)

$$\frac{\partial z_i(w,q)}{\partial w_j} = \frac{\partial z_j(w,q)}{\partial w_i}$$

(iii)

$$\frac{\partial MC(w,q)}{\partial w_i} = \frac{\partial z_i(w,q)}{\partial q} \implies \begin{cases} > 0 & \textit{Normal Input} \\ < 0 & \textit{Inferior Input} \end{cases}$$

where $MC(w,q) = \frac{\partial C(w,q)}{\partial q}$.

Proof. Using Shepard's lemma, write the Hessian of C as

$$D^2C(w,q) = \begin{bmatrix} \frac{\partial MC(w,q)}{\partial q} & \frac{\partial MC(w,q)}{\partial w_1} & \cdots & \frac{\partial MC(w,q)}{\partial w_n} \\ \frac{\partial z_1(w,q)}{\partial q} & \frac{\partial z_1(w,q)}{\partial w_1} & \cdots & \frac{\partial z_1(w,q)}{\partial w_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial z_n(w,q)}{\partial q} & \frac{\partial z_n(w,q)}{\partial w_1} & \cdots & \frac{\partial z_n(w,q)}{\partial w_n} \end{bmatrix}$$

Then, (ii) and (iii) follow from the symmetry of the second derivatives. Since C is concave in w, the sub-matrix

$$\begin{bmatrix} \frac{\partial z_1(w,q)}{\partial w_1} & \cdots & \frac{\partial z_1(w,q)}{\partial w_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial z_n(w,q)}{\partial w_1} & \cdots & \frac{\partial z_n(w,q)}{\partial w_n} \end{bmatrix}$$

is negative semidefinite and its diagonal entries must be nonpositive. This proves (i). $\square$

3.5 Comparative Statics

Assumptions 3.24.

(i) Two inputs (x_1, x_2)

(ii) One output $q = f(x)$

(iii) $f \in C^2$ and the Hessian, H_f, is negative definite

(iv) $f(0, x_2) = f(x_1, 0) = 0$, *i.e.*, both inputs necessary

(v) Inada conditions on x_1, x_2

(vi) Output price $p > 0$

(vii) Input price $w \gg 0$

Consider the profit maximization problem:

$$\max_{x \in \mathbb{R}^2_{++}} pf(x) - w \cdot x$$

The first order conditions are

$$\begin{aligned} pf_1(x) - w_1 = 0 \\ pf_2(x) - w_2 = 0 \end{aligned}$$

Since the Lagrangian is strictly concave, the first order conditions are sufficient. To determine the sign of $\frac{\partial x_1(p,w)}{\partial w_1}$, we apply the Implicit Function Theorem. Since H_f is negative definite,

$$H(x) = \begin{bmatrix} pf_{11}(x) & pf_{12}(x) \\ pf_{21}(x) & pf_{22}(x) \end{bmatrix}$$

has strictly positive determinant. This satisfies the condition for the IFT, so there exists an implicit function

$$x(p, w) = (x_1(p, w), x_2(p, w))$$

which is C^1 near (x, p, w). Writing x as an implicit function of (p, w), we have

$$\begin{aligned} pf_1(x(p, w)) - w_1 = 0 \\ pf_2(x(p, w)) - w_2 = 0 \end{aligned}$$

Taking the derivative with respect to w_1 gives

$$\begin{aligned} pf_{11}\frac{\partial x_1}{\partial w_1} + pf_{12}\frac{\partial x_2}{\partial w_1} - 1 &= 0 \\ pf_{21}\frac{\partial x_1}{\partial w_1} + pf_{22}\frac{\partial x_2}{\partial w_1} &= 0 \end{aligned}$$

Writing it in matrix form,

$$\begin{bmatrix} pf_{11} & pf_{12} \\ pf_{21} & pf_{22} \end{bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial w_1} \\ \frac{\partial x_2}{\partial w_1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Since the first matrix is non-singular, invert it and find that

$$\frac{\partial x_1}{\partial w_1} = \frac{pf_{22}}{|H(x)|} < 0$$
$$\frac{\partial x_2}{\partial w_1} = \frac{-pf_{12}}{|H(x)|}$$

where the inequality is from the negative definiteness of H_f. Thus we have shown that the demand for an input always decreases with its price.

To determine the effect of a price change on output, *i.e.*, the sign of $\frac{\partial q}{\partial w_1}$, we write

$$q(p,w) = f(x(p,w))$$

and take the derivative with respect to w_1:

$$\begin{aligned} \frac{\partial q}{\partial w_1} &= f_1 \frac{\partial x_1}{\partial w_1} + f_2 \frac{\partial x_2}{\partial w_1} \\ &= \frac{p(f_1 f_{22} - f_2 f_{12})}{|H(x)|} \end{aligned}$$

where the sign depends on the term $f_1 f_{22} - f_2 f_{12}$. To find this, we consider the cost minimization problem:

$$\begin{aligned} &\min_{x \in \mathbb{R}^2_{++}} \; w \cdot x \\ &\text{st } f(x) = q \end{aligned}$$

The first order conditions are

$$\begin{aligned} -w_1 + \lambda f_1(x) &= 0 \\ -w_2 + \lambda f_2(x) &= 0 \\ q - f(x) &= 0 \end{aligned}$$

where $\lambda(w,q)$ is the Lagrangian multiplier. Taking the derivative with respect to q gives:

$$\begin{aligned} \frac{\partial \lambda}{\partial q} f_1 + \lambda \frac{\partial f_1}{\partial x_1} \frac{\partial x_1}{\partial q} + \lambda \frac{\partial f_1}{\partial x_2} \frac{\partial x_2}{\partial q} &= 0 \\ \frac{\partial \lambda}{\partial q} f_2 + \lambda \frac{\partial f_2}{\partial x_1} \frac{\partial x_1}{\partial q} + \lambda \frac{\partial f_2}{\partial x_2} \frac{\partial x_2}{\partial q} &= 0 \\ 1 - \frac{\partial f_1}{\partial x_1} \frac{\partial x_1}{\partial q} - \frac{\partial f_1}{\partial x_2} \frac{\partial x_2}{\partial q} &= 0 \end{aligned}$$

Writing it in matrix form,

$$\begin{bmatrix} \lambda f_{11} & \lambda f_{12} & f_1 \\ \lambda f_{21} & \lambda f_{22} & f_2 \\ f_1 & f_2 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial q} \\ \frac{\partial x_2}{\partial q} \\ \frac{\partial \lambda}{\partial q} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

where the first matrix is the Hessian of the Lagrangian, $H_c(x)$, and is thus invertible. By Cramer's Rule,

$$\begin{aligned} \frac{\partial x_1}{\partial q} &= \frac{\begin{vmatrix} 0 & \lambda f_{12} & f_1 \\ 0 & \lambda f_{22} & f_2 \\ 1 & f_2 & 0 \end{vmatrix}}{|H_c(x)|} \\ &= \frac{\lambda(f_{12}f_2 - f_{22}f_1)}{|H_c(x)|} \quad \begin{cases} > 0 & \text{Normal Input} \\ < 0 & \text{Inferior Input} \end{cases} \end{aligned}$$

where λ and $|H_c(x)|$ are strictly positive. Combined with the result from the profit maximization problem, we conclude:

(i) If input 1 is normal, $\frac{\partial x_1}{\partial q} > 0$, then $f_{12}f_2 - f_{22}f_1 > 0$ and $\frac{\partial q}{\partial w_1} < 0$.

$$w_1 \uparrow \implies q \downarrow \implies x_1 \downarrow$$

(ii) If input 1 is inferior, $\frac{\partial x_1}{\partial q} < 0$, then $f_{12}f_2 - f_{22}f_1 < 0$ and $\frac{\partial q}{\partial w_1} > 0$.

$$w_1 \uparrow \implies q \uparrow \implies x_1 \downarrow$$

In either case, this reinforces the substitution effect where x_1 necessarily decreases when w_1 increases, keeping output level q fixed.

3.6 Duality

Fix an output level q and suppose we observe $C(w, q)$ for all $w \gg 0$. We can recover an "outer bound" of the (unobserved) input requirement set,

$$V^*(q) := \{x \in \mathbb{R}_+^{L-1} \mid w \cdot x \geq C(w, q) \text{ for all } w \in \mathbb{R}_{++}^{L-1}\}$$

Proposition 3.25. *$V^*(q)$ is convex.*

Proof. Suppose $x, x' \in V^*(q)$. Let $\alpha \in [0,1]$ and $x'' := \alpha x + (1-\alpha)x'$. We want to show that $x'' \in V^*(q)$. Since $x \in V^*(q)$, $w \cdot x \geq C(w,q)$ for all $w \in \mathbb{R}^{L-1}_{++}$ and similarly $x' \in V^*(q)$ implies $w \cdot x' \geq C(w,q)$ for all $w \in \mathbb{R}^{L-1}_{++}$. Then

$$w \cdot x'' = \alpha w \cdot x + (1-\alpha) w \cdot x' \geq C(w,q)$$

Thus $x'' \in V^*(q)$. □

Remark 3.26. This doesn't imply that the true input requirement set $V(q)$ is convex, but it does imply that the non-convex part of $V(q)$ is not economically relevant since a cost-minimizing firm would never choose something in that region of $V(q)$.

Proposition 3.27 (Relationship between $V(q)$ and $V^*(q)$)**.**

(i) $V(q) \subseteq V^*(q)$*.*

(ii) If $V(q)$ *is closed, convex and comprehensive upward,*[13] *then* $V(q) = V^*(q)$*.*

**Proof.*

(i) Suppose $x \notin V^*(q)$. We want to show $x \notin V(q)$. If $x \notin V^*(q)$ then there exists some $w \in \mathbb{R}^{L-1}_{++}$ such that $w \cdot x < C(w,q)$. If $x \in V(q)$ then $C(w,q)$ is not the minimum, contradicting the definition of C.

(ii) Suppose not. In particular, suppose $x \in V^*(q)$ and $x \notin V(q)$. $V(q)$ and $\{x\}$ are both closed, convex, disjoint, nonempty subsets of $\mathbb{R}^{L-1}$ and $\{x\}$ is compact. Applying a version of the separating hyperplane theorem,[14] we obtain $w^* \neq 0$ such that $w^* \cdot x < w^* \cdot x'$ for all $x' \in V(q)$. In particular, $w^* \cdot x < C(w^*, q)$, which contradicts the definition of $V^*(q)$. We also want to show that $w^* \geq 0$. Suppose instead that for some i, $w^*_i < 0$. Because $V(q)$ is comprehensive upward, this implies we can choose $x' \in V(q)$ with x_i sufficiently large that $w^* \cdot x' < w^* \cdot x$, contradicting our choice of w^*.

□

Now, let

$$C^*(w,q) := \min_{x \in V^*(q)} w \cdot x$$

[13] $V(q)$ is comprehensive upward if $x \in V(q)$ and $x' \geq x$ imply $x' \in V(q)$. That is, the same output can always be produced using more input. If Y has the free disposal property, then for all q, $V(q)$ is comprehensive upward. The converse is not true. See Kreps Proposition 9.23c.

[14] Covered in the math class.

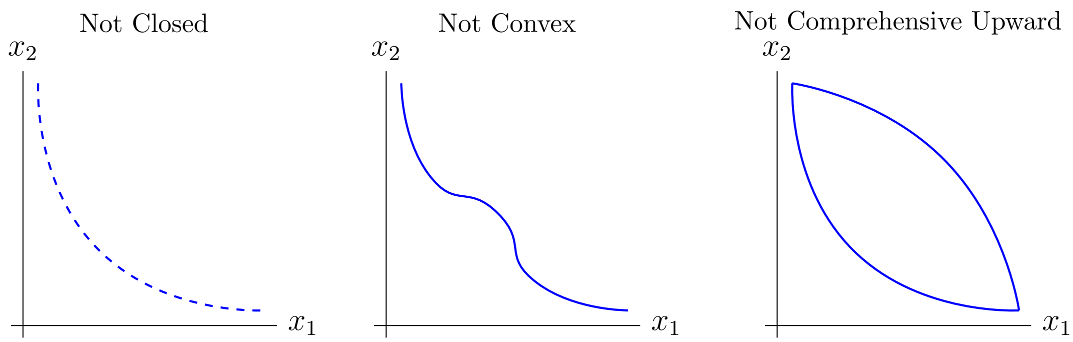

Figure 1: Three classes of $V(q)$ for which $V(q) \neq V^*(q)$

Proposition 3.28.
$$C^*(w, q) = C(w, q)$$

Proof. $V(q) \subseteq V^*(q)$ implies $C(w, q) \geq C^*(w, q)$. Suppose that for some $\bar{w} \in \mathbb{R}^{L-1}_{++}$, we have $C^*(\bar{w}, q) = \bar{w} \cdot \bar{x} < C(\bar{w}, q)$. Then $\bar{x} \notin V^*(q)$ which contradicts the definition of C^*. This implies $C^*(w, q) \geq C(w, q)$ for all $w \in \mathbb{R}^{L-1}_{++}$. Combining both inequalities, we have $C^*(w, q) = C(w, q)$. $\square$

4 Risk, Uncertainty, Ambiguity

4.1 Decision Making Under Uncertainty

Assumptions 4.1.

(i) Finite set of prizes, $X := \{x_1, x_2, \ldots, x_n\}$.

(ii) Set of probability distributions, P, on X. That is, P is the subset of $\mathbb{R}^n_+$ consisting of all $p := (p_1, \ldots, p_n)$ such that $\sum_{i=1}^n p_i = 1$. We interpret p_i as $\Pr(x_i)$ for $i = 1, \ldots, n$. Elements of P are also called simple probability distributions or simple lotteries.

(iii) We can also have *compound* lotteries – probability distributions over elements of P. These are regarded as identical to the simple lotteries to which they can be reduced. For example, if $X := \{x_1, x_2\}$, $p := (\frac{1}{2}, \frac{1}{2})$ and $q := (\frac{1}{4}, \frac{3}{4})$, then the compound lottery $c := \left[\Pr(p) = \frac{2}{3}, \Pr(q) = \frac{1}{3}\right]$ is treated as the simple lottery $s := (\frac{5}{12}, \frac{7}{12})$. We often write $c = \frac{2}{3}p + \frac{1}{3}q$.

(iv) Preferences, $\succsim$, over P.

Definition 4.2. The preference relation $\succsim$ on P is **rational** if it is complete and transitive.

Definition 4.3. The preference relation $\succsim$ on P is **continuous** if for any $p, q, r \in P$, the sets $\{\alpha \in [0,1] \mid \alpha p + (1-\alpha)q \succsim r\}$ and $\{\alpha \in [0,1] \mid r \succsim \alpha p + (1-\alpha)q\}$ are closed.

Definition 4.4. The preference relation $\succsim$ on P satisfies **independence** if for all $p, q, r \in P$ and $\alpha \in (0,1)$,

$$p \succsim q \iff \alpha p + (1-\alpha)r \succsim \alpha q + (1-\alpha)r$$

Lemma 4.5. *If $\succsim$ is complete and continuous, then for any $p, q, r \in P$ such that $p \succ r \succ q$, there exists $\alpha \in (0,1)$ such that $\alpha p + (1-\alpha)q \sim r$.*

**Proof.* Continuity implies that the sets $\{\alpha \in [0,1] \mid \alpha p + (1-\alpha)q \succsim r\}$ and $\{\alpha \in [0,1] \mid r \succsim \alpha p + (1-\alpha)q\}$ are closed. Completeness implies that they are nonempty and partition $[0,1]$, which is connected. Connectedness implies that $[0,1]$ cannot be partitioned into two nonempty, closed, and disjoint subsets. Therefore, the sets cannot be disjoint. Thus, their intersection $\{\alpha \in [0,1] \mid \alpha p + (1-\alpha)q \sim r\}$ is nonempty. $\square$

Lemma 4.6. *If $\succsim$ satisfies independence, then for any $p, q \in P$ such that $p \succ q$, and for any $\alpha, \beta \in [0,1]$,*

$$\beta p + (1-\beta)q \succ \alpha p + (1-\alpha)q \iff \beta > \alpha \tag{4.1}$$

Proof. ($\Longleftarrow$) Let $\gamma := \frac{\beta-\alpha}{1-\alpha}$

$$\begin{aligned}
\beta p + (1-\beta)q &= \beta\left(\frac{1-\alpha}{1-\alpha}\right)p + (1-\beta)\left(\frac{1-\alpha}{1-\alpha}\right)q \\
&= \left(\frac{\beta-\alpha}{1-\alpha}\right)p + \left(\frac{(1-\beta)\alpha}{1-\alpha}\right)p + (1-\gamma)(1-\alpha)q \\
&= \gamma p + (1-\gamma)[\alpha p + (1-\alpha)q] \\
&\succ \gamma[\alpha p + (1-\alpha)q] + (1-\gamma)[\alpha p + (1-\alpha)q] \\
&= \alpha p + (1-\alpha)q
\end{aligned}$$

where the strict preference step follows from repeated application of independence.

($\Longrightarrow$) Say $\alpha \geq \beta$. Then, by hypothesis,

$$\begin{aligned}
&\beta p + (1-\alpha)q + (\alpha-\beta)q \succ \beta p + (1-\alpha)q + (\alpha-\beta)p \\
\Longrightarrow\ & q \succ (1+\beta-\alpha)q + (\alpha-\beta)p \\
\Longrightarrow\ & q \succ p
\end{aligned}$$

□

Proposition 4.7 (**Expected utility theorem/ Von Neumann–Morgenstern utility theorem**)**.** *A preference relation $\succsim$ on P satisfies rationality, continuity, and independence if and only if there exists a Bernoulli utility function $u\colon X \to \mathbb{R}$ and a utility function, $U\colon P \to \mathbb{R}$ given by $U(p) = \sum_{x\in X} u(x)p(x)$, representing the preference relation. In other words, for any $p, q \in P$,*

$$p \succsim q \iff \sum_{x\in X} u(x)p(x) \geq \sum_{x\in X} u(x)q(x)$$

where $p(x)$ and $q(x)$ are the probabilities of x under p and q.

Proof. With a finite set of prizes, we claim that there are best and worst probabilities. Call them $B, W \in P$ and let $B \succ W$.

Step 1 By Lemma 4.5, for any $p \in P$ there is an $\alpha^p \in [0,1]$ such that

$$p \sim \alpha^p B + (1-\alpha^p)W \tag{4.2}$$

Step 2 Lemma 4.6 implies that α^p is unique.

Proof. Suppose β^p also satisfies (4.2). Negating both sides of (4.1), we get $\beta^p \geq \alpha^p$ and $\alpha^p \geq \beta^p$, or equivalently $\alpha^p = \beta^p$. $\square$

Step 3

$$p \succsim q \iff \alpha^p \geq \alpha^q$$

Proof. ($\Longleftarrow$)

$$p \sim \alpha^p B + (1-\alpha^p)W \succsim \alpha^q B + (1-\alpha^q)W \sim q$$

where indifference relations follow from Step 1 and the weak preference relation follows from Lemma 4.6.

($\Longrightarrow$)

$$\alpha^p B + (1-\alpha^p)W \sim p \succsim q \sim \alpha^q B + (1-\alpha^q)W$$

By Lemma 4.6, this implies $\alpha^p \geq \alpha^q$. $\square$

Step 4 Let $U \colon P \to \mathbb{R}$ be given by $U(p) := \alpha^p$. Then U represents $\succsim$.

Proof. This follows immediately from the previous step. $\square$

Step 5 $U \colon P \to \mathbb{R}$ is linear in convex combinations. That is, for any $p, q \in P$ and $\alpha \in [0,1]$

$$U(\alpha p + (1-\alpha)q) = \alpha U(p) + (1-\alpha)U(q)$$

Proof. We know

$$\begin{aligned} p &\sim \alpha^p B + (1-\alpha^p)W \\ q &\sim \alpha^q B + (1-\alpha^q)W \end{aligned}$$

So,

$$\begin{aligned} \alpha p + (1-\alpha)q &\sim \alpha[\alpha^p B + (1-\alpha^p)W] + (1-\alpha)[\alpha^q B + (1-\alpha^q)W] \\ &= [\alpha\alpha^p + (1-\alpha)\alpha^q]B + [\alpha(1-\alpha^p) + (1-\alpha)(1-\alpha^q)]W \\ &= [\alpha\alpha^p + (1-\alpha)\alpha^q]B + [1 - \alpha\alpha^p - (1-\alpha)\alpha^q]W \end{aligned}$$

Thus,

$$\begin{aligned} U(\alpha p + (1-\alpha)q) &= \alpha\alpha^p + (1-\alpha)\alpha^q \\ &= \alpha U(p) + (1-\alpha)U(q) \end{aligned}$$

$\square$

Step 6
$$U(p) = \sum_{x \in X} p(x)u(x)$$

Proof. Note that any $p \in P$ can be written as
$$\sum_{x \in X} p(x)\delta_x = p$$
where $\delta_x \in P$ has singleton support $\{x\}$ and puts $\Pr(x) = 1$. Thus,
$$U(p) = U\left(\sum_{x \in X} p(x)\delta_x\right) = \sum_{x \in X} p(x)U(\delta_x) \qquad \square$$

Proposition 4.8. *Expected utility representations are unique up to a positive affine transformation: Suppose U provides an expected utility representation of $\succsim$ in the sense of the previous proposition. Then V also provides an expected utility representation of the same preferences if and only if there exists constants $A, B \in \mathbb{R}$ such that $A > 0$ and $V := AU + B$.*

**Proof.* We prove the "if" direction. Because V is a positive monotonic transformation of U, we know that it represents the same preferences (Proposition 2.17). Moreover, for $p, q \in P$ and $\alpha \in [0,1]$,
$$\begin{aligned} V(\alpha p + (1-\alpha)q) &= AU(\alpha p + (1-\alpha)q) + B \\ &= A\alpha U(p) + A(1-\alpha)U(q) + B \\ &= \alpha[AU(p) + B] + (1-\alpha)[AU(q) + B] \\ &= \alpha V(p) + (1-\alpha)V(q) \end{aligned}$$
For the proof of the converse, see pages 94 and 95 of Kreps. $\square$

Example 4.9. Suppose $X := \{x, y, z\}$, $p, q \in P$, and $\succsim$ on P satisfies rationality, continuity, and independence. By the expected utility theorem,
$$U(p) = u(x)p(x) + u(y)p(y) + u(z)p(z)$$
is a valid utility representation of $\succsim$. Suppose $p \sim q$, then $U(p) = U(q)$. Consider $\alpha \in (0,1)$ and a linear combination of the two probabilities $r =$

$\alpha p + (1-\alpha)q \in P$. Then

$$\begin{aligned} U(r) &= u(x)[\alpha p(x) + (1-\alpha)q(x)] + u(y)[\alpha p(y) + (1-\alpha)q(y)] \\ &\quad + u(z)[\alpha p(z) + (1-\alpha)q(z)] \\ &= \alpha U(p) + (1-\alpha)U(q) \\ &= U(p) = U(q) \end{aligned}$$

This implies that the indifference curves over probabilities are straight lines. In fact, on an indifference curve with utility level $\bar{u}$,

$$\begin{aligned} \bar{u} &= u(x)p(x) + u(y)p(y) + u(z)[1 - p(x) - p(y)] \\ \implies p(y) &= \frac{u(z) - \bar{u}}{u(z) - u(y)} + \left[\frac{u(x) - u(z)}{u(z) - u(y)}\right] p(x) \end{aligned}$$

The slope of the indifference curve is given by $\frac{u(x)-u(z)}{u(z)-u(y)}$. Suppose $z = 0$, $x = -1$, $y = 1$. If the slope is bigger than 1, then the Bernoulli utility function is concave: the person is risk averse.

4.1.1 Money Prizes

We now look at the case where $x \in X$ is a quantity of money. We assume that (Bernoulli) utility is strictly increasing in money.

Proposition 4.10. *For $p \in P$, define $\mathbb{E}_p(x) = \sum_{x \in X} x p(x)$. Suppose that for all $p \in P$, $\delta_{\mathbb{E}_p} \succsim p$. This holds if and only if u is concave.*

**Proof.* Suppose u is not concave. Then

$$U(\delta_{\mathbb{E}_p}) = u(\alpha x + (1-\alpha)x') < \alpha u(x) + (1-\alpha)u(x') = U(p)$$

for some $p \in P$ and some $\alpha \in [0,1]$. That is, $p \succ \delta_{\mathbb{E}_p}$. Conversely, suppose that u *is* concave. Then

$$U(\delta_{\mathbb{E}_p}) = u\left(\sum_x p(x)x\right) \geq \sum_x p(x)u(x) = U(p)$$

for all $p \in P$, where the inequality is an application of Jensen's inequality. We have, then, $\delta_{\mathbb{E}_p} \succsim p$. $\square$

Definition 4.11. A **certainty equivalent** for a lottery p is a prize c satisfying $\delta_c \sim p$. If utility is strictly concave, every lottery has a unique certainty equivalent, which we denote by $c(p)$.

Definition 4.12. If the certainty equivalent is unique, the **risk premium** of a lottery p is given by

$$r(p) := \mathbb{E}\,p - c(p)$$

Definition 4.13. The **coefficient of absolute risk aversion** is given by

$$\lambda(x) := -\frac{u''(x)}{u'(x)}$$

Definition 4.14. The **coefficient of relative risk aversion** is given by

$$\lambda(x) := -\frac{u''(x)x}{u'(x)}$$

Example 4.15. Insurance problem.

A consumer starts with wealth W. Any one of the following mutually exclusive events can happen:

(i) She incurs loss L_1, with probability p_1.

(ii) She incurs loss L_2, with probability p_2.

(iii) She incurs no loss, with probability $1 - p_1 - p_2$.

For each i, let the insurance pay π_i if loss L_i is incurred. The cost of this insurance is $q_i\pi_i$. The consumer needs to choose (π_1, π_2). Assume that $u' > 0$ and $u'' < 0$. We can write the consumer's problem as

$$\begin{aligned}&\max_{\pi_1,\pi_2} \{p_1 u(W_{L_1}) + p_2 U(W_{L_2}) + (1 - p_1 - p_2)U(W_{NL})\}\\ &\text{st } 0 \le \pi_1 \le L_1 \text{ and } 0 \le \pi_2 \le L_2\end{aligned}$$

where

$$\begin{aligned}W_{L_1} &:= W - (q_1\pi_1 + q_2\pi_2) - L_1 + \pi_1\\ W_{L_2} &:= W - (q_1\pi_1 + q_2\pi_2) - L_2 + \pi_2\\ W_{NL} &:= W - (q_1\pi_1 + q_2\pi_2)\end{aligned}$$

The first-order conditions are

$$\begin{aligned}p_1 u'(W_{L_1})(1 - q_1) + p_2 u'(W_{L_2})(-q_1) + (1 - p_1 - p_2)u'(W_{NL})(-q_1) = 0\\ p_1 u'(W_{L_1})(-q_2) + p_2 u'(W_{L_2})(1 - q_2) + (1 - p_1 - p_2)u'(W_{NL})(-q_2) = 0\end{aligned}$$

implying

$$p_1 u'(W_{L_1}) \left(\frac{1-q_1}{q_1} \right) - p_2 u'(W_{L_2}) = -p_1 u'(W_{L_1}) + p_2 u'(W_{L_2}) \left(\frac{1-q_2}{q_2} \right)$$

Simplifying, we get

$$\frac{p_1}{q_1} u'(W_{L_1}) = \frac{p_2}{q_2} u'(W_{L_2})$$

Suppose $p_1/q_1 = p_2/q_2 = \alpha$. In other words, the price of insurance is a constant markup over the probability of incurring the loss. Then because $u'' > 0$,

$$\begin{aligned} & u'(W_{L_1}) = u'(W_{L_2}) \\ \implies & W_{L_1} = W_{L_2} \\ \implies & \pi_1 - L_1 = \pi_2 - L_2 \end{aligned}$$

The last line implies that in this case, it is optimal to have the same "deductible" for different losses. Suppose in particular, that $\alpha = 1$. That is, $q_1 = p_1$ and $q_2 = p_2$. Plugging this and the equality above back into the first-order conditions gives

$$\begin{aligned} & p_1 u'(W_L)(1-p_1) + p_2 u'(W_L)(-p_1) + (1-p_1-p_2)u'(W_{NL})(-p_1) = 0 \\ \implies & u'(W_L)[p_1(1-p_1) - p_1 p_2] = u'(W_{NL}) p_1 (1-p_1-p_2) \\ \implies & u'(W_L) = u'(W_{NL)} \\ \implies & W_L = W_{NL} \\ \implies & \pi_1 - L_1 = 0 \end{aligned}$$

where the last line uses the definitions of W_{L_1} and W_{NL}. Therefore, $\pi_1 = L_1$ and $\pi_2 = L_2$: the consumer buys full insurance.

4.1.2 Infinite set of prizes

We generalize the definition of a simple probability distribution to infinite X:

Definition 4.16. A **simple probability distribution** on X is a $p \in P$ that has a finite support. That is, it puts positive probability only on some finite subset of X. Denote by P_S the subset of P consisting of all simple probability distributions on X.

Remark 4.17. All the results we proved for finite X also hold for infinite X when we restrict ourselves to simple probability distributions. In particular, the expected utility theorem holds.

Now we consider non-simple probability distributions on infinite X.

Assumptions 4.18.

(i) Infinite set of prizes, $X := \mathbb{R}^n$.

(ii) Set of all probability distributions, P, on X.

(iii) Preferences, $\succsim$, over P.

Definition 4.19. We say that a sequence of probability distributions $(p_n)_{n\in\mathbb{N}}$ in P **converges weakly** to p if

$$\int_X f(x)dp_n(x) \to \int_X f(x)dp(x)$$

for all bounded continuous functions $f\colon X \to \mathbb{R}$.

Definition 4.20. Preferences, $\succsim$, are **continuous in the weak topology** on P if for any $p \in P$, the sets $\{q \in P \mid q \succsim p\}$ and $\{q \in P \mid p \succsim q\}$ are both closed.

Proposition 4.21. *A preference relation $\succsim$ on P is rational, satisfies the independence axiom and is continuous in the weak topology on P if and only if there is a bounded, continuous (Bernoulli) utility function $u\colon X \to \mathbb{R}$ such that*

$$p \succsim q \iff \int_X u(x)dp(x) \geq \int_X u(x)dq(x)$$

Proof. Omitted. □

Example 4.22. Asset allocation problem.

An investor must allocate her initial wealth, x_0, between two assets:

(i) A risk-free asset giving certain return equal to the amount invested.

(ii) A risky asset giving random return $r \sim \mathcal{N}(\bar{r}, \sigma_r^2)$. If she invests s now at price p, she will get rs in the future. Assume $\bar{r} \geq p$.

The investor aims to maximize the expected utility of her future wealth, x. Given s, her future wealth is a random variable, $x \sim \mathcal{N}(\bar{x}, \sigma_x^2)$, where $\bar{x} = x_0 - ps + \bar{r}s$ and $\sigma_x^2 = s^2\sigma_r^2$. Her Bernoulli utility function is given by[15]

$$u(x) := -\exp(-\lambda x)$$

[15] This is a constant absolute risk aversion (CARA) utility function. It can easily be shown that the coefficient of absolute risk aversion is λ.

where $\lambda > 0$. It can be shown that

$$\mathbb{E}[u(x)] = -\exp\left[-\lambda\left(\bar{x} - \frac{\lambda}{2}\sigma_x^2\right)\right]$$

We can therefore write her maximization problem as

$$\max_s \left\{\bar{x} - \frac{\lambda}{2}\sigma_x^2\right\} = \max_s \left\{x_0 - ps + \bar{r}s - \frac{\lambda}{2}s^2\sigma_r^2\right\}$$

The first-order condition is

$$\begin{aligned} \bar{r} - p - \lambda s\sigma_r^2 &= 0 \\ \implies s^* &= \frac{\bar{r} - p}{\lambda\sigma_r^2} \end{aligned}$$

4.2 Subjective Expected Utility

4.2.1 Anscombe-Aumann Acts

Assumptions 4.23.

(i) Finite set of the states of the world, S.

(ii) Set of prizes, X.

(iii) Set of simple probability distributions on prizes, P.

(iv) Set of acts $H := P^S$: the set of all $h\colon S \to P$. For each state s, h specifies an objective lottery $h(s) \in P$ with $h(s)(x)$ an objective probability of x conditional on s.[16] For clarity we will use the (abuse of) notation $h(x \mid s)$ for $h(s)(x)$.

(v) Preferences, $\succsim$, on H.

Definition 4.24. For $h, g \in H$ and $\alpha \in [0, 1]$, a **compound act** $\alpha h + (1 - \alpha)g \in H$ is defined by

$$\alpha h(\cdot \mid s) + (1 - \alpha)g(\cdot \mid s)$$

for each $s \in S$.

[16] In Kreps, elements of P are called *roulette lotteries* and those of H are called *horse-race lotteries*. This alludes to the idea that roulette wheels yield outcomes according to commonly-known objective probabilities, whereas individuals may hold distinct subjective probabilities over horse races.

Definition 4.25. The preference relation $\succsim$ on H satisfies **state independence** if for any $p, q \in P$, any state $s^* \in S$ and acts $h, g \in H$ defined by

$$h(\cdot \mid s) = \begin{cases} p & \text{if } s = s^* \\ r & \text{if } s \neq s^* \end{cases} \text{ and } g(\cdot \mid s) = \begin{cases} q & \text{if } s = s^* \\ r & \text{if } s \neq s^* \end{cases}$$

$p \succsim q$ if and only if $h \succsim g$.[17]

Proposition 4.26 (**Subjective expected utility theorem/Anscombe-Aumann utility theorem**). *A preference relation $\succsim$ on H satisfies rationality, continuity, independence,[18] and state independence if and only if $\succsim$ is represented by a function $U\colon H \to R$ defined by*

$$U(h) = \sum_s \pi(s) \sum_x h(x \mid s) u(x)$$

where $u\colon X \to R$ is a Bernoulli utility function and π is a probability distribution on S with $\pi(s) > 0$ for all $s \in S$. In this case, (u, π) is called a subjective expected utility (SEU) representation of $\succsim$.

Definition 4.27. For $f, h \in H$, define

$$f_A h(s) := \begin{cases} f(s) & \text{if } s \in A \\ h(s) & \text{if } s \notin A \end{cases}$$

Definition 4.28. Define a preference relation, $\succsim_A$ on H, such that $f \succsim_A g$ if for all $h \in H$:

$$f_A h \succsim g_A h$$

We say that f is at least as good as g given A.

Proposition 4.29. *Suppose $\succsim$ has a SEU representation (u, π). Then for any $A \subseteq S$ with $\pi(A) > 0$,*

$$f \succsim_A g$$

if and only if

$$\sum_{s \in A} \pi(s \mid A) \sum_{x \in X} f(x \mid s) u(x) \geq \sum_{s \in A} \pi(s \mid A) \sum_{x \in X} g(x \mid s) u(x)$$

[17] To be rigorous, since $\succsim$ is defined on acts, p and q here should be the acts h_p and h_q that give – regardless of the state of the world – the lotteries p and q respectively.

[18] Here, rationality, continuity, and independence are defined as in Section 4.1, but with acts f, g, h replacing objective probabilities p, q, r.

Proof. By definition, $f \succsim_A g$ is equivalent to $f_A h \succsim g_A h$ for all $h \in H$. With the SEU representation of $\succsim$, this implies that

$$\sum_{s\in A}\pi(s)\sum_{x\in X} f(x \mid s)u(x) + \sum_{s\notin A}\pi(s)\sum_{x\in X} h(x \mid s)u(x) \geq \sum_{s\in A}\pi(s)\sum_{x\in X} g(x \mid s)u(x) + \sum_{s\notin A}\pi(s)\sum_{x\in X} h(x \mid s)u(x)$$

Cancelling the common term,

$$\begin{aligned} &\sum_{s\in A}\pi(s)\sum_{x\in X} f(x \mid s)u(x) \geq \sum_{s\in A}\pi(s)\sum_{x\in X} g(x \mid s)u(x) \\ \implies &\sum_{s\in A}\frac{\pi(s)}{\pi(A)}\sum_{x\in X} f(x \mid s)u(x) \geq \sum_{s\in A}\frac{\pi(s)}{\pi(A)}\sum_{x\in X} g(x \mid s)u(x) \\ \implies &\sum_{s\in A}\pi(s \mid A)\sum_{x\in X} g(x \mid s)u(x) \geq \sum_{s\in A}\pi(s \mid A)\sum_{x\in X} h(x \mid s)u(x) \end{aligned}$$

where the last implication uses Bayes' rule. □

4.2.2 Value of Information

Proposition 4.30. *If the decision maker is an SEU maximizer, ex-ante information is always (weakly) valued.*

Proof. Suppose the information is whether the state is an element of A. That is, the information is $s \in A$ or $s \in A^{\mathsf{C}}$ where $A \subseteq S$. Suppose without the information $h^* \in H$ is optimal, h^*_A is optimal given A, and $h^*_{A^{\mathsf{C}}}$ given A^{C}. From the SEU representation of $\succsim$,

$$\sum_{s\in A}\pi(s \mid A)\sum_{x\in X} h^*_A(x \mid s)u(x) \geq \sum_{s\in A}\pi(s \mid A)\sum_{x\in X} h^*(x \mid s)u(x)$$
$$\sum_{s\in A^{\mathsf{C}}}\pi(s \mid A^{\mathsf{C}})\sum_{x\in X} h^*_{A^{\mathsf{C}}}(x \mid s)u(x) \geq \sum_{s\in A^{\mathsf{C}}}\pi(s \mid A^{\mathsf{C}})\sum_{x\in X} h^*(x \mid s)u(x)$$

Multiply both sides of the first inequality by $\pi(A)$ and both sides of the second by $\pi(A^{\mathsf{C}})$, and summing them gives

$$\pi(A)\sum_{s\in A}\pi(s \mid A)\sum_{x\in X} h^*_A(x \mid s)u(x) + \pi(A^{\mathsf{C}})\sum_{s\in A^{\mathsf{C}}}\pi(s \mid A^{\mathsf{C}})\sum_{x\in X} h^*_{A^{\mathsf{C}}}(x \mid s)u(x) \geq \sum_{s}\pi(s)\sum_{x\in X} h^*(x \mid s)u(x)$$

where the left-hand-side is the ex-ante optimal expected utility of having the information and the right-hand-side is the optimal expected utility without information. Therefore, having the information is weakly preferred. □

Example 4.31. Betting on a coin flip.

The decision-maker will win \$30 if she is correct and lose \$50 if she is wrong. She is risk neutral with $u(x) = x$. Suppose the coin is either fair, has heads on both sides ($2H$) or neither side ($2T$) with equal probability, 1/3. Without any information, she would not bet because the expected utility of betting is

$$U(\text{bet}) = \frac{1}{2}u(30) + \frac{1}{2}u(-50) < 0 = U(\text{don't bet})$$

Suppose now the person has the choice to observe one flip and then bet. How much would she be willing to pay to receive this information? Without loss of generality, suppose the first flip gives heads, and denote this event by H. Then, by Bayesian updating, we can summarize the prior and posterior probabilities:

States	$\Pr(S)$	$\Pr(H \mid S)$	$\Pr(S \mid H)$
2H	$1/3$	1	$2/3$
Fair	$1/3$	$1/2$	$1/3$
2T	$1/3$	0	0

where

$$\Pr(2H \mid H) = \frac{\Pr(H \mid 2H) \cdot \Pr(2H)}{\Pr(H)} = \frac{1 \cdot \frac{1}{3}}{1 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} + 0 \cdot \frac{1}{3}} = \frac{2}{3}$$

Thus, the conditional probability of getting heads in the second flip is

$$\Pr(H_2 \mid H) = \frac{2}{3} \cdot 1 + \frac{1}{3} \cdot \frac{1}{2} + 0 \cdot 0 = \frac{5}{6}$$

and that of getting tails is

$$\Pr(T_2 \mid H) = \frac{1}{6}$$

Therefore, the person will bet on heads if she observes heads on the first toss, and her subjective expected utility is

$$U(\text{bet on heads} \mid H) = \frac{5}{6}u(30) + \frac{1}{6}u(-50) = \frac{50}{3}$$

By symmetry, observing tails on the first flip and betting on tails in the following flip will yield the same expected utility. Hence, the value of observing one flip is $\frac{50}{3}$.

4.2.3 Learning

Example 4.32. Two-armed bandit problem.

Suppose at each date $t = 1, \ldots, T$, the decision-maker can choose to open one of two boxes, A and B. Box A gives a payoff of 1 dollar for sure, while box B gives a payoff of 2 dollars with probability θ and nothing with probability $1 - \theta$. The value of θ depends on the states of the world. In the good state, $\theta = p > \frac{1}{2}$ and in the bad state, $\theta = q < \frac{1}{2}$. The decision-maker's prior probability of the good state being the true state is λ_1. She discounts future profits by β where $0 \leq \beta < 1$. She has a Bernoulli utility function $u(x) = x$ and is an SEU maximizer.

Consider the simple case where $T = 1$: a single-period decision problem. She will open box A if and only if

$$\begin{aligned} & 1 \geq 2\lambda_1 p + 2(1 - \lambda_1)q \\ \iff & \frac{1 - 2q}{2p - 2q} \geq \lambda_1 \end{aligned}$$

Otherwise, she will open box B. Let $\lambda^* := \frac{1-2q}{2p-2q}$. The value function at $t = 1$ is:

$$V(\lambda_1) = \begin{cases} 1 & \text{if } \lambda^* \geq \lambda_1 \\ 2\lambda_1 p + 2(1 - \lambda_1)q & \text{if } \lambda^* \leq \lambda_1 \end{cases}$$

Now, consider $T = 2$ and denote her choice in period t by

$$a_t := \begin{cases} 0 & \text{if choose } A \text{ in period } t \\ 1 & \text{if choose } B \text{ in period } t \end{cases}$$

Denote her observation in period 1 by

$$\sigma_1 := \begin{cases} \varnothing & \text{if } a_1 = 0 \\ 0 & \text{with probability } \lambda_1(1 - p) + (1 - \lambda_1)(1 - q) \text{ if } \ a_1 = 1 \\ 2 & \text{with probability } \lambda_1 p + (1 - \lambda_1)q \text{ if } \ a_1 = 1 \end{cases}$$

Then, by applying Bayes' rule to update her beliefs about the true state, given the observation σ_1,

$$\begin{aligned}
\lambda_2(\sigma_1) &= \Pr(\theta = p \mid \sigma_1) \\
&= \begin{cases} \lambda_1 & \text{if } \sigma_1 = \varnothing \\ \frac{\Pr(\sigma_1=0|\theta=p)\Pr(\theta=p)}{\Pr(\sigma_1=0)} & \text{if } \sigma_1 = 0 \\ \frac{\Pr(\sigma_1=2|\theta=p)\Pr(\theta=p)}{\Pr(\sigma_1=2)} & \text{if } \sigma_1 = 2 \end{cases} \\
&= \begin{cases} \lambda_1 & \text{if } \sigma_1 = \varnothing \\ \frac{(1-p)\lambda_1}{\lambda_1(1-p)+(1-\lambda_1)(1-q)} & \text{if } \sigma_1 = 0 \\ \frac{p\lambda_1}{\lambda_1 p+(1-\lambda_1)q} & \text{if } \sigma_1 = 2 \end{cases}
\end{aligned}$$

We solve the problem by backward induction. At $t = 2$, her decision problem is maximizing the one-period expected payoff given her updated belief $\lambda_2(\sigma_1)$, which can be seen as the same problem as in the $T = 1$ case. Therefore, the value function is analogous:

$$V(\lambda_2(\sigma_1)) = \begin{cases} 1 & \text{if } \lambda^* \geq \lambda_2(\sigma_1) \\ 2\lambda_2(\sigma_1)p + 2(1 - \lambda_2(\sigma_1))q & \text{if } \lambda^* \leq \lambda_2(\sigma_1) \end{cases}$$

Then, at $t = 1$, her decision problem is to maximize the two-periods expected utility:

$$\max_{a_1 \in \{0,1\}} (1 - a_1) + a_1 \left(2\lambda_1 p + 2(1 - \lambda_1)q\right) + \beta\, \mathbb{E}[V(\lambda_2(\sigma_1)) \mid \lambda_1, a_1]$$

Remark 4.33. If $a_t = 0$ for some t (opening box A in period t), then $\lambda_{t+1}(\sigma_t) = \lambda_t$. The belief is not updated and the optimal action at $t + 1$ will still be $a_{t+1} = 0$. In other words, once the decision-maker opens the safe box, she will continue to do so for all future periods.

Remark 4.34. Even if the problem has an infinite horizon, it may not be optimal to open box B and learn the true state. As long as $\beta < 1$, there exists initial beliefs such that the decision-maker will open box A in the first period. In fact, if $T = \infty$, it can be shown that as long as

$$\frac{1 - 2q(1 - p) - \beta}{2p - 2q(1 - \beta) - \beta} > \lambda_1$$

the decision-maker will always open box A.

Remark 4.35. Note that the value function at each period is a convex function of λ. And in the example above,

$$\begin{aligned}\mathbb{E}[\lambda_2(\sigma_1)] &= \lambda_1 \Pr(a_1 = 0) + ((1-p)\lambda_1 + p\lambda_1) \Pr(a_1 = 1) \\ &= \lambda_1 \Pr(a_1 = 0) + \lambda_1 \Pr(a_1 = 1) \\ &= \lambda_1\end{aligned}$$

More generally,

$$\mathbb{E}[\text{posterior probability}] = \text{ prior probability}$$

This can be readily seen from the law of iterated expectations:

$$\mathbb{E}_D[\mathbb{E}[X|D]] = \mathbb{E}[X]$$

where D denotes observation (information) and the first expectation is taken over possible values of D. This, combined with the convexity of $V(\lambda)$ and Jensen's inequality, gives us

$$\mathbb{E}[V(\text{posterior probability})] \geq V(\mathbb{E}[\text{posterior probability}]) = V(\text{prior probability})$$

This shows that the convexity of V makes information valuable.

4.2.4 Ambiguity Aversion

Definition 4.36. A preference relation $\succsim$ on H satisfies **certainty independence** if for all $f, g \in H$, $h \in H$ a constant act,[19] and $\alpha \in (0,1)$

$$f \succ g \iff \alpha f + (1-\alpha)h \succ \alpha g + (1-\alpha)h$$

Definition 4.37. A preference relation $\succsim$ on H satisfies **uncertainty aversion** if for all $f, g \in H$ and $\alpha \in (0,1)$

$$f \sim g \implies \alpha f + (1-\alpha)g \succsim f \sim g$$

Proposition 4.38 (Max-min expected utility – Gilboa & Schmeidler)**.** *A preference relation $\succsim$ on H satisfies rationality, continuity, state independence, certainty independence, and uncertainty aversion if and only if there exist a*

[19] A constant act maps every state to the same objective probability distribution.

utility function $u\colon X \to R$ and a closed and convex set Π of probability distributions over states such that the function

$$V(h) := \min_{\pi \in \Pi} \sum_{s \in S} \pi(s) \sum_{x \in X} h(x \mid s) u(x)$$

represents $\succsim$ on H. In other words, the individual's decision problem is the max-min problem

$$\max_{h \in H} \min_{\pi \in \Pi} \sum_{s \in S} \pi(s) \sum_{x \in X} h(x \mid s) u(x)$$

Example 4.39. Consider an asset allocation example, similar to Example 4.22. As before, an investor allocates her initial wealth x_0 between two assets:

(i) A risk-free asset that returns the amount invested.

(ii) A risky asset with price p and return $r \sim N(\bar{r}, \sigma_r^2)$.

She has Bernoulli utility on wealth, $u(x) := -\exp(-x)$. There are two states of the world, one in which the expected return is $\bar{r}_L$ and another in which it is $\bar{r}_H > \bar{r}_L$. Let s be the amount of the risky asset that the individual purchases. Using the result from Example 4.22, the expected utility equals

$$(\bar{r} - p)s - \frac{1}{2}\sigma_r^2 s^2 + x_0$$

If the investor is a G-S type decision-maker, her decision problem is

$$\max_{s} \min_{\bar{r} \in \{\bar{r}_L, \bar{r}_H\}} \left\{ (\bar{r} - p)s - \frac{1}{2}\sigma_r^2 s^2 \right\}$$

That is, for each s she evaluates the expected utility under the worst state and then chooses s to maximize this minimum. It is easy to see that the expected utility function attains minimum at $\bar{r}_L$ for $s > 0$ and at $\bar{r}_H$ for $s < 0$. Using this observation and taking first order conditions, we have

$$s^* = \begin{cases} \frac{\bar{r}_L - p}{\sigma_r^2} & \text{if } p < \bar{r}_L \\ 0 & \text{if } \bar{r}_L < p < \bar{r}_H \\ \frac{\bar{r}_H - p}{\sigma_r^2} & \text{if } p > \bar{r}_H \end{cases}$$

Remark 4.40. We could replace the state space $\{\bar{r}_L, \bar{r}_H\}$ with the interval $[\bar{r}_L, \bar{r}_H]$ without affecting the result.

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