ECON 6130

Chahrour Exam Questions

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1 Qual 2024

1.1 Questions

Consider an economy populated by identical consumer-worker households with preferences for consumption and labor effort given by

$$\mathop{\mathbb{E}}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{C_t^{1-\sigma} - 1}{1-\sigma} - \gamma_s S_t - \gamma_n N_t \right\}$$

Notice that now the household supplies search effort S_t and work effort N_t at constant marginal disutilities γ_s and γ_n respectively. The household earns income from the wages, W_t, paid to currently-employed workers N_t and any profits π_t earned by the firm, which it owns. In making their decisions, both the household and the firm take the wage as exogenous.

Unlike in class, assume that workers who match with an employer today start productive work in the following period. From the household perspective, this means that N_t evolves according to

$$N_{t+1} = (1 - \delta_n)N_t + p_t S_t$$

where δ_n is the worker separation rate and p_t is the household probability that a unit of search effort results in a job.

Firms hire workers to produce output, according to the production function

$$Y_t = A_t N_t$$

In order to hire a worker, firms must post a vacancy at cost ϕ. A vacancy results in a match (and a future employed worker) with probability q_t, so that from the firm's perspective

$$N_{t+1} = (1 - \delta_n)N_t + q_t V_t$$

The firm's objective is to maximize the discounted present value of profits, given by

$$V_0 = \mathop{\mathbb{E}}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{\lambda_{1t}}{\lambda_{10}} \pi_t \right\}$$

where λ_t is the marginal utility of consumption in period t.

Equilibrium matches are determined by an aggregate matching function $M(V_t, S_t) = \chi V_t^{\varepsilon} S_t^{1-\varepsilon}$. Each vacancy has an equal chance of being matched, so that in equilibrium

$$q_t = M(V_t, S_t)/V_t = \chi(V_t/S_t)^{\varepsilon-1}$$

Conversely,

$$p_t = M(V_t, S_t)/S_t = \chi(V_t/S_t)^{\varepsilon}$$

Finally, technology is purely exogenous and evolves according to an AR(1) process in logs,

$$\log(A_t) = \rho \log(A_{t-1}) + \varepsilon_t$$

1. In the language of the course, list separately the endogenous jump variables, the endogenous state variables, and the exogenous state variables in this model. Finally, make a list of all of the exogenous parameters of this economy. (5 points)
2. Write the <u>household</u>'s Lagrangian optimization problem and find the first order necessary conditions

for optimality of the household. Denote the multipliers on the household budget constraint and labor evolution constraint with λ_{1t} and λ_{2t} respectively. (10 points)

3. Write the firm's Lagrangian optimization problem and find the first order necessary conditions for optimality. Denote the multipliers on constraints (2) and (3) with $\theta_{1,t}$ and $\theta_{2,t}$ respectively. (10 points)
4. Now write the Bellman equation that corresponds to the social planner's optimization problem in this economy and find the first order necessary conditions for optimality using the envelope theorem. (10 points)
5. Log-linearize the vacancy posting condition (first order condition for V_t) from the firm problem. You should proceed from first principals (*i.e.* replace V with $\exp(v)$, etc and compute a first-order Taylor approximation). You should log-linearize the equations around the steady-state, and you may treat the steady-state values of endogenous variables, like N, C, V, etc, as parameters. (5 points)
6. Suppose I wanted to see what would happen over time to employment in the economy in response to temporary increase in the level of risk-aversion (σ) in the economy. Decide which of the four solution algorithms (linearization, shooting, value function, or projection) that we studied you would use to answer this question, and explain why you chose that method. Your response should include a brief description of the method you choose, and an explanation of why other methods are not as well-suited to the question. (20 points)

1.2 Solutions

(Ryan's Solutions)

1. We have that the one exogenous state variable is A_t, the one endogenous state variable is N_t, and the jump variables are C_t, V_t, Y_t, q_t, p_t, and S_t. The parameters are W_t, β, σ, γ_s, γ_n, δ_n, χ, ε, ρ, and ε_t.
2. The household's problem is

$$\max_{\{C_t,S_t,N_{t+1}\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{C_t^{1-\sigma}-1}{1-\sigma} - \gamma_s S_t - \gamma_n N_t + \lambda_{1,t}\left(\pi_t + W_t N_t - C_t\right) + \lambda_{2,t}\left((1-\delta_n)N_t + p_t S_t - N_{t+1}\right) \right\}$$

   Which admits the first order conditions

$$\begin{aligned} C_t^{-\sigma} &= \lambda_{1,t} && (C_t) \\ \gamma_s &= p_t \lambda_{2,t} && (S_t) \\ \lambda_{2,t} &= \beta \mathbb{E}_t \left[\lambda_{1,t+1} W_{t+1} - \gamma_n + \lambda_{2,t+1}(1-\delta_n)\right] && (N_{t+1}) \end{aligned}$$

3. The firm's problem is

$$\max_{\{Y_t,V_t,N_{t+1}\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{\lambda_{1,t}}{\lambda_{1,0}} \underbrace{(Y_t - W_t N_t - \phi V_t)}_{\pi_t} + \theta_{1,t}(A_t N_t - Y_t) + \theta_{2,t}((1-\delta_n)N_t + q_t V_t - N_{t+1}) \right\}$$

Which admits the first order conditions

$$\frac{\lambda_{1,t}}{\lambda_{1,0}} = \theta_{1,t} \tag{Y_t}$$

$$\phi\frac{\lambda_{1,t}}{\lambda_{1,0}} = \theta_{2,t} q_t \tag{V_t}$$

$$\theta_{2,t} = \beta\,\mathbb{E}\left[\theta_{1,t+1}A_{t+1} - \frac{\lambda_{1,t+1}}{\lambda_{1,0}}W_{t+1} - \theta_{2,t+1}(1-\delta_n)\right] \tag{N_{t+1}}$$

Combining, we get that

$$\frac{\phi}{q_t}\lambda_{1,t} = \beta\,\mathbb{E}\left[\lambda_{1,t+1}(A_{t+1}-W_{t+1}) - (1-\delta_n)\frac{\phi}{q_{t+1}}\lambda_{1,t+1}\right]$$

4. The planner's value function is

$$V(N_t, A_t) = \max_{\{C_t, S_t, V_t, N_{t+1}\}} \frac{C^{1-\sigma}-1}{1-\sigma} - \gamma_s S_t - \gamma_n N_t + \beta\,\mathbb{E}\left[V(N_{t+1}, A_{t+1})\right]$$

subject to

$$N_{t+1} = (1-\delta_n)N_t + \chi V_t^{\varepsilon} S_t^{1-\varepsilon} \quad \text{and} \quad C_t = A_t N_t - \phi V_t$$

Since

$$V_t = \left(\frac{N_{t+1}-(1-\delta_n)N_t}{\chi S_t^{1-\varepsilon}}\right)^{\frac{1}{\varepsilon}}$$

we have that

$$V(N_t, A_t) = \max_{\{S_t, N_{t+1}\}} u\left(A_t N_t - \phi\left(\frac{N_{t+1}-(1-\delta_n)N_t}{\chi S_t^{1-\varepsilon}}\right)^{\frac{1}{\varepsilon}}\right) - \gamma_s S_t - \gamma_n N_t + \beta\,\mathbb{E}\left[V(N_{t+1}, A_{t+1})\right]$$

The first order conditions are

$$-\gamma_s + u'(\cdot)\phi\frac{1}{\varepsilon}\left(\frac{N_{t+1}-(1-\delta_n)N_t}{\chi S_t^{1-\varepsilon}}\right)^{\frac{1}{\varepsilon}-1}(\varepsilon-1)\frac{N_{t+1}-(1-\delta_n)N_t}{\chi}S_t^{\varepsilon-2} = 0 \tag{S_t}$$

$$-u'(\cdot)\phi\frac{1}{\varepsilon}\left(\frac{N_{t+1}-(1-\delta_n)N_t}{\chi S_t^{1-\varepsilon}}\right)^{\frac{1}{\varepsilon}-1}\frac{1}{\chi S_t^{1-\varepsilon}} + \beta\,\mathbb{E}\left[V_N(N_{t+1}, A_{t+1})\right] = 0 \tag{N_{t+1}}$$

By the Envelope condition, we have that

$$V_N(N_t, A_t) = u'(\cdot)\phi\frac{1}{\varepsilon}\left(\frac{N_{t+1}-(1-\delta_n)N_t}{\chi S_t^{1-\varepsilon}}\right)^{\frac{1}{\varepsilon}-1}\frac{1}{\chi S_t^{1-\varepsilon}}(1-\delta_n) + A_t u'(\cdot)$$

Combining, we get the Bellman Equation

$$u'(\cdot)\frac{\phi}{\varepsilon}\left(\frac{N_{t+1}-(1-\delta_n)N_t}{\chi S_t^{1-\varepsilon}}\right)^{\frac{1}{\varepsilon}-1}\frac{1}{\chi S_t^{1-\varepsilon}} = \beta\,\mathbb{E}\left[u'(\cdot)\left\{A_{t+1} + \frac{(1-\delta_n)\phi}{\varepsilon\chi S_{t+1}^{1-\varepsilon}}\left(\frac{N_{t+2}-(1-\delta_n)N_{t+1}}{\chi S_{t+1}^{1-\varepsilon}}\right)^{\frac{1}{\varepsilon}-1}\right\}\right]$$

5. We will log-linearize the combined posting condition from above:

$$\frac{\phi}{q_t}\lambda_{1,t} = \beta\,\mathbb{E}\left[\lambda_{1,t+1}(A_{t+1}-W_{t+1}) - (1-\delta_n)\frac{\phi}{q_{t+1}}\lambda_{1,t+1}\right]$$

For each variable, denote $\tilde{x} = \log(x)$. We have that this becomes

$$\frac{\phi}{\exp(\tilde{q}_t)} = \beta\,\mathbb{E}\left[\frac{\exp(\tilde{\lambda}_{1,t+1})}{\exp(\tilde{\lambda}_{1,t})}\left(\exp(\tilde{a}_{t+1}) - \exp(\tilde{w}_{t+1}) - (1-\delta_n)\frac{\phi}{\exp(\tilde{q}_{t+1})}\right)\right]$$

Using the first order Taylor expansion about the steady state, where the steady state variables are denoted by subscripts, the left side is approximated by

$$\frac{\phi}{\exp(q_{ss})(\hat{q}_t)} \equiv \frac{-\phi\hat{q}_t}{\exp(q_{ss})}$$

where $\hat{q}_t = \tilde{q}_t - q_{ss}$. The right hand side is approximated by

$$\beta\,\mathbb{E}\left[\exp(a_{ss})(\hat{a}_t) - \exp(w_{ss})(\hat{w}_t) + (1-\delta_n)\frac{\phi\hat{q}_{t+1}}{\exp(q_{ss})}\right] \equiv \beta\,\mathbb{E}\left[A_{ss}\hat{a}_t - W_{ss}\hat{w}_t + (1-\delta_n)\frac{\phi\hat{q}_{t+1}}{\exp(q_{ss})}\right]$$

6. We should use projection, which captures both uncertainty and non-linearities. We would approximate policy functions with weights on basis functions, so for policy $h(x)$, where we know $\{x_i\}_{i=1}^{I}$, we would approximate

$$\hat{h}(x) = \sum_{i=1}^{I} a_i b_i(x)$$

where b_i are basis functions and a_i are weights.

The other methods each have issues. Log-Linearization would not capture the effects of risk aversion, since it assumes agents are implicitly risk-neutral. Shooting requires perfect foresight, so we would not see the effects of uncertainty, and value function iteration is slow, since we have smooth choice sets.

2 Final 2023

2.1 Questions

1. **Model with production externality**. Consider an economy populated by identical producer-consumer households with preferences for consumption given by

$$\mathop{\mathbb{E}}_{0} \sum_{t=0}^{\infty} \beta^t \log(C_t)$$

Households are endowed with an initial stock of capital K_0. Household-level capital evolves according to the equation

$$K_{t+1} = (1-\delta)K_t + I_t$$

Household-level consumption is equal to output minus investment

$$C_t = A_t K_t^{\alpha} - I_t$$

where $Y_t = A_t K_t^{\alpha}$ is the production function and $0 < \alpha < 1$.

Finally, assume that total factor productivity, A_t, is taken as exogenous by each household, but depends on the aggregate level of current capital according to

$$\log(A_t) = a_0 \log(\bar{K}_t)$$

with $a_0 \geq 0$ and $a_0 + \alpha < 1$. In equilibrium, $K_t = \bar{K}_t$.

(a) In the language of the course, list separately the endogenous jump variables, the endogenous state variables, and the exogenous state variables in this model. Finally, make a list of all of the exogenous parameters of this economy. (5 points)

(b) Write the household's Lagrangian optimization problem and find the first order necessary conditions for optimality of the household. Denote the multipliers on constraints (1) and (2) with $\lambda_{1,t}$ and