Econ 6190 Problem Set 8

Fall 2024

1. [Hansen] A Bernoulli random variable X is

$$P(X = 0) = 1 - p$$
$$P(X = 1) = p$$

Given a random sample $\{X_i, i = 1 \dots n\}$ from X,

(a) Find the MLE estimator $\hat{p}_{MLE}$ for p.

(b) Find the asymptotic distribution of $\hat{p}_{MLE}$.

(c) Propose an estimator for the asymptotic variance V of $\hat{p}_{MLE}$.

(d) Show the variance estimator you proposed in (c) is consistent.

(e) Calculate the information for p by taking the variance of the efficient score.

(f) Calculate the information for p by taking the expectation of (minus) the second derivative. Did you obtain the same answer?

(g) Thus find the Cramér-Rao lower bound (CRLB) for p.

(h) Let $\text{var}(\hat{p}_{MLE})$ be the asymptotic variance of $\hat{p}_{MLE}$. Compare $\text{var}(\hat{p}_{MLE})$ with the CRLB.

(i) Propose a Method of Moment Estimator $\hat{p}_{MME}$ for p.

2. Suppose X follows a uniform distribution $[0, \theta]$ with $\theta > 0$. Given a random sample $\{X_i, i = 1 \dots n\}$ drawn from X, find the MLE estimator for θ.

3. Suppose X follows a normal distribution with unknown mean μ and variance $\sigma^2 > 0$. The density of X is

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(x-\mu)^2}{2\sigma^2}).$$

Given a random sample $\{X_i, i = 1 \dots n\}$ drawn from X, find the MLE estimator for (μ, σ^2).

4. Based on the notation in the slides on *Estimation*, let us prove the Information Matrix Equality

$$\mathbb{E}\left[\frac{\partial^2 \log f(X|\theta_0)}{\partial\theta\partial\theta'}\right] = -\mathbb{E}\left[\frac{\partial \log f(X|\theta_0)}{\partial\theta}\frac{\partial \log f(X|\theta_0)}{\partial\theta'}\right].$$

Let $f = f(x|\theta_0)$, ∇_j means derivative with respect to the j-th element $\theta^{(j)}$, and ∇_{jk} mean 2nd-order derivative with respect to $\theta^{(j)}$ and $\theta^{(k)}$. Suppose we can exchange the integral "$\int$" and derivatives "∇_j".

(a) By differentiating $\int f dx = 1$ with respect to $\theta^{(j)}$, show that $\mathbb{E}[\nabla_j \log f] = 0$.

(b) By differentiating $\mathbb{E}[\nabla_j \log f] = 0$ with respect to $\theta^{(k)}$, show that

$$\mathbb{E}[\nabla_{jk} \log f] + \mathbb{E}\left[(\nabla_j \log f)(\nabla_k \log f)\right] = 0,$$

which yields the Information Matrix Equality.

5. [Hansen 10.16] Let $g(x)$ be a density function of a random variable with mean μ and variance σ^2. Let X be a random variable with density function

$$f(x|\theta) = g(x)(1 + \theta(x - \mu)).$$

Assume $g(x)$, μ and σ^2 are known. The unknown parameter is θ. Assume that X has bounded support so that $f(x|\theta) \geq 0$ for all x.

(a) Verify that $\int_{-\infty}^{\infty} f(x|\theta)dx = 1$.

(b) Calculate $\mathbb{E}[X]$.

(c) Find the information $\mathcal{F}_\theta$ for θ when true parameter is θ_0. Write your expression as an expectation of some function of X

(d) Find a simplified expression for $\mathcal{F}_\theta$ when $\theta_0 = 0$.

(e) Given a random sample $\{X_1, ..., X_n\}$, write the log-likelihood function for θ.

(f) Find the first-order-condition for the MLE $\hat{\theta}$ for θ_0.

(g) Using the known asymptotic distribution for maximum likelihood estimators, find the asymptotic distribution for $\sqrt{n}(\hat{\theta} - \theta_0)$ as $n \to \infty$

(h) How does the asymptotic distribution simplify when $\theta_0 = 0$?

6. Complete the proof of Cramér-Rao Lower Bound on page 20 of the slides on *Estimation* by showing

$$\operatorname{var}\left(\frac{\partial}{\partial\theta} \log f(\mathbf{X}|\theta_0)\right) = n\mathscr{F}_\theta$$

7. Let $\hat{F}_n(x)$ denote the empirical distribution function of a random sample. For each fixed x, show that

$$\sqrt{n}(\hat{F}_n(x) - F(x)) \xrightarrow{d} N(0, F(x)(1 - F(x))),$$

where $F(x) = P\{X \leq x\}$ is the cdf function evaluated at x.

8. [Hansen] Let X follows an exponential distribution with pdf $f(x) = \theta\exp(-\theta x), x \geq 0, \theta > 0$. The expected value of X is given by $\mathbb{E}X = \frac{1}{\theta}$

(a) Find the Cramér-Rao lower bound for θ.

(b) Find the Method of Moment Estimator $\hat{\theta}_{MME}$ for θ.

(c) Find the asymptotic distribution of $\hat{\theta}_{MME}$ by delta method.

Q1

(a) The probability mass function of X is $f(x) = p^x(1-p)^{1-x}, x = 0, 1$. Hence the likelihood function is

$$L_n(p) = \Pi_{i=1}^n p^{X_i}(1-p)^{1-X_i}.$$

The log-likelihood is

$$\begin{aligned}\ell_n(p) &= \sum_{i=1}^n \log\left(p^{X_i}(1-p)^{1-X_i}\right) \\ &= \log(p)\sum_{i=1}^n X_i + \log(1-p)\sum_{i=1}^n (1-X_i)\end{aligned}$$

$\hat{p}_{MLE}$ should satisfy the FOC:

$$\frac{\partial}{\partial p}\ell_n(p)|_{p=\hat{p}_{MLE}} = \frac{1}{\hat{p}_{MLE}}\sum_{i=1}^n X_i - \frac{1}{1-\hat{p}_{MLE}}\sum_{i=1}^n (1-X_i) = 0,$$

which yields $\hat{p}_{MLE} = \frac{1}{n}\sum_{i=1}^n X_i$. The SOC is

$$\begin{aligned}\frac{\partial^2}{\partial p^2}\ell_n(p)|_{p=\hat{p}_{MLE}} &= -\frac{\sum_{i=1}^n X_i}{\hat{p}_{MLE}^2} - \frac{\sum_{i=1}^n (1-X_i)}{(1-\hat{p}_{MLE})^2} \\ &= -\frac{n^2}{\sum_{i=1}^n X_i} - \frac{n^2}{(n-\sum_{i=1}^n X_i)} < 0\end{aligned}$$

since $\sum_{i=1}^n X_i \geq 0$ and $n - \sum_{i=1}^n X_i \geq 0$.

(b) Since $\hat{p}_{MLE} = \frac{1}{n}\sum_{i=1}^n X_i$, $\mathbb{E}X_i = p$, $\mathbb{E}X_i^2 = p < \infty$, it follows by Lindeberg Levy CLT:

$$\sqrt{n}(\hat{p}_{MLE} - p) \xrightarrow{d} \mathrm{N}(0, \mathrm{var}(X_i)),$$

where $\mathrm{var}(X_i) = \mathbb{E}X_i^2 - (\mathbb{E}X_i)^2 = p - p^2 = p(1-p)$.

(c) $V = p(1-p)$. A plug-in estimator of V is $\hat{V} = \hat{p}_{MLE}(1-\hat{p}_{MLE})$.

(d) Note $\hat{p}_{MLE} = \frac{1}{n}\sum_{i=1}^n X_i$ and $\mathbb{E}X_i = p < \infty$, it follows by Khinchin's WLLN $\hat{p}_{MLE} \xrightarrow{p} p$. Moreover, it is clear $f(x) = x(1-x)$ is a continuous function of x. It follows by continuous mapping theorem that

$$\hat{V} = f(\hat{p}) \xrightarrow{p} f(p) = V.$$

Note the probability mass function of X is $f(x) = p^x(1-p)^{1-x}, x = 0, 1$.

(e) Since expectation of efficient score is 0,

$$\begin{aligned}
\mathscr{F}_\theta &= \mathbb{E}\left[\left(\frac{\partial}{\partial p}\log f(X|p)\right)^2\right] \\
&\mathbb{E}\left[\left(\frac{\partial}{\partial p}\log\left(p^X(1-p)^{1-X}\right)\right)^2\right] \\
&= \mathbb{E}\left[\left(\frac{X}{p} - \frac{(1-X)}{1-p}\right)^2\right] \\
&= \frac{\mathbb{E}\left[X^2\right]}{p^2} + 2\mathbb{E}\left[\frac{X}{p}\frac{(1-X)}{1-p}\right] + \frac{\mathbb{E}\left[(1-X)^2\right]}{(1-p)^2} \\
&= \frac{1}{p} + \frac{1}{1-p} = \frac{1}{p(1-p)}.
\end{aligned}$$

where the last equality follows from: (1) $X^2 = X$, (2) $X(1-X) = 0$ (3) $(1-X)^2 = (1-X)$.

(f) $\mathscr{F}_\theta = -\mathbb{E}\left[\left(\frac{\partial^2}{\partial p^2}\log f(X|p)\right)\right]$. Since $\frac{\partial}{\partial p}\log f(X|p) = \frac{X}{p} - \frac{(1-X)}{1-p}$,

$$\frac{\partial^2}{\partial p^2}\log f(X|p) = -\frac{X}{p^2} - \frac{(1-X)}{(1-p)^2}.$$

It follows

$$\begin{aligned}
\mathscr{F}_\theta &= \mathbb{E}\left[\frac{X}{p^2} + \frac{(1-X)}{(1-p)^2}\right] = \frac{\mathbb{E}[X]}{p^2} + \frac{1-\mathbb{E}X}{(1-p)^2} \\
&= \frac{1}{p} + \frac{1}{(1-p)} = \frac{1}{p(1-p)}.
\end{aligned}$$

So yes we obtain the same answer.

(g) $CRLB = (n\mathscr{F}_\theta)^{-1} = \frac{p(1-p)}{n}$.

(h) Recall

$$\sqrt{n}(\hat{p}_{MLE} - p) \xrightarrow{d} \mathrm{N}(0, p(1-p)),$$

that is, the asymptotic variance of $\sqrt{n}(\hat{p}_{MLE} - p)$ is $p(1-p)$. That is to say, the asymptotic variance of $\hat{p}_{MLE}$ when n is large is approximately $\frac{p(1-p)}{n}$, which is equivalent to CRLB.

(i) Since $\mathbb{E}X = p$, $\hat{p}_{MME} = \frac{1}{n}\sum_{i=1}^n X_i$.

2. Note the density of X is $f(x|\theta) = \frac{1}{\theta}$, $0 \le x \le \theta$. The log density is

$$\log f(x|\theta) = \begin{cases} -\log\theta & 0 \le x \le \theta \\ -\infty & \text{otherwise} \end{cases}$$

Thus the log-likelihood is

$$\begin{aligned} \ell_n(\theta) &= \frac{1}{n}\sum_{i=1}^{n} \log f(X_i|\theta) \\ &= \begin{cases} -\log\theta & 0 \le X_i \le \theta \text{ for all } i = 1 \dots n \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

That is, $\ell_n(\theta)$ is not $-\infty$ if and only if $0 \le X_i \le \theta$ for all $i = 1 \dots n$, or equivalently, $\theta \ge \max_{i \le n} X_i$. And when $\theta \ge \max_{i\le n} X_i$, $\ell_n(\theta) = -\log\theta$ is a decreasing function of θ. Thus the log-likelihood is maximized at $\max_{i\le n} X_i$. This means $\hat{\theta}_{MLE} = \max_{i\le n} X_i$.

Note in this example, the likelihood is not differentiable at the maximum. Thus the MLE does not satisfy a first order condition. Hence the MLE cannot be found by solving first order conditions.

Q3 [Sketch]

The log-likelihood is

$$\ell_n(\theta) = -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i - \mu)^2$$

MLE estimator $(\hat{\mu}, \hat{\sigma}^2)$ should satisfy FOC

$$\begin{aligned} \frac{\partial \ell_n(\mu,\sigma^2)}{\partial \mu}\Big|_{\mu=\hat{\mu},\sigma^2=\hat{\sigma}^2} &= \frac{1}{\hat{\sigma}^2}\sum_{i=1}^{n}(X_i - \hat{u}) = 0 \\ \frac{\partial \ell_n(\mu,\sigma^2)}{\partial \sigma^2}\Big|_{\mu=\hat{\mu},\sigma^2=\hat{\sigma}^2} &= -\frac{n}{2\hat{\sigma}^2} + \frac{1}{2\left(\hat{\sigma}^2\right)^2}\sum_{i=1}^{n}(X_i - \hat{u})^2 = 0. \end{aligned}$$

It follows $\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i$, $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{u})^2$.

Let $\theta = (\mu, \sigma^2)$ and $\hat{\theta} = (\hat{\mu}, \hat{\sigma}^2)$. The SOC should be such that

$$\frac{\partial^2 \ell_n(\theta)}{\partial\theta\partial\theta'}\Big|_{\theta=\hat{\theta}} \text{ is negative definite.}$$

Note

$$\begin{aligned} \frac{\partial^2 \ell_n(\theta)}{\partial\theta\partial\theta'} &= \begin{pmatrix} \frac{\partial^2 \ell_n(\mu,\sigma^2)}{\partial\mu^2} & \frac{\partial^2 \ell_n(\mu,\sigma^2)}{\partial\mu\partial\sigma^2} \\ \frac{\partial^2 \ell_n(\mu,\sigma^2)}{\partial\sigma^2\partial\mu} & \frac{\partial^2 \ell_n(\mu,\sigma^2)}{\partial(\sigma^2)^2} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{n}{\sigma^2} & -\frac{1}{\sigma^4}\sum_{i=1}^{n}(X_i - \mu) \\ -\frac{1}{\sigma^4}\sum_{i=1}^{n}(X_i - \mu) & \frac{n}{2\sigma^4} - \frac{1}{\sigma^6}\sum_{i=1}^{n}(X_i - \mu)^2 \end{pmatrix} \end{aligned}$$

Thus

$$\frac{\partial^2 \ell_n(\theta)}{\partial\theta\partial\theta'}\Big|_{\theta=\hat{\theta}} = \begin{pmatrix} -\frac{n}{\hat{\sigma}^2} & 0 \\ 0 & -\frac{n}{2\hat{\sigma}^4} \end{pmatrix}$$

which is negative definite.

Q4

(a) $\forall j$, differentiating $\int f dz = 1$ with respect to $\theta^{(j)}$, and exchanging "$\int$" and derivatives "∇_j ", we get:

$$\int \nabla_j f dz = 0$$

Thus:

$$\begin{aligned} 0 = \int \nabla_j f dz &= \int (\nabla_j f)\frac{1}{f} f dz \\ &= \int [\nabla_j \log f] f dz \\ &= \mathbb{E}[\nabla_j \log f] \end{aligned}$$

(b) Take one more derivative with respect to $\theta^{(k)}$ yields

$$\begin{aligned} 0 &= \nabla_k \mathbb{E}[\nabla_j \log f] \\ &= \int \nabla_k [(\nabla_j \log f) f] dz (\text{exchange integral and derivative}) \\ &= \int \{(\nabla_{jk} \log f) f + (\nabla_j \log f)\nabla_k f\} dz (\text{chain rule}) \\ &= \underbrace{\int \{(\nabla_{jk} \log f) f\} dz}_{(1)} + \underbrace{\int \{(\nabla_j \log f)\nabla_k f\} dz}_{(2)} \end{aligned}$$

$$\begin{aligned} (1) &= \mathbb{E}(\nabla_{jk} \log f) \\ (2) &= \int (\nabla_j \log f)\left(\nabla_k f \frac{1}{f}\right) f dz \\ &= \int (\nabla_j \log f)(\nabla_k \log f) f dz \\ &= \mathbb{E}[(\nabla_j \log f)(\nabla_k \log f)] \end{aligned}$$

Q5

(a)

$$\begin{aligned}\int_{-\infty}^{\infty} f(x|\theta)dx &= \int_{-\infty}^{\infty} g(x)(1+\theta(x-\mu))dx \\ &= \int_{-\infty}^{\infty} g(x)dx + \int_{-\infty}^{\infty} g(x)\theta(x-\mu)dx \\ &= 1+\theta\int_{-\infty}^{\infty} g(x)(x-\mu)dx \\ &= 1+\theta\left(\int_{-\infty}^{\infty} g(x)xdx-\mu\right) = 1\end{aligned}$$

where the third equality is because $\int_{-\infty}^{\infty} g(x)dx = 1$ since $g(x)$ is a density, and the fourth equality uses $\int_{-\infty}^{\infty} g(x)dx = 1$ again. Final equality follows from $\int_{-\infty}^{\infty} g(x)xdx = \mu$ by assumption.

(b)

$$\begin{aligned}\mathbb{E}X &= \int xf(x|\theta)dx \\ &= \int_{-\infty}^{\infty} g(x)(1+\theta(x-\mu))xdx \\ &= \underbrace{\int_{-\infty}^{\infty} g(x)xdx}_{\mu} + \theta\int_{-\infty}^{\infty} g(x)x(x-\mu)dx \\ &= \mu+\theta\underbrace{\int_{-\infty}^{\infty} g(x)(x-\mu)^2dx}_{\sigma^2} + \theta\mu\underbrace{\int_{-\infty}^{\infty} g(x)(x-\mu)dx}_{0} \\ &= \mu+\theta\sigma^2.\end{aligned}$$

(c) The log likelihood for a single observation X is

$$\begin{aligned}\log f(X|\theta) &= \log\left[g(X)(1+\theta(X-\mu))\right] \\ &= \log\left[g(X)\right] + \log\left[(1+\theta(X-\mu))\right].\end{aligned}$$

Efficient score is

$$\frac{\partial}{\partial\theta}\log f(X|\theta_0) = \frac{X-\mu}{1+\theta_0(X-\mu)}.$$

So

$$\begin{aligned}\mathcal{F}_\theta &= \mathbb{E}\left[\left(\frac{\partial}{\partial\theta}\log f(X|\theta_0)\right)^2\right] \\ &= \mathbb{E}\left[\left(\frac{X-\mu}{1+\theta_0(X-\mu)}\right)^2\right]\end{aligned}$$

where the expectation is taken with respect to density $f(x|\theta_0)$.

(d) when $\theta_0 = 0$,

$$\mathcal{F}_\theta = \mathbb{E}\left[(X-\mu)^2\right]$$

(e)

$$\ell_n(\theta) = \sum_{i=1}^{n} \log f(X_i|\theta) = \sum_{i=1}^{n} \log [g(X_i)] + \sum_{i=1}^{n} \log [(1 + \theta(X_i - \mu))]$$

(f) Note

$$\frac{\partial}{\partial \theta} \ell_n(\theta) = \sum_{i=1}^{n} \frac{X_i - \mu}{1 + \theta(X_i - \mu)}$$

So the MLE estimator $\hat{\theta}$ should satisfy FOC:

$$\sum_{i=1}^{n} \frac{X_i - \mu}{1 + \hat{\theta}(X_i - \mu)} = 0$$

(g) The asymptotic distribution of $\sqrt{n}(\hat{\theta} - \theta_0)$ should be

$$\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} \mathrm{N}(0, \mathcal{F}_\theta^{-1}),$$

where $\mathcal{F}_\theta = \mathbb{E}\left[\left(\frac{X-\mu}{1+\theta_0(X-\mu)}\right)^2\right]$.

(h) When $\theta_0 = 0$,

$$\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} \mathrm{N}(0, \mathbb{E}\left[(X - \mu)^2\right]^{-1}).$$

6. Recall from slides: $\mathbf{x} = (x_1, \dots x_n)'$, $\mathbf{X} = (X_1, \dots X_n)'$. By definition

$$\begin{aligned}\operatorname{var}\left(\frac{\partial}{\partial\theta}\log f(\mathbf{X}|\theta_0)\right) &= \mathbb{E}\left[\frac{\partial}{\partial\theta}\log f(\mathbf{X}|\theta_0)\frac{\partial}{\partial\theta'}\log f(\mathbf{X}|\theta_0)\right] \\ &\quad - \mathbb{E}\left[\frac{\partial}{\partial\theta}\log f(\mathbf{X}|\theta_0)\right]\mathbb{E}\left[\frac{\partial}{\partial\theta}\log f(\mathbf{X}|\theta_0)\right] \\ &= \mathbb{E}\left[\frac{\partial}{\partial\theta}\log f(\mathbf{X}|\theta_0)\frac{\partial}{\partial\theta'}\log f(\mathbf{X}|\theta_0)\right]\end{aligned}$$

since we have shown in class that $\mathbb{E}\left[\frac{\partial}{\partial\theta}\log f(\mathbf{X}|\theta_0)\right] = 0$. It remains to find

$$T = \mathbb{E}\left[\frac{\partial}{\partial\theta}\log f(\mathbf{X}|\theta_0)\frac{\partial}{\partial\theta'}\log f(\mathbf{X}|\theta_0)\right].$$

Note again by iid assumption

$$\frac{\partial}{\partial\theta}\log f(\mathbf{X}|\theta_0) = \frac{\partial}{\partial\theta}\log f(X_1, \dots X_n|\theta_0) = \sum_{i=1}^n \frac{\partial}{\partial\theta}\log f(X_i|\theta_0).$$

Thus

$$\begin{aligned}T &= \mathbb{E}\left[\sum_{i=1}^n \frac{\partial}{\partial\theta}\log f(X_i|\theta_0)\sum_{i=1}^n \frac{\partial}{\partial\theta'}\log f(X_i|\theta_0)\right] \\ &= \mathbb{E}\left[\sum_{i=1}^n \frac{\partial}{\partial\theta}\log f(X_i|\theta_0)\frac{\partial}{\partial\theta'}\log f(X_i|\theta_0) + \sum_{i\neq j}^n \frac{\partial}{\partial\theta}\log f(X_i|\theta_0)\frac{\partial}{\partial\theta'}\log f(X_i|\theta_0)\right] \\ &= \left[\underbrace{\sum_{i=1}^n \mathbb{E}\frac{\partial}{\partial\theta}\log f(X_i|\theta_0)\frac{\partial}{\partial\theta'}\log f(X_i|\theta_0)}_{A} + \underbrace{\sum_{i\neq j}^n \mathbb{E}\frac{\partial}{\partial\theta}\log f(X_i|\theta_0)\frac{\partial}{\partial\theta'}\log f(X_i|\theta_0)}_{B}\right],\end{aligned}$$

where the third equality we used linearity of expectation.

Now note $\mathbb{E}\frac{\partial}{\partial\theta}\log f(X_i|\theta_0)\frac{\partial}{\partial\theta'}\log f(X_i|\theta_0) = \mathscr{F}_\theta$ for each $i = 1 \dots n$ by identical assumption. Thus $A = n\mathscr{F}_\theta$. And $B = 0$ since for each $i \neq j$:

$$\mathbb{E}\frac{\partial}{\partial\theta}\log f(X_i|\theta_0)\frac{\partial}{\partial\theta'}\log f(X_i|\theta_0) = \mathbb{E}\left[\frac{\partial}{\partial\theta}\log f(X_i|\theta_0)\right]\mathbb{E}\left[\frac{\partial}{\partial\theta'}\log f(X_i|\theta_0)\right] = 0$$

where the first equality is by independence and the second equality is by property of efficient score. Thus we have shown

$$\begin{aligned}T &= \mathbb{E}\left[\frac{\partial}{\partial\theta}\log f(\mathbf{X}|\theta_0)\frac{\partial}{\partial\theta'}\log f(\mathbf{X}|\theta_0)\right] \\ &= n\mathscr{F}_\theta\end{aligned}$$

as required.

7. Note for each fixed point x on the real line, we have

$$F(x) = P\{X \le x\} = E[\mathbf{1}\{X \le x\}],$$

while

$$\hat{F}(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}\{x_i \le x\}$$

Therefore $\hat{F}(x) \xrightarrow{p} F(x)$ by Khinchine's LLN for iid data. Moreover,

$$\sqrt{n}(\hat{F}(x) - F(x)) = \sqrt{n}\frac{1}{n}\sum_{i=1}^{n} \{\mathbf{1}\{x_i \le x\} - \mathbb{E}[\mathbf{1}\{X \le x\}]\}$$

We check conditions for Lindeberg-Levy CLT, which requires second moment of $\mathbf{1}\{x_i \le x\}$ to be finite. This is apparent. Thus, we have

$$\sqrt{n}(\hat{F}(x) - F(x)) \xrightarrow{d} N(0, \sigma^2)$$

where

$$\begin{aligned}
\sigma^2 &= Var(\mathbf{1}\{X \le x\}) \\
&= \mathbb{E}\left[\mathbf{1}^2\{X \le x\}\right] - \mathbb{E}^2[\mathbf{1}\{X \le x\}] \\
&= \mathbb{E}[\mathbf{1}\{X \le x\}] - \mathbb{E}^2[\mathbf{1}\{X \le x\}] \\
&= F(x) - F^2(x) \\
&= F(x)(1 - F(x))
\end{aligned}$$

8. (a) $\mathscr{F}_\theta = -\mathbb{E}\left[\left(\frac{\partial^2}{\partial\theta^2}\log f(X|\theta)\right)\right]$. Since $\frac{\partial}{\partial\theta}\log f(X|\theta) = \frac{1}{\theta} - X$,

$$\frac{\partial^2}{\partial\theta^2}\log f(X|\theta) = -\frac{1}{\theta^2}.$$

Hence $\mathscr{F}_\theta = \frac{1}{\theta^2}$. And $CRLB = (n\mathscr{F}_\theta)^{-1} = \frac{\theta^2}{n}$.

(b) Since $\mathbb{E}X = \frac{1}{\theta}$, $\hat{\theta}_{MME}$ should be such that $\frac{1}{\hat{\theta}_{MME}} = \frac{1}{n}\sum_{i=1}^{n} X_i$. That is, $\hat{\theta}_{MME} = \frac{1}{\bar{X}_n}$, where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

(c) By CLT, $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i - \mathbb{E}[X]) \xrightarrow{d} \mathrm{N}(0, \mathrm{var}(X))$, where $\mathrm{var}(X)$ $= \frac{1}{\theta^2}$. That is,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i - \mathbb{E}[X]). \xrightarrow{d} \mathcal{N}(0, \tfrac{1}{\theta^2}) \tag{1}$$

Now, note

$$\hat{\theta}_{MME} = f(\bar{X}_n), \theta = \frac{1}{\mathbb{E}[X]} = f(\mathbb{E}[X]),$$

where $f(a) = \frac{1}{a}$. By Taylor expansion,

$$\sqrt{n}(f(\bar{X}_n) - f(\mathbb{E}[X])) = \sqrt{n}f'(\tilde{X}_n)(\bar{X}_n - \mathbb{E}[X]), \tag{2}$$

where $f'(a) = -\frac{1}{a^2}$, and $\tilde{X}_n$ is between $\bar{X}_n$ and $\mathbb{E}[X]$. Since $\bar{X}_n \xrightarrow{p} \mathbb{E}[X]$,

$$f'(\tilde{X}_n) \xrightarrow{p} f'(\mathbb{E}X) = -\frac{1}{(\mathbb{E}X)^2}. \tag{3}$$

Combining (1), (2), and (3), and by continuous mapping theorem,

$$\sqrt{n}(f(\bar{X}_n) - f(\mathbb{E}[X])) \xrightarrow{d} \mathrm{N}(0, \frac{\mathrm{var}(X)}{(\mathbb{E}X)^4}).$$

Note $\mathbb{E}X = \frac{1}{\theta}$, and $\mathrm{var}(X) = \frac{1}{\theta^2}$, we have

$$\sqrt{n}(\hat{\theta}_{MME} - \theta) \xrightarrow{d} \mathrm{N}(0, \theta^2).$$