About TA sections:

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Our plan for today:[1]

- General equilibrium in pure exchange economy (model from class)
- Arrow-Debreu economy
- Pareto efficiency and SPP
- Practice question (Q 2022)
- Math appendix

[1]Materials adapted from notes provided by a previous Teaching Assistant, Zhuoheng Xu.

1 General Equilibrium in a Pure Exchange Economy

1.1 Model from class

Agent $i \in \{1, 2\}$ has the utility function ($0 < \beta < 1$):

$$u(c^i) = \sum_{t=0}^{\infty} \beta^t \log(c_t^i)$$

Endowments are given by

$$e_t^1 = \begin{cases} 2 & \text{if } t \text{ is even} \\ 0 & \text{if } t \text{ is odd} \end{cases}$$

$$e_t^2 = \begin{cases} 0 & \text{if } t \text{ is even} \\ 2 & \text{if } t \text{ is odd} \end{cases}$$

Compare this new model with the one we discussed last time (partial equilibrium):

- Concavity of the utility function imply agents want to smooth consumption in both models.
- In partial equilibrium, we assumed the existence of some bank, so agent could freely save and borrow in each period, as long as the borrowing constraint is not reached. Hence, consumption smoothing could be achieved by intertemporal reallocation of endowments over periods.
- In the new model of general equilibrium, there is no such bank. As both agents experience good and bad days, they share an incentive to smooth consumption. In essence, each agent plays the role of a "bank" for the other. Here, for each set of prices $\{p_t\}_{t=0}^{\infty}$, while adhering to their budget constraints, each agent optimizes their consumption across periods, which involves making decisions about borrowing and saving. However, these borrowing and saving decisions are bounded by a constraint: the amount that agent 1 is willing to save/borrow must equate to the amount that agent 2 is willing to borrow/save. Adjustments to the price must be made to ensure that net saving equals 0 in each period. The lesson here:

never consider the market clearing condition a constraint when solving the agent's utility maximization problem because it is a separate equilibrium condition.

1.2 Solving the model

1. Write the Lagrangian function for each agent and find first order conditions:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t ln(c_t^i) + \lambda^i [\sum_{t=0}^{\infty} p_t(e_t^i - c_t^i)]$$

   Intuition: Before the world begins, agents know the whole sequence of their endowments and optimize over all periods (note that the Lagrange multiplier is not indexed by t here).

2. Write consumption Euler equation for each agent:

$$\beta p_t c_t^i = p_{t+1} c_{t+1}^i$$

3. By summing this equation over all agents and using market clearing condition, find prices (use normalization: $p_0 \equiv 1$):

$$\hat{p}_t = \beta^t$$

   Remark: Note that only relative prices matter.

4. Substituting back into Euler equation, we obtain the desired result - consumption smoothing:

$$\forall i, \forall t \quad c_t^i = c_{t+1}^i = c^i$$

5. Substituting prices and consumption back into budget constraints, we get the solution - sequence of consumption allocations for each agent and the vector of equilibrium prices (prices found in step 3).

2 Arrow-Debreu Economy

2.1 About the economy

Here we consider Arrow-Debreu Equilibrium (ADE) within the setting of an economy with I agents who receive **stochastic** endowments.

Key ideas:

- Trade, at least for now, takes place at the very beginning, before anyone learns anything about the evolution of uncertainty (but agents know what states are possible!).

- Individuals trade state-contingent commodities, i.e., promises to deliver or receive different amounts of goods as time and uncertainty evolve.

- Uncertainty is described by a finite set of states, S. An element (or subset) s_t is called a dated **event**.

- Each **history** s^t is associated with the individual endowment $y_t^i(s^t)$ for agent i.

- If s^t is a history, then $\pi_t(s^t)$ is the perceived probability of the history s^t happening at time t by all agents at the beginning of time, namely at $t = -1$ (or, at the very beginning of time t=0 before anything is realized). This is public information.

- Agents aim to maximize expected utility.

Remark: Even though realizations of individual endowments $y_t^i(s^t)$ are stochastic, there is no aggregate uncertainty because possible endowments and the perceived probability of each history are known by all agents from the beginning.

2.2 Equilibrium

Two components:

(a) $\{\{c_t^i(s^t)\}_{t=0}^{\infty}\}_{i\in I}$ are the consumption stream of every individual i at each time t with every possible stochastic realization history s^t;

(b) $\{q_t^0(s^t)\}_{t=0}^{\infty}$ are the time-0 prices of time-t consumption given history s^t under AD trading arrangements. The subscript t indicates the timing of the consumption. The superscript 0 indicates the unit of account. The s^t in parenthesis indicates the history.

All decisions are made "before the world begins". Individuals exchange promises to deliver or receive amounts of the commodities in pre-specified dated events. This is called a contingent claim.

In equilibrium:

- Given prices, for each $i \in I$, household i maximizes its utility:

$$\max_{\{c_t^i(s^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u\left[c_t^i(s^t)\right] \pi_t(s^t)$$

  Interpretation: Each household seeks to maximize their expected lifetime utility. This is done by choosing a consumption plan $\{c_i^t(s^t)\}_{t=0}^{\infty}$ that maximizes the sum of discounted utility over all time periods t and across all possible histories s^t.

  subject to

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) y_t^i(s^t) \quad \forall i$$

  Interpretation: The present value of what a household consumes over time and across all possible scenarios must not exceed the present value of their total endowments.

$$c_t^i(s^t) \geq 0$$

- Allocation is feasible:

$$\sum_{i \in I} c_t^i(s^t) \leq \sum_{i \in I} y_t^i(s^t) \quad \forall t, \quad \forall s^t$$

  Interpretation: At any point in time and for any given state of the world, the total consumption by all agents must not exceed the total endowment.

Remark: When you solve this problem, you will notice that each agent's consumption depends on his own endowment only through the aggregate endowment, i.e., agents insure themselves from the entire idiosyncratic risk and only face the aggregate risk.

3 Pareto Efficiecy and Social Planner's Problem (SPP)

Intuition: Social planner - "benevolent dictator" who has **complete information** about the economy and aims to **maximize social welfare**. One the contrary, individual agents do not always have complete information and only seek to maximize their own utility.

Pareto efficiency - a measure of the efficiency of resource allocations. In the Pareto efficient allocation, there does not exist another set of consumption streams in which someone is strictly better off without anyone worse off.

Economy is the same (and we will eventually get the same allocation as in ADE) but in SPP some things change:

1. Only one component: $\{\{c_t^i(s^t)\}_{t=0}^{\infty}\}_{i\in I}$
   Important to remember: In SPP, there is **no price** $\{p_t\}_{t=0}^{\infty}$. Social planner only allocates goods according to the "social utility". Intuitively, any allocation that social planner suggests is Pareto efficient, so nobody wants to trade. We simply do not need prices.

2. Characterization:
   Proposition from class: An allocation is efficient if and only if it solves the social planner's problem.

$$\max_{\{\{c_t^i(s^t)\}_{t=0}^{\infty}\}} \sum_{i=1}^{I} \lambda_i U\left(c_t^i(s^t)\right) \quad s.t.$$

$$\sum_{i\in I} c_t^i(s^t) \leq \sum_{i\in I} y_t^i(s^t) \quad \forall t, \quad \forall s^t$$

   for some non-negative λ^i for all i. The λ's are some Pareto weights.

   Interpretation: a higher Pareto weight on an individual indicates that the social planner cares more about them.

   Remark: In SPP, instead of budget constraint, we have feasibility (or, resource) constraint.

4 Practice Question: Arrow-Debreu Economy without Uncertainty (Q 2022)

Consider an endowment economy with a single non-storable good. There are two types of agents each of whom lives forever and has a time-additive utility function. There is a measure of $1/2$ of each type. Both types of agents have the same lifetime discounted utility

$$\sum_{t=0}^{\infty} \beta^t u(c_t^i)$$

Where $0 < \beta < 1$. Agents receive consumption goods in every period through an endowment process ω_t^i. Agents of type $i = 1$ have the endowment process $\omega_t^1 = \lambda^t$. Agents of type $i = 2$ have the endowment process $\omega_t^1 = 1 - \lambda^t$. Where $0 < \lambda < 1$.

1. Define an Arrow-Debreu equilibrium in this environment.

2. From now on, assume that $u(c) = \log(c)$. Show that in an Arrow-Debreu equilibrium we must have
$$\frac{c_t^1}{c_t^2} = \frac{\alpha_2}{\alpha_1}$$
What are α_1 and α_2?

3. Find c_t^1 and c_t^2 as functions of α_1 and α_2. Are agents smoothing consumption? Why?

4. Normalize $q_0 = 1$. Find the prices q_t.

5. Find α_1 and α_2, and c_t^1 and c_t^2. Intuitively describe what happens to these quantities as λ moves from 0 to 1.

6. Write down the problem of a social planner that puts weight κ on agent 1 and $1 - \kappa$ on agent 2. Derive c_t^1 and c_t^2 in the planner's optimal allocation. Is there a value of κ such that the planner's allocation coincides with the Arrow-Debreu equilibrium? Why or why not?

7. Going back to the equilibrium, suppose that agents cannot trade with each other. What are c_t^1 and c_t^2? Is there a value of κ such that the planner's problem coincides with this no-trade equilibrium? Why or why not?

5 Math Appendix

Definition: For any scalar k, a real-valued function $f(x_1, \ldots, x_n)$ is homogeneous of degree k if

$$f(tx_1, \ldots, tx_n) = t^k f(x_1, \ldots, x_n) \text{ for all } x_1, \ldots, x_n \text{ and all } t > 0.$$

Euler's theorem: Let $f(x)$ be a C^1 homogeneous function of degree k on $\mathbb{R}^n_+$. Then for all x,

$$x_1 \frac{\partial f}{\partial x_1}(x) + x_2 \frac{\partial f}{\partial x_2}(x) + \ldots + x_n \frac{\partial f}{\partial x_n}(x) = k f(x)$$

or, in gradient notation,

$$\mathbf{x} \cdot \nabla f(x) = k f(x)$$

For an HOD1 utility function of the form $U(c_1, c_2, \ldots, c_N)$ for N goods/sectors, the Euler's Theorem suggests that:

$$c_1 \frac{\partial U}{\partial c_1}(c) + c_2 \frac{\partial U}{\partial c_2}(c) + \ldots + c_n \frac{\partial U}{\partial c_n}(c) = U(c_1, c_2, \ldots, c_N)$$

Definition: Let I be an interval on the real line. Then $g : I \to \mathbb{R}$ is a monotonic transformation of I if g is a strictly increasing function on I. Furthermore, if g is a monotonic transformation and u is a real-valued function of n variables, then we say that

$$g \circ u : \mathbf{x} \mapsto g(u(\mathbf{x}))$$

is a monotonic transformation of u.

A function $v : \mathbb{R}^n_+ \to \mathbb{R}$ is called homothetic if it is a monotone transformation of a homogeneous function (of degree one), that is, if there is a monotonic transformation $x \mapsto g(z)$ of $\mathbb{R}_+$ and a homogeneous function $u : \mathbb{R}^n_+ \to \mathbb{R}_+$ such that $v(x) = g(u(x))$ for all x in the domain.