ECON 6170 Section 7

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Upper Hemicontinuity

Definition 1. The upper inverse image of a set S under the correspondence F is given by

$$F^{-1}(S) := \{x \in X \mid F(x) \subseteq S\}$$

Definition 2 (Characterisations of upper hemicontinuity)**.** The following imply upper hemicontinuity of F:[1]

(i) For any $x \in X$ and any open $U \subseteq Y$ such that $F(x) \subseteq U$, there exists $\varepsilon > 0$ such that $F(B_\varepsilon(x)) \subseteq U$.

(ii) $F^{-1}(U) \subseteq X$ is open for every open $U \subseteq Y$.

(iii) For any $x \in X$, if (x_n) is a sequence in X converging to x and (y_n) is a sequence in Y such that $y_n \in F(x_n)$ for all n, then some subsequence $y_{n_k} \to y \in F(x)$.

(iv) $\operatorname{Gr} F$ is closed relative to $X \times Y$, if $F(X) \subseteq K$ for some compact K.[2]

Upper hemicontinuity implies the following:

(i)

(ii)

(3) If F is compact-valued then (iii) must hold.

(4) If F is closed-valued then $\operatorname{Gr} F$ must be closed relative to $X \times Y$.

Section Exercise 1. In each case, provide an example correspondence $F : X \subseteq \mathbb{R} \rightrightarrows Y \subseteq \mathbb{R}$ satisfying the statement:

- ➢ F is upper hemicontinuous but does not satisfy the sequence characterisation, (iii).

[1] Here, open means open *relative* to the containing space (the domain or codomain of F) and $B_\varepsilon(x)$ is shorthand for $B_\varepsilon(x) \cap X$.

[2] (iv) is a slight generalisation of Proposition 5 (i) that follows from a trivial change to the proof in the lecture notes (replacing $y_n \in Y$ with $y_n \in K$).

Take $F(x) := (0,1]$ for all x. F is upper hemicontinuous because

$$F^{-1}(U) = \begin{cases} X & \text{if } (0,1] \subseteq U \\ \varnothing & \text{otherwise} \end{cases}$$

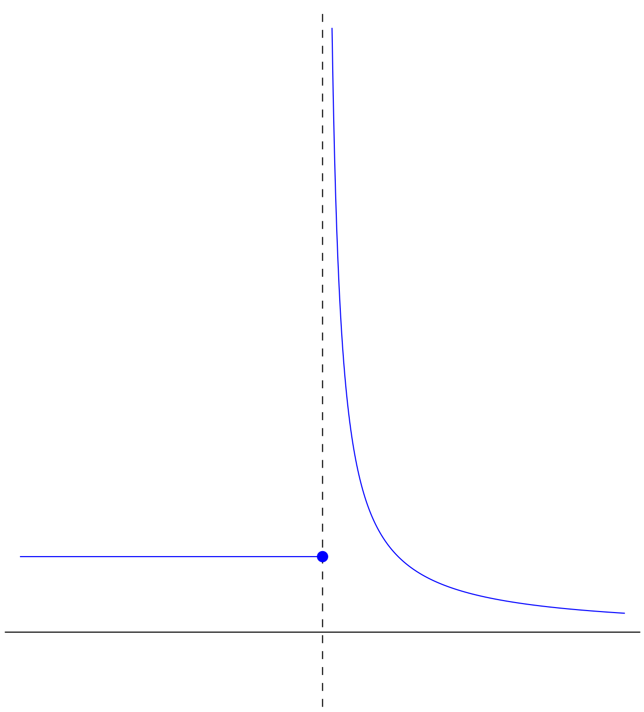
both of which are open relative to X. But it doesn't satisfy the sequence characterisation, as the constant sequence $x, x, x, \ldots$ converges to x and the sequence defined by $y_n := 1/n$ satisfies $y_n \in F(x)$ for all n, but $y_n \to 0 \notin F(x)$. In particular every subsequence of (y_n) fails to converge to an element of $F(x)$.

- F is upper hemicontinuous but does not have a closed graph.
  Consider again F defined above. $(x, 1/n)_{n=1}^{\infty}$ is a sequence in the graph that converges to $(x, 0)$ which lies outside the graph.

- F has a closed graph but is not upper hemicontinuous.
  Define the correspondence $F : \mathbb{R} \rightrightarrows \mathbb{R}$ by

$$F(x) := \begin{cases} \{1\} & \text{if } x \leq 0 \\ \{1/x\} & \text{if } x > 0 \end{cases}$$

This is graphed below (compare Figure 1 (e) on Problem Set 6). The curve $\{(x,1) \in \mathbb{R}^2 \mid x \leq 0\}$ is closed. So too is the curve $\{(x,y) \mid y = 1/x \text{ and } x > 0\}$, by a similar argument to that used in Problem 1 of the previous section. The union of two closed sets is again closed, so the graph of F is closed. But the preimage of $(1-\varepsilon, 1+\varepsilon)$ is $(-\infty, 0] \cup (\frac{1}{1+\varepsilon}, \frac{1}{1-\varepsilon})$, which is not open. Therefore, F is not upper hemicontinuous. It is also sufficient to note that F is compact-valued, but we can take a sequence $x_n \to 0$ with $y_n \in F(x_n)$ for all n such that $y_n \to \infty$, so (y_n) does not have a subsequence converging to $y \in \{1\} = F(0)$.



Lower Hemicontinuity

Definition 3. The lower inverse image of a set S under the correspondence F is given by

$$F_{-1}(S) := \{x \in X \mid F(x) \cap S \neq \emptyset\}$$

Definition 4 (Characterisations of lower hemicontinuity)**.** The following are equivalent to lower hemicontinuity of F:

(I) For any $x \in X$ and any open $U \subseteq Y$ such that $F(x) \cap U \neq \emptyset$, there exists $\varepsilon > 0$ such that $F(z) \cap U \neq \emptyset$ for all $z \in B_\varepsilon(x)$.

(II) $F_{-1}(U)$ is open for every open $U \subseteq Y$.

(III) For any $x \in X$, if (x_n) is a sequence in X converging to x, and $y \in F(x)$, then there is some sequence $y_n \to y$, with $y_n \in F(x_n)$ for sufficiently large n.[3]

Section Exercise 2. Define a constant correspondence as one in which $F(x) = F(x')$ for all $x, x' \in X$. Prove that a constant correspondence is both upper and lower hemicontinuous.
Let $S := F(x)$ and let $U \subseteq Y$ be open relative to Y. Then

$$F^{-1}(U) = \begin{cases} X & \text{if } S \subseteq U \\ \emptyset & \text{otherwise} \end{cases}$$

is open, and so is

$$F_{-1}(U) = \begin{cases} X & \text{if } S \cap U \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases}$$

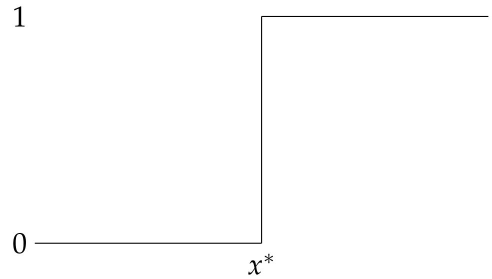
Section Exercise 3. Define the correspondence $F : \mathbb{R} \rightrightarrows \mathbb{R}$ by (compare Figure 1 (a) on Problem Set 6)

$$F(x) := \begin{cases} \{0\} & \text{if } x < x^* \\ [0,1] & \text{if } x = x^* \\ \{1\} & \text{if } x > x^* \end{cases}$$

Prove that F is not lower hemicontinuous, first using lower inverse images, and then using sequences.

$F_{-1}(0,1) = \{x^*\}$, which is not open. Alternatively, let $x_n \to x^*$ with $x_n \neq x^*$ for all n. Then any sequence of y_n satisfying $y_n \in F(x_n)$ for large n, must eventually consist only of 0's and 1's. Therefore, no such (y_n) sequence converges to $\frac{1}{2} \in [0,1] = F(x^*)$.

[3]"Sufficiently large n" because if we said "all n" and we had $F(z) = \emptyset$ for some z then F would not be lower hemicontinuous anywhere.



Section Exercise 4. Suppose that $F : \mathbb{R} \rightrightarrows \mathbb{R}$, has $F(x) = \emptyset$ for some, but not all, $x \in \mathbb{R}$. Show that F cannot be both upper and lower hemicontinuous (compare Figures 1 (b) and (c) on Problem Set 6).

Suppose F is upper hemicontinuous. Then $F^{-1}(\emptyset)$ is open, so $\left[F^{-1}(\emptyset)\right]^{\mathsf{C}} = \{x \mid F(x) \neq \emptyset\}$ is closed. By assumption, this set is nonempty and has a nonempty complement, so it has a nonempty boundary.[4] Because it is closed, it must contain this boundary. That is, there is an x with $F(x) \neq \emptyset$ such that every open interval centred at x contains a z with $F(z) = \emptyset$. This means we can define a sequence $z_n \to x$ such that $F(z_n) = \emptyset$ for all n. It follows that for any $y \in F(x)$, there is no $y_n \to y$ satisfying $y_n \in F(z_n)$ for large n. Therefore, F is not lower hemicontinuous.[5]

Section Exercise 5 (Midterm 2 Q1, 2022)**.** Let $F, G : [0,1] \rightrightarrows \mathbb{R}$ be continuous correspondences. Prove that $H : [0,1] \rightrightarrows \mathbb{R}$ defined by $H(x) := F(x) \cup G(x)$ for all $x \in [0,1]$ is also continuous.

Let $U \subseteq \mathbb{R}$ be open. Observe that $x \in H^{-1}(U) \iff H(x) \subseteq U \iff F(x) \cup G(x) \subseteq U \iff F(x) \subseteq U$ and $G(x) \subseteq U \iff x \in F^{-1}(U)$ and $x \in G^{-1}(U) \iff x \in F^{-1}(U) \cap G^{-1}(U)$. Therefore, $H^{-1}(U) = F^{-1}(U) \cap G^{-1}(U)$ must be open, by upper hemicontinuity of F and G. It follows that H is upper hemicontinuous.

Similarly, $x \in H_{-1}(U) \iff H(x) \cap U \neq \emptyset \iff [F(x) \cap U] \cup [G(x) \cap U] \neq \emptyset \iff x \in F_{-1}(U) \cup G_{-1}(U)$. Thus, $H_{-1}(U) = F_{-1}(U) \cup G_{-1}(U)$ must be open, by lower hemicontinuity of F and G. It follows that H is also lower hemicontinuous.

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[4] A set is closed if it contains its boundary and open if it doesn't intersect its boundary. If its boundary is empty then both of these are true, so the set is clopen, implying it must be $\mathbb{R}^d$ or $\emptyset$.

[5] Note that we are proving a "P or Q" statement here: F is not upper hemicontinuous or F is not lower hemicontinuous. Such statements can be proven by showing (not P) implies Q: in this case, we showed F is (not not) upper hemicontinuous implies F is not lower hemicontinuous.