ECON 6090 - Solutions to PS5

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6.C.9

(a) *Proof.* Define the objective function with uncertainty:

$$V(x) = u(w - x) + \mathbb{E}[v(x + y)]$$

Since $u(\cdot)$ and $v(\cdot)$ are both concave functions, the objective function is also a concave function, $V'(x) > 0$ and $V''(x) < 0$.
Given x^* solves the maximization problem,

$$V'(x^*) = \mathbb{E}[v'(x^* + y)] - u'(w - x^*) = 0$$

Since x_0 solves the optimization problem without uncertainty, x_0 satisfies the FOC:

$$u'(w - x_0) = v'(x_0)$$

Therefore,

$$\begin{aligned} V'(x_0) &= \mathbb{E}[v'(x_0 + y)] - u'(w - x_0) \\ &= \mathbb{E}[v'(x_0 + y)] - v'(x_0) \\ &> 0 \\ &= V'(x^*) \end{aligned}$$

we have $x_0 > x^*$. $\square$

(b) Define $\mu_1(x) = -v_1'(x)$ and $\mu_2(x) = -v_2'(x)$. Therefore, $\mu_1(\cdot)$ and $\mu_2(\cdot)$ are increasing and concave. Therefore, $\mu_1(\cdot)$ and $\mu_2(\cdot)$ are valid utility functions. By Proposition 6.C.2,

$$\mathbb{E}[\mu_1(x_0 + y)] < \mu_1(x_0) \Rightarrow \mathbb{E}[\mu_2(x_0 + y)] < \mu_2(x_0)$$

which means

$$\mathbb{E}[v_2'(x_0 + y)] \geq v_2'(x_0)$$

The implications of this fact in the context of part (a) is that if the coefficient of absolute prudence of person 1 is not larger than that of person 2, and a risk y makes person 1 save more, it will also make person 2 save more. In fact, the coefficient of absolute prudence measure how much more money the person is willing to save when facing possible future risk.

(c) If $v'''(\cdot) > 0$, $\mu''(x) = -v'''(x) < 0$. Therefore, $\mu(\cdot)$ is risk averse, which implies

$$\mathbb{E}[\mu(x+y)] < \mu(x)$$

which means

$$\mathbb{E}[v'(x+y)] > v'(x)$$

(d) Since the coefficient of absolute risk aversion of $v(\cdot)$ is decreasing with wealth,

$$\begin{aligned}
&\partial(-\frac{v''(x)}{v'(x)})/\partial x \\
&= -\frac{v'''(x)v'(x) - (v''(x))^2}{(v'(x)^2)} \\
&=(-\frac{v'''(x)}{v''(x)} + \frac{v''(x)}{v'(x)})(\frac{v'(x)}{v''(x)}) \\
< 0
\end{aligned}$$

Since $v'(x) > 0$ and $v''(x) < 0$,

$$-\frac{v'''(x)}{v''(x)} + \frac{v''(x)}{v'(x)} > 0$$

for all x and $v'''(x) > 0$.

6.C.14

(a) By definition, if $u^*(\cdot)$ is strongly more risk averse than $u(\cdot)$, then there exists a $k > 0$ and non-decreasing concave function $v(\cdot)$, such that $u^*(x) = ku(x) + v(x)$ for all x.

$$\begin{aligned}
-\frac{u^{*''}(x)}{u^{*'}(x)} &= -\frac{ku''(x) + v''(x)}{ku'(x) + v'(x)} \\
&\geq -\frac{ku''(x)}{ku'(x) + v'(x)} \quad \text{by } u^{*'} > 0 \\
&\geq -\frac{ku''(x)}{ku'(x)} \quad \text{by } u'' < 0,\ v' < 0 \ \text{ and } u' > 0 \\
&= -\frac{u''(x)}{u'(x)}
\end{aligned}$$

Therefore, $u^*(\cdot)$ is more risk averse than $u(\cdot)$ in the usual Arrow-Pratt sense.

(b) For a strictly decreasing, concave $v(\cdot)$, $v(x+1) - v(x) < 0$ and decreasing in x. Since $u(\cdot)$ is increasing, concave and bound, $u(x+1) - u(x) > 0$, decreasing with limit 0. Since

$$u^*(x+1) - u^*(x) = k(u(x+1) - u(x)) + v(x+1) - v(x)$$

for x large enough, $u^*(x+1) - u^*(x)$ is negative, which contradicts to the fact that $u^*(\cdot)$ is a Bernoulli utility function, which should be increasing in x.

(c) To argue that the concept of a strongly more risk-averse utility function is stronger than the Arrow-Pratt concept of a more risk-averse utility function is equivalent to argue the following two statements are true:

(1) if $u^*(\cdot)$ is strongly more risk-averse than $u(\cdot)$, $u^*(\cdot)$ is more risk averse in AP sense;

(2) $u^*(\cdot)$ being more risk averse in AP sense than $u(\cdot)$ doesn't necessarily mean that $u^*(\cdot)$ is strongly more risk-averse than $u(\cdot)$.

Statement (1) has been proved in part (a). For statement (2), define $u(x) = -\exp(-\alpha x)$ and $u(x) = -\exp(-\beta x)$ where $0 < \alpha < \beta$. $u^*(\cdot)$ and $u(\cdot)$ have constant absolute risk aversion β and α. Therefore, $u^*(\cdot)$ is more risk averse in AP sense than $u(\cdot)$. However, given the conclusion in part (b), $u(\cdot)$ is upper bounded, it's not true that for any x, $u^*(\cdot)$ is strongly more risk-averse than $u(\cdot)$. This concludes the proof that the concept of a strongly more risk-averse utility function is stronger than the Arrow-Pratt concept of a more risk-averse utility function.

6.C.15

(a) A simple necessary condition for the demand for the riskless asset to be strictly positive is

$$\min a, b < 1$$

(b) A simple necessary condition for the demand for the risky asset to be strictly positive is

$$\pi a + (1-\pi)b > 1$$

(c) The maximization problem can be written as

$$\begin{aligned}
&\max_{x_1,x_2} EU(x_1,x_2) \text{ s.t. } x_1 + x_2 = 1 \\
=&\max_{x_1,x_2} \pi u(x_1 + ax_2) + (1-\pi)u(x_1 + bx_2) \text{ s.t. } x_1 + x_2 = 1 \\
=&\max_{x_1} \pi u((1-a)x_1 + a) + (1-\pi)u((1-b)x_1 + b)
\end{aligned}$$

FOC can be written as:

$$\frac{\partial EU}{\partial x_1} = \pi(1-a)u'((1-a)x_1+a) + (1-\pi)(1-b)u'((1-b)x_1+b) = 0$$

(d) Apply Implicit function theorem to the FOC, we get:

$$\frac{\partial x_1}{\partial a} = \frac{\pi u'(A) - \pi u''(A)(1-x_1)(1-a)}{\pi(1-a)^2 u''(A) + (1-\pi)(1-b)^2 u''(B)}$$

where $A = (1-a)x_1 + a$ and $B = (1-b)x_1 + b$. Given the conditions $a < 1$, $u'(\cdot) \geq 0$ and $u''(\cdot) \leq 0$, we have

$$\frac{\partial x_1}{\partial a} \leq 0$$

(e) The sign of $dx_1/d\pi$ should be positive. When the probability of having a bad outcome (π) increases, the expected payoff of the risky assets decreases. The decision maker should invest less in the risky assets and invest more in the riskless assets, which means x_1 increases.

(f) Apply Implicit function theorem to the FOC, we get:

$$\frac{\partial x_1}{\partial \pi} = \frac{u'(B)(1-b) - u'(A)(1-a)}{\pi u''(A)(1-a)^2 + (1-\pi)u''(B)(1-b)^2} > 0$$

6.C.16

(a) Assume that the individual owns the lottery and the price for the lottery is x. If he sells the lottery, his utility is $u(w+x)$. If he doesn't sell the lottery, his expected utility is $pu(w+G) + (1-p)u(w+B)$. Therefore, the individual will only sell the lottery when

$$u(w+x) \geq pu(w+G) + (1-p)u(w+B)$$

(b) Assume that the individual doesn't own the lottery and the price for the lottery is y. If he doesn't buy the lottery, his utility is $u(w)$. If he buys the lottery, his expected utility is $pu(w-y+G) + (1-p)u(w-y+B)$. Therefore, the individual will only buy the lottery when

$$pu(w-y+G) + (1-p)u(w-y+B) \geq u(w)$$

(c) Generally, x and y are different. However, if the Bernoulli utility function $u(\cdot)$ is CARA, which means $\omega\frac{u''(\omega)}{u'(\omega)} = c$ for any ω, $x = y$.

(d) Plug the parameters into the function of x and y, we have:

$$x = (\sqrt{20}p + \sqrt{15}(1-p))^2 - 10$$

and y is a solution to the function

$$p\sqrt{20-y} + (1-p)\sqrt{15-y} = \sqrt{10}$$

6.C.17

(1) First consider if his utility function exhibits constant relative risk aversion. The utility function is in the form $u(x) = x^{1-\gamma}$[1]. First show the individual's optimal choice of α_1 is independent of the wealth level.

Assume the realization of return x_1 is fixed and the corresponding wealth is w_1. The individual solves the maximization problem:

$$\max_{\alpha_1} \int u[((1-\alpha_1)R + \alpha_1 x_2)w_1]dF(x_2)$$

The FOC is:

$$\int u'[((1-\alpha_1)R + \alpha_1 x_2)w_1](x_2 - R)w_1 dF(x_2) = 0$$

Apply implicit function theorem and denote $A = (1-\alpha_1)R + \alpha_1 x_2$,

$$\begin{aligned}
\frac{\partial \alpha_1}{\partial w_1} &= -\frac{\int (u''[Aw_1]Aw_1 + u'[Aw_1])(x_2 - R)dF(x_2)}{\int u''[Aw_1](x_2-R)^2 w_1^2 dF(x_2)} \\
&= -\frac{\int (1-\gamma)u'[Aw_1](x_2 - R)dF(x_2)}{\int u''[Aw_1](x_2-R)^2 w_1^2 dF(x_2)} \quad \text{by CRRA: } -\frac{u''(w)}{u'(w)}w = \gamma \\
&= -(1-\gamma)\frac{\int u'[Aw_1](x_2 - R)dF(x_2)}{\int u''[Aw_1](x_2-R)^2 w_1^2 dF(x_2)} \\
&= 0 \qquad \text{by FOC, the numerator is 0}
\end{aligned}$$

Therefore, α_1 is independent of the wealth level of w_1.

Then consider the optimal choice of α_0. The maximization problem at the first stage is:

$$\max_{\alpha_0} \iint u[((1-\alpha_1)R + \alpha_1 x_2)((1-\alpha_0)R + \alpha_0 x_1)w_0]dF(x_1)dF(x_2)$$

Given the function form of the utility function, the objective function can be re-written as:

$$[\int ((1-\alpha_1)R + \alpha_1 x_2)^{1-\gamma} dF(x_2)][\int u(((1-\alpha_0)R + \alpha_0 x_1)w_0)dF(x_1)]$$

Since α_1 is a constant, the item in the first bracket is a constant. Note that the item in the second bracket is the same as the individual's maximization problem at the second stage. Therefore,

[1] In general, the function form of a constant relative risk aversion utility function is $u(x) = \beta x^{1-\gamma} + c$ or $u(x) = \beta \ln(x) + c$. The argument for these cases are similar to the simplified case.

the optimal α_0 should satisfy:

$$\alpha_0^* = \alpha_1^*$$

(2) Consider the case for constant absolute risk aversion case. The function form of a CARA utility is $u(x) = -\exp(-\rho x)$. Still consider the second stage problem first. The FOC condition given by the maximization problem is the same as the previous case except the specific functional form of utility functions are different. Plug in the functional form of CARA utility function, the FOC condition is:

$$\int \rho \exp[-\rho((1-\alpha_1)R + \alpha_1 x_2)w_1](x_2 - R)w_1 dF(x_2) = 0$$

which means that the optimal choice of α_1 depends on the realization of x_2. For any given α_0, the optimal choice of α_1 can be different in different states. Therefore, the individual will not always set $\alpha_0 = \alpha_1$.

6.C.18

(a) By definition,

$$\text{CARA}(x) = -\frac{u''(x)}{u'(x)} = \frac{1}{2x}$$

and

$$\text{CRRA}(x) = -x\frac{u''(x)}{u'(x)} = \frac{1}{2}$$

Therefore, the Arrow-Pratt coefficients of absolute and relative risk aversion at the level of wealth $w = 5$ are 0.1 and 0.5 respectively.

(b) The CE in this case satisfies:

$$u(CE) = \frac{1}{2}u(16) + \frac{1}{2}u(4)$$

Therefore, the certainty equivalent is 9.

The probability premium ρ satisfies

$$(\frac{1}{2} + \rho) * u(16) + (\frac{1}{2} - \rho) * u(4) = u(10)$$

and

$$\rho = (\sqrt{10} - 3)/2$$

(c) The CE in this case satisfies:

$$u(CE) = \frac{1}{2}u(36) + \frac{1}{2}u(16)$$

Therefore, the certainty equivalent is 25.

The probability premium ρ satisfies

$$(\frac{1}{2} + \rho) * u(36) + (\frac{1}{2} - \rho) * u(16) = u(26)$$

and

$$\rho = (\sqrt{26} - 5)/2$$

In both cases, the difference between the expected outcome and certainty equivalent is 1. However, for (b), the probability premium is higher. This is because the absolute risk aversion of the utility function is decreasing.

6.C.19

For each n, define β_n to be the wealth invested in the risky asset n. Therefore, the wealth invested in risk free asset is $w - \sum_n \beta_n$.
For $\beta = (\beta_1, ..., \beta_n) \in \mathbb{R}^N$, the random return will be

$$x = (w - \sum_n \beta_n)r + \sum_n \beta_n z_n$$

when z_n is the random return of risky asset n. Since $z = (z_1, .., z_n) \sim N(\mu, V)$, $x \sim N((w - \sum_n \beta_n)r + \beta \cdot \mu, \beta' V \beta)$. Therefore,

$$\mathbb{E}[-\exp(-\alpha x)] = -\exp[(w - \sum_n \beta_n)r + \beta \cdot \mu(-\alpha) - (\beta' V \beta)\frac{\alpha^2}{2}]$$

Solving the FOC,

$$\beta^* = \frac{1}{\alpha} V^{-1}(\mu - re)$$

where $e \in \mathbb{R}^N$ and $e_i = 1$ for all $i = 1, .., N$.