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Our plan for today:[1]

- Welfare Theorems and Negishi's Method
- Asset Pricing
- Sequential Trading Equilibrium
- Practice Question
- Intertemporal Substitution
- Income and Substitution Effects

[1] Materials adapted from notes provided by a previous Teaching Assistant, Zhuoheng Xu.

1 Welfare Theorems

Welfare theorems aim to answer the following questions: Can market deliver good allocation? Can good allocation be achieved via market? By "good", we usually mean efficient, Pareto optimal.

Assumptions: No market frictions (market power, information asymmetry, bargaining, etc.), convexity of preferences and production sets.

First Welfare Theorem: Each market-based outcome corresponds to some Pareto Optimum with certain weights.

Second Welfare Theorem: For some given social planner weights, there exists a system of transfers $\{\tau_i\}_{i\in I}$ such that Pareto efficient allocation is achievable by market.

Intuition: Basically, SWT says that if we have an ideal allocation for some particular SP's weights, we can find a system of transfers such that if we give those transfers to agents and allow them to trade, market will eventually achive that ideal allocation.

Example: Consider an endowment economy with two agents: $e_1 = 2$, $e_2 = 0$. And $u_i = \ln(c_i)$.

$\rightarrow$ First Welfare Theorem: Without any intervention, no trade can be made. We are left with $c_i = e_i$. So agent 2 "starves to death". This corresponds to a social planner problem with weights $\lambda_1 = 1$, $\lambda_2 = 0$.

$\rightarrow$ Second Welfare Theorem: If instead, the social planner has weights $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{1}{2}$, then a transfer from endowment $\tau_1 = -1$, $\tau_2 = 1$, such that $c_1 = 1$, $c_2 = 1$, would achieve the Pareto optimum.

1.1 Connection between Arrow-Debreu and Pareto Optimum

To find Pareto optimal allocation, we need to solve the SPP:

$$\mathcal{L}_{SPP} = \sum_{t=0}^{\infty}\sum_{s^t}\left\{\sum_{i\in I}\lambda_i\beta^t u\left(c_i^t(s^t)\right)\pi_t(s^t) - \theta_t(s^t)\left[\sum_{i\in I}c_i^t(s^t) - \sum_{i\in I}y_i^t(s^t)\right]\right\}$$

$$= \sum_{t=0}^{\infty} \sum_{s^t} \left\{ \sum_{i \in I} \lambda_i \beta^t u\left(c_i^t(s^t)\right) \pi_t(s^t) - \theta_t(s^t) \left[c_i^t(s^t) - y_i^t(s^t)\right] \right\}$$

The first order condition for agent i's allocation is:

$$\lambda_i \beta^t u'\left(c_i^t(s^t)\right) \pi_t(s^t) = \theta_t(s^t)$$

The same condition holds for agent $i = 1$ (and all other agents), so we can get rid of the Lagrange multiplier and discount factor:

$$\frac{u'\left(c_i^t(s^t)\right)}{u'\left(c_1^t(s^t)\right)} = \frac{\lambda_1}{\lambda_i}$$

Intuition: Note that higher weights correspond to lower marginal utility. As we said before, if the SP cares about some agents more, they will be assigned higher Pareto weights, so they will get higher utility and higher consumption (utility function is increasing). At the same time, higher consumption means lower marginal utility (utility function is concave, diminishing marginal utility).

We derive similar equation for ADE following the same steps:

$$\mathcal{L}_i = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u\left(c_i^t(s^t)\right) \pi_t(s^t) - \mu_i^0 q_t^0(s^t) \left[c_i^t(s^t) - y_i^t(s^t)\right]$$

The first order condition is:

$$\beta^t u'\left(c_i^t(s^t)\right) \pi_t(s^t) = \mu_i q_t^0(s^t)$$

We divide the FOC across agents:

$$\frac{u'\left(c_i^t(s^t)\right)}{u'\left(c_1^t(s^t)\right)} = \frac{\mu_i}{\mu_1}$$

Comparing with similar expression for the SPP above, we conclude that when inverse of social planner weights equal the shadow value of relaxing individual i's budget constraint ($\lambda_i = 1/\mu_i$), the ADE and PO allocations exactly coincide.

Using $\lambda_i = 1/\mu_i$, plug in FOCs and we can see the Arrow-Debreu prices are given by

$$q_t^0(s^t) = \theta_t(s^t).$$

$$q_t^0(s^t) = \beta^t u'\left(c_i^t(s^t)\right)\pi_t(s^t)\frac{1}{\mu_i} = \beta^t u'\left(c_i^t(s^t)\right)\pi_t(s^t)\lambda_i = \theta_t(s^t)$$

Interpretation of the condition $q_t^0(s^t) = \theta_t(s^t)$:

- $q_t^0(s^t)$: the price of the consumption good that measures scarcity.
- $\theta_t(s^t)$: the value at which the aggregate utility will increase if the aggregate endowment increases by 1 unit. It is also a scarcity indicator.

1.2 Negishi's Method

- The solution to the Social Planner's Problem is Pareto Efficient. With different sets of Pareto weights $\{\lambda_i\}_{i\in I}$, we obtain various Social Planner's Problems, each with distinct solutions. This yields a set of efficient allocations.
- By the First Welfare Theorem, the ADE allocation is in this set.
- To isolate the ADE allocation, we need to find the allocation that is consistent with each agent's budget constraint.

Solution algorithm:

1. Specify Pareto weights.
2. Solve the Social Planner's Problem, find FOC conditions and substitute them into the resource constraint.
3. Express consumption in terms of Pareto weights and calculate shadow prices, denoted as θ_t.
4. Incorporate budget constraints to solve for consumption.

2 Asset Pricing

Be careful with the superscript and the subscript!

- We denote $p^0_\tau(s^\tau)$ as the time-0 price of an asset that grants its owner a stream of dividends $\{d_t(s^t)\}_{t\geq\tau}$ if history s^τ is realized. Dividends are $d_t(s^t)$ units of time-t consumption goods.

- $q^\tau_t(s^t)$ is the time-τ price of one unit of consumption at time t given history s^t.

So, we can express $p^0_\tau(s^\tau)$ as

$$p^0_\tau(s^\tau) = \sum_{t\geq\tau}\sum_{s^t|s^\tau} q^0_t(s^t)d_t(s^t)$$

Note the difference:

- $p^0_\tau(s^\tau)$ - we evaluate the value of the asset *in terms of time-0 consumption goods.*

- $p^\tau_\tau(s^\tau)$ - we evaluate the value of the asset *in terms of time-τ consumption goods.*

We treat the time-0 consumption good as the numeraire (i.e., $q^0_0(s_0) = 1$), so the value of one unit of consumption good in time τ contingent on history s^τ equals to the value of $q^0_\tau(s^\tau)$ units of consumption goods in time 0. In other words, $q^0_\tau(s^\tau)$ can be treated as the value of an asset that only pays you 1 unit of consumption good in period τ contingent on history s^τ in terms of period 0 consumption goods.

If we want to price time-t consumption goods in terms of time-τ consumption goods, we use the following formula:

$$q^\tau_t(s^t) = \frac{q^0_t(s^t)}{q^0_\tau(s^\tau)}.$$

The left hand side is the time-τ price of one unit of consumption good at time t given history s^t. To find such price, we go all the way back to time 0. The numerator on the right hand side is the time-0 value of the consumption goods that we want to convert. The denominator on the right hand side is the unit of account that we use. Here, the denominator is the time-0 value of one unit of time-τ consumption goods.

To convert the time-0 price of an asset into units of account at time-τ, we can use the following formula:

$$p^{\tau}_{\tau}(s^{\tau}) = \frac{p^{0}_{\tau}(s^{\tau})}{q^{0}_{\tau}(s^{\tau})}.$$

The idea is the same as before. Here, we changed the numerator on the right hand side as it is the value that we want to convert.

Substituting the FOC (Euler equation) into the equation for $q^{\tau}_{t}(s^{t})$, we have:

$$q^{\tau}_{t}(s^{t}) = \frac{q^{0}_{t}(s^{t})}{q^{0}_{\tau}(s^{\tau})} = \frac{\beta^{t} u'(c^{i}_{t}(s^{t}))\pi_{t}(s^{t})}{\beta^{\tau} u'(c^{i}_{\tau}(s^{\tau}))\pi_{\tau}(s^{\tau})} = \beta^{t-\tau}\frac{u'(c^{i}_{t}(s^{t}))}{u'(c^{i}_{\tau}(s^{\tau}))}\pi_{t}(s^{t} \mid s^{\tau}).$$

Special case: when $t = \tau + 1$, we have:

$$q^{\tau}_{\tau+1}(s^{\tau+1}) = \beta\frac{u'(c^{i}_{\tau+1}(s^{\tau+1}))}{u'(c^{i}_{\tau}(s^{\tau}))}\pi_{t}(s^{\tau+1} \mid s^{\tau}).$$

This is the "pricing kernel" in the Arrow-Debreu (ADE) framework. Intuitively, the higher the probability of a specific event s^{t+1} occurring from the perspective of time t with realized events s^{t}, the higher the value it will have at time t if s^{t} is realized. (Example: After an earthquake, insurance for surviving consumers will be valued higher.)

With the random payoff $\omega(s_{\tau+1})$, we can write:

$$p^{\tau}_{\tau} = E_{\tau}\left[\beta\frac{u'(c_{\tau+1})}{u'(c_{\tau})}\omega(s^{\tau+1})\right]$$

$$1 = E_{\tau}\left[\beta\frac{u'(c_{\tau+1})}{u'(c_{\tau})}\frac{\omega(s_{\tau+1})}{p^{\tau}_{\tau}}\right] = E_{\tau}\left[m_{\tau+1}R_{\tau+1}\right]$$

where $m_{\tau+1}$ is the stochastic discount factor (SDF). The interpretation of the stochastic discount factor is that the value of the asset p^{T}_{τ} can be calculated by discounting the future stochastic return $\omega(s_{\tau+1})$.

Remark: Any asset must satisfy the equation above.

3 Sequential Trading Equilibrium

3.1 Difference between AD and SM Settings

Arrow-Debreu Trading

- The market opens and closes before the world begins (typically at period $t = -1$ or $t = 0$, depending on where you start).
- Agents trade contingent claims that promise to deliver one unit of consumption in a future period t, provided that a specific history s^t occurs, i.e., agents trade claims that are contingent on *the entire future history.*

Sequential Trading

- The market opens at the beginning of each period t and closes after trading for that period is complete.
- Agents trade contingent claims that promise to deliver one unit of consumption in the next period, $t+1$, if a specific state s^{t+1} occurs, given the current realized history s^t, i.e., agents trade claims that are contingent on *the state in the next period.*

3.2 The Basic Setup

1. Components

   (a) $\tilde{a}^i_{t+1}(s_{t+1}, s^t)$: a distribution of assets for all i and t

   - Positive $\tilde{a}^i_{t+1}(s_{t+1}, s^t)$: holding Arrow securities issued by other agents.
   - Negative $\tilde{a}^i_{t+1}(s_{t+1}, s^t)$: issuing Arrow securities to other agents.

   (b) $\{\tilde{c}^i_t(s^t)\}$: consumption for all i and t

   (c) $\tilde{Q}_t(s_{t+1}|s^t)$ (pricing kernels): how I view the value of consumption goods in $t+1$, contingent on the possible events s_{t+1}, from the perspective of time t with realized events s^t. (No need to discount all the way back to $t = 0$.)

2. Characterization:

    (a) For all i, $\tilde{c}_t^i(s^t)$ solves household i's problem, that is

    $$\max_{\tilde{c}_t^i(s^t), \{\tilde{a}_{t+1}^i(s_{t+1}, s^t)\}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u[\tilde{c}_t^i(s^t)] \pi_t(s^t)$$

    subject to

    $$\tilde{c}_t^i(s^t) + \sum_{s_{t+1}} \tilde{a}_{t+1}^i(s_{t+1}, s^t) \tilde{Q}_t(s_{t+1}|s^t) \leq y_t^i(s^t) + \tilde{a}_t^i(s^t)$$

    $$-\tilde{a}_{t+1}^i(s^{t+1}) \leq A_{t+1}^i(s^{t+1})$$

    (b) For all $\{s^t\}_{t=0}^{\infty}$, we have $\sum_i \tilde{c}_t^i(s^t) = \sum_i y_t^i(s^t)$ and $\sum_i \tilde{a}_{t+1}^i(s_{t+1}, s^t) = 0$

3.3 Interpretation of Equations

Budget Constraint

$$\tilde{c}_t^i(s^t) + \sum_{s_{t+1}} \tilde{a}_{t+1}^i(s_{t+1}, s^t) \tilde{Q}_t(s_{t+1}|s^t) \leq y_t^i(s^t) + \tilde{a}_t^i(s^t)$$

LHS: Expenditures include current period consumption and the total value of Arrow securities that the agent chooses to hold for each future state s_{t+1}.

RHS: Income consists of the value of the agent's endowment and previously traded assets.

Remark: Note that there are no explicit prices because everything is expressed in terms of current period units of consumption.

No Ponzi Scheme Condition

$$\tilde{a}_{t+1}^i(s^{t+1}) \geq -A_{t+1}^i(s^{t+1})$$

Intuition: Agents cannot borrow more than they can repay. There is a borrowing limit, but it is assumed not to be binding (Inada conditions).

Resource Constraint

$$\sum_i \tilde{c}_t^i(s^t) = \sum_i y_t^i(s^t)$$

Intuition: The goods market clears, meaning total consumption equals total output.

Zero Net Supply of Arrow Securities (Asset market clearing condition)

$$\sum_i \tilde{a}_{t+1}^i(s_{t+1}, s^t) = 0$$

Intuition: For every security bought (positive position), there must be an equivalent security sold (negative position), ensuring that the market for Arrow securities clears.

3.4 Equilibrium

$$\mathcal{L}^i = \sum_{t=0}^{\infty} \sum_{s^t} \left(\beta^t u\left[\tilde{c}_t^i(s^t)\right] \pi_t(s^t)\right) +$$

$$+\eta_t^i(s^t)\left(y_t^i(s^t) + \tilde{a}_t^i(s^t) - \tilde{c}_t^i(s^t) - \sum_{s_{t+1}} \tilde{a}_{t+1}^i(s_{t+1}, s^t)\tilde{Q}_t(s_{t+1}|s^t)\right) +$$

$$+\sum_{s^{t+1}} \nu_t^i(s^t, s_{t+1})\left(A_{t+1}^i(s^{t+1}) + \tilde{a}_{t+1}^i(s^{t+1})\right)$$

The FOC's are:

$$\beta^t u'\left[\tilde{c}_t^i(s^t)\right] \pi_t(s^t) - \eta_t^i(s^t) = 0$$

$$-\eta_t^i(s^t)\tilde{Q}_t(s_{t+1}|s^t) + \nu_t^i(s^t, s_{t+1}) + \eta_{t+1}^i(s^{t+1}, s^t) = 0$$

Using complementary slackness, we can set all the $\nu_t^i(s^t, s_{t+1})$ equal to 0, since borrowing constraints are not binding here. Combining the FOCs, we have:

$$\tilde{Q}_t(s_{t+1}|s^t) = \beta \frac{u'\left[\tilde{c}_{t+1}^i(s^{t+1})\right]}{u'\left[\tilde{c}_t^i(s^t)\right]} \pi(s^{t+1}|s^t)$$

which is exactly the same as the pricing kernel of ADE.

3.5 Connection between ADE and SQE

	ADE	**SQE**
Components	$(\{c_t^i(s^t)\}_{t=0}^{\infty})_{i\in I}$ $\{q_t^0(s^t)\}_{t=0}^{\infty}$	$\left(\{\tilde{a}_{t+1}^i(s_{t+1}, s^t), \tilde{c}_t^i(s^t)\}_{t=0}^{\infty}\right)_{i\in I}$ $\{\tilde{Q}_t(s_{t+1}\|s^t)\}_{t=0}^{\infty}$
Characterization	Agent i optimization problem Market clearing	Agent i optimization problem Market clearing

Theorem: Equivalence relation between ADE and SQE.

1. Same allocation (consumption): $\tilde{c}_t^i(s^t) = c_t^i(s^t)$
2. Prices relationship: $q_{t+1}^0(s^{t+1}) = \tilde{Q}_t(s_{t+1}|s^t)q_t^0(s^t)$

Intuition: both market structures allow agents to move resources across all histories.

4 Practice Question: ADE with aggregate uncertainty

Consider an exchange economy with two infinitely-lived consumers with identical preferences given by:

$$E\left(\sum_{t=0}^{\infty}\beta^t u\left(c_t^i\right)\right) = \sum_{t=0}^{\infty}\sum_{s^t}\beta^t u\left(c_t^i(s^t)\right)\pi_t(s^t)$$

where $u(c_t^i(s^t)) = \log(c_t^i(s^t))$. Both consumers have random endowments that depend on an exogenous sequence of state variables $\{s_t\}_{t=0}^{\infty}$. The s_t are statistically independent random variables with identical probability distributions. Specifically, for each t, $s_t = H$ with probability π and $s_t = L$ with probability $1-\pi$, where π does not depend on time or on the previous realization of states. If $s_t = H$, then the first consumer's endowment is 2 and the second consumer's endowment is 1; if $s_t = L$, then the first consumer's endowment is 1 and the second consumer's endowment is 0. Markets are complete.

1. Define an Arrow-Debreu competitive equilibrium for this economy.

2. Determine the competitive allocation and price system for this economy. (Hint: use Negishi's method)

5 Appendix: Intertemporal Substitution in Consumption

The elasticity of intertemporal substitution (EIS) in consumption characterizes a consumer's willingness to either advance or delay consumption in response to changes in investment opportunities. A consumer who saves more when interest rates are high has a higher EIS. More formally, the EIS is defined as the negative ratio of the change in log consumption growth to the change in the log growth of the marginal utility of consumption, that is,

$$\text{EIS} = -\partial \log\left(\frac{C_{t+1}}{C_t}\right) / \partial \log\left(\frac{\partial U(\cdot)/\partial C_{t+1}}{\partial U(\cdot)/\partial C_t}\right)$$

where $U(\cdot)$ represents the utility function of the consumer.

5.1 The EIS and the Real Interest Rate

Recall the classical problem we are trying to solve:

$$U = \sum_{t=0}^{T} \beta^t u(c_t)$$

In this setting, the gross real interest rate R will be determined by the Euler Equation:

$$u'(c_t) = \beta R u'(c_{t+1}) \implies R = \frac{u'(c_t)}{\beta u'(c_{t+1})}$$

In logs, we have

$$\ln(R) = \ln(1+r) \approx r = -\ln\left[\frac{u'(c_{t+1})}{u'(c_t)}\right] - \ln\beta$$

Since $\ln(1+r) \approx r$ for small r (logs are very close to percentage changes), we have:

$$r \approx -\ln\left[\frac{u'(c_{t+1})}{u'(c_t)}\right] - \ln\beta$$

Therefore, the elasticity of intertemporal substitution can be equivalently defined as the percent change in consumption growth per percent increase in the net interest rate:

$$-\frac{d\ln(c_{t+1}/c_t)}{d\ln\left(u'(c_{t+1})/u'(c_t)\right)} \approx \frac{d\ln(c_{t+1}/c_t)}{dr}$$

5.2 Example: CRRA Utility Function

Let utility of consumption in period t be given by the CRRA (Constant Relative Risk Aversion) form:

$$u(c_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma}.$$

Since this utility function belongs to the family of CRRA utility functions we have $u'(c_t) = c_t^{-\sigma}$. Thus,

$$\ln\left[\frac{u'(c_{t+1})}{u'(c_t)}\right] = -\sigma \ln\left[\frac{c_{t+1}}{c_t}\right].$$

This can be rewritten as

$$\ln\left[\frac{c_{t+1}}{c_t}\right] = -\frac{1}{\sigma}\ln\left[\frac{u'(c_{t+1})}{u'(c_t)}\right].$$

Hence, applying the above derived formula

$$-\frac{\partial\ln(c_{t+1}/c_t)}{\partial\ln(u'(c_{t+1})/u'(c_t))} = -\left[-\frac{1}{\sigma}\right] = \frac{1}{\sigma}.$$

Intuition: If the agent has low EIS, then σ is high. A high σ implies the marginal utility decreases quickly (recall that $u'(c_t) = c_t^{-\sigma}$). Then the agent does not want the consumption to grow too fast. The growth rate responds less to the real interest rate.

6 Appendix: Income Effect and Substitution Effect

In microeconomics, the total effect induced by a good's price change on consumer choices can be decomposed into the income effect and the substitution effect. However, these two terms have very specific meaning in macro context.

Intertemporal Decision:

As indicated by the Euler equation, changes in the interest rate affect both current consumption c_t and future consumption c_{t+1}.

- **Income effect:** Higher interest rates increase the future income of the consumer due to higher returns on savings. This increase in overall wealth may lead consumers to spend more today, an effect known as the "income effect."
- **Substitution effect:** Higher interest rates make future consumption more attractive relative to current consumption because saving today yields greater returns. This leads consumers to save more today and consume less, an effect known as the "substitution effect."

Consumers with a high EIS are more willing to substitute consumption over time, which has a direct impact on the substitution effect.[2]

Intratemporal Decision:

Consider a scenario with only one period, where the agent must decide how much labor to supply for production, with a maximum limit on the amount of time available for labor, denoted as L. The agent derives utility not only from consumption but also from leisure. Thus, the agent faces a trade-off: increasing labor supply results in higher consumption but reduces leisure time. Formally, we can write:

$$\max_{c,l} U(c, l)$$

subject to

$$c = w(L - l)$$

[2]Material adapted from notes provided by a previous Teaching Assistant, [Name].

The first-order condition (FOC) for this problem is:

$$U_c(c, l)w = U_l(c, l)$$

The effect of a change in the wage rate w can be separated into:

- **Income effect:** A higher wage rate increases the agent's wealth, making it optimal to enjoy more consumption and leisure, which can lead to a decrease in labor supply.

- **Substitution effect:** A higher wage increases the opportunity cost of leisure (making leisure more expensive), inducing the agent to work more, leading to an increase in labor supply.

In a more complex dynamic macro model, such as the Real Business Cycle Model, the agent must make both intertemporal and intratemporal decisions. When analyzing such a model, it is crucial to clearly identify which specific substitution and income effects you are addressing.