Econ 6170: Mid-Term 2

24 October 2023

Write your name on every sheet that you answer.

You have the full class time to complete the following problems. You are to work alone. This test is not open book. Please write out your answer neatly below each question, and use a new sheet of paper if you need more space than provided. When using extra sheets, make sure to write out your name and the relevant question number. In your answers, you are free to cite results that you can recall from class or previous problem sets unless explicitly stated otherwise. The exam is out of 20 points, and there are four bonus points. The highest possible score is 24/20.

Question 1 (5 points) Suppose $F_1, F_2 : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$ are two non-empty valued correspondences such that $F_1(x) \cap F_2(x) \neq \varnothing$ for all $x \in \mathbb{R}^d$. Define $F : \mathbb{R}^d \to \mathbb{R}^m$ as $F(x) := F_1(x) \cap F_2(x)$. For each of the following statement, prove either that the statement is true or false.

(i) (1.5 points) If F_1 and F_2 are both compact-valued and upper hemi-continuous, then F is upper hemi-continuous.

(ii) (2 points) If F_1 is compact-valued and, for some $x_0 \in \mathbb{R}^d$, F_1 is upper hemi-continuous at x_0 and F_2 is closed at x_0, then F is upper hemi-continuous at x_0.

(iii) (1.5 points) If F_1 and F_2 are both compact-valued and lower hemi-continuous, then F is lower hemi-continuous.

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Question 1 continued

Question 2 (5 points)

(i) (2 points) Suppose $f : \mathbb{R} \to \mathbb{R}$ is concave and $g : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function. Prove that $g \circ f$ is quasiconcave.

(ii) (1 point) Define $h : \mathbb{R}_+ \to \mathbb{R}$ be given by

$$h(x) := \begin{cases} 0 & x \in [0,1] \\ (x-1)^2 & \text{if } x > 1 \end{cases}.$$

Show that h is quasiconcave.

(iii) (2 points) Show that h, defined above, is not a strictly increasing function of a concave function.

Hint: Prove by contradiction and use the fact that every local maximum of a concave function is a global maximum.

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Question 2 continued

Question 3 (5 points)

(i) (3 points) Suppose f and g are continuous on $[a, b] \subseteq \mathbb{R}$ and differentiable in (a, b), and that $g'(x) \neq 0$ for all $x \in (a, b)$. Show that there exists a number $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Hint: Use the Rolle's theorem. Rearrange above to an equation with 0 on one side might give you an idea on the function you should apply the Rolle's theorem.

(ii) (1 point) How does this result relate to the mean value theorem?

(iii) (1 point) Why can't we apply the mean value theorem on f and g separately to prove the result above?

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Question 3 continued

Question 4 (5 points + 1 bonus point)

(i) (2 points) State a separating hyperplane theorem and a supporting hyperplane theorem.

(ii) (3 points) Fix some $I \in \mathbb{N}$ with $I \geq 2$ and suppose that, for each $i \in I$, the function $u_i : \mathbb{R}^d \to \mathbb{R}$ is continuous, concave and strictly increasing in each argument. Fix some $\mathbf{e} \in \mathbb{R}^d_{++}$. Define

$$X := \left\{ \left(\mathbf{x}^i\right)_{i=1}^{I} \in \mathbb{R}^{dI}_{+} : \sum_{i=1}^{I} \mathbf{x}^i = \mathbf{e} \right\},$$

where $\mathbf{x}^i = (x^i_1, \ldots, x^i_d) \in \mathbb{R}^d_+$. Say that a vector $\mathbf{x} \in X$ is dominated by $\mathbf{y} \in X$ if $u_i(\mathbf{y}^i) \geq u_i(\mathbf{x}^i)$ for all $i \in \{1, \ldots, I\}$ and the inequality is strict for at least one $i' \in \{1, 2, \ldots, I\}$. Say that a vector $\mathbf{x} \in X$ is undominated if no other vector dominates it. Show that if $\overline{\mathbf{x}}$ is undominated, then there exist a vector $\boldsymbol{\lambda} = (\lambda_i)_{i=1}^I \in [0,1]^I$ with $\sum_{i=1}^I \lambda_i = 1$ such that

$$\overline{\mathbf{x}} \in \operatorname*{arg\,max}_{\mathbf{x} \in X} \sum_{i=1}^{I} \lambda_i u_i \left(\mathbf{x}^i\right).$$

Hint: Define

$$A := \left\{ (w_i)_{i=1}^I \in \mathbb{R}^I : \exists \mathbf{x} \in X,\ w_i \leq u_i \left(\mathbf{x}^i\right)\ \forall i \in \{1, \ldots, I\} \right\}.$$

Argue that A is nonempty and convex and that if $\mathbf{x}$ is undominated, then $(u_i(\mathbf{x}))_{i=1}^I$ is not in the interior of A. Then, appeal to the supporting hyperplane theorem to find a vector that can subsequently be normalised to obtain $\boldsymbol{\lambda}$.

(iii) (1 bonus point) What did we just show?

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Question 4 continued

Question 5 (3 bonus points)

(i) (1 bonus point) Consider the function $f : [-1, 1] \to \mathbb{R}$ defined as $f(x) := \sqrt{1 - x^2}$. Verify that there exists $c \in (-1, 1)$ such that $f'(c) = 0$. Is f differentiable at $x = 1$ or at $x = -1$? Why does this not violate the Rolle's theorem?

(ii) (1 bonus point) State the Berge's theorem of the maximum. What more can we say about the solution correspondence if we additionally assume that the objective function is strictly concave?

(iii) (1 bonus point) Suppose $f : \mathbb{R} \to \mathbb{R}$ is convex. Show that, for any $x_1 < x_2 < x_3$,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}.$$

Hint: It's enough to draw a graph *and* only prove formally one of the inequalities.

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Question 5 continued