ECON 6090 - Solutions to PS1

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Exercise 1

a. Let x denote wage, y the amount of money during retirement, $u()$ a utility function defined on earnings, and δ a discount factor. The expected utility of not participating in the retirement plan yield a utility of $u(x) + \delta u(y)$, and the expected utility of participating in the retirement plan is $u(0.95x) + \delta u(0.05x + y)$. If in scenario 1 a person chooses to contribute, then by using a revealed preference argument, we must have $u(0.95x) + \delta u(0.05x + y) > u(x) + \delta u(y)$. But if the same person were facing scenario 2 and chose to not contribute, then $u(0.95x) + \delta u(0.05x + y) < u(x) + \delta u(y)$, a contradiction. Any similar setup works.

b. Consider set $X = \{$(not object,5), (object, 0), (object, 5), (not object, 0)$\}$. A preference relation such that (not object,5) $\succ$ (object, 0) (from scenario I) and (not object, 0)$\succ$ (object, 5) (from scenario II) need not violate completeness or transitivity. The observed choices could indeed be generated by

$$(\text{not object, } 5) \succeq (\text{not object, } 0) \succ (\text{object }, 0) \succeq (\text{object }, 5)$$

where $\succeq$ is a rational preference relation.

c. The choice set is now X = { (not object, 5), (object, 0), (object, 5), (not object, 0), (not object,10), (object,20)}. We have:

(1) $C((\text{not object, } 5), (\text{object, } 0)) = (\text{not object, } 5)$

(2) $C((\text{not object, } 0), (\text{object, } 5)) = (\text{not object, } 0)$

(3) $C((\text{not object, } 10), (\text{object, } 5)) = (\text{object, } 5)$

(4) $C((\text{not object, } 10), (\text{object, } 20)) = (\text{not object, } 10)$

(5) $C((\text{not object, } 0), (\text{object, } 20)) = (\text{object, } 20)$

Supposed these choices were generated by a rational preference relation denoted $\succeq$. Combining (4),(5), and (2) we have, by a revealed preference argument,

$$(\text{not object, } 10) \succ^* (\text{ object, } 20) \succ^* (\text{ not object, } 0) \succ^* (\text{object, } 5)$$

By transitivity, we should have (not object,10) $\succ^*$ (object,5). But the choice in (3) contradicts this result, thus transitivity is violated. Hence, no rational preference relation could have generated these choices.

Exercise 2

No, it's not rational because it's not complete.
Counterexample: Let $x = 1$ and $y = 1$. Then it's not $[x \succeq y]$ because $x = 1 < 2 = 2y$ and it's not $[y \succeq x]$ because $y = 1 < 2 = 2x$. Therefore, it's not complete thus it's not rational.

Exercise 3

a. We will show that $C^*(B, \succeq) = C^*(B - \{x\}, \succeq)$.
   Since $y \in B$ and $y \succ x$, we have that $x \notin C^*(B, \succeq)$. Also, since $x \notin (B - \{x\})$, then $x \notin C^*(B - \{x\}, \succeq)$.

   First, we want to show: $C^*(B, \succeq) \subseteq C^*(B - \{x\}, \succeq)$.
   Take $z \in C^*(B, \succeq)$
   $\Rightarrow z \succ w$ for all $w \in B$ and $z \neq x$
   $\Rightarrow z \succ w$ for all $w \in B - \{x\}$
   $\Rightarrow z \in C^*(B - \{x\}, \succeq)$

   Then, we want to show: $C^*(B - \{x\}, \succeq) \subseteq C^*(B, \succeq)$.
   Take $z \in C^*(B - \{x\}, \succeq)$
   $\Rightarrow z \succ w$ for all $w \in B - \{x\}$
   Also, since $z \succeq y \succ x$, $x, y \in B$, and the preference is transitive, we have $z \succ x$.
   $\Rightarrow z \succ w$ for all $w \in B$
   $\Rightarrow z \in C^*(B, \succeq)$

   Therefore, $C^*(B, \succeq) = C^*(B - \{x\}, \succeq)$.

b. They need not be the same and not necessarily there is a $\subseteq$ relation between both sets. For example, suppose $X = \{x, y, z\}$ and $y \succ x \succ z$. If $B = \{x, z\}$ then $C^*(B, \succeq) = \{x\}$ and $C^*(B - \{x\}, \succeq) = \{z\}$.
   Also, notice that in general $C^*(B, \succeq)$ and $C^*(B - \{x\}, \succeq)$ need not be singletons.

Exercise 4

Because X is finite and $\mathcal{B}$ is all subsets of X, then given that WARP is satisfied, the revealed preference $\succeq^*$ is rational. From $C(\{a,b,c\}) = \{a\}$, we know that $a \succeq^* b$ and $a \succeq^* c$. For the remaining $A \subset \mathcal{B}$, the one element case is trivial. For the two elements cases, we can say something about $C(A)$ if $A = \{a,b\}$ or $A = \{a,c\}$. To see why, consider $A = \{a,b\}$. We must have $C(A) = \{a\}$ because WARP is violated if $b \in C(A)$. The case for $A = \{a,c\}$ is similar. But we can't say something about $C(A)$ if $A = \{b,c\}$:

Exercise 5

For a choice rule defined on all nonempty subsets of a finite set of alternatives X, WARP $\Rightarrow$ Sen's β.

Proof. To prove: for any $A, B \in \beta$, if $x, y \in C(A)$, $A \subset B$ and $y \in C(B)$, then $x \in C(B)$.
For any $A,\ B \in \beta$, s.t. $A \subset B$, $A \cap B = A$. By WARP, if $x, y \in A = A \cap B$, $y \in C(B)$, $x \in C(A)$, then $x \in C(B)$. □

Exercise 6

Yes it is. Since f is a monotonic transformation (strictly increasing), $v(x) = f(u(x))$ represents the same preferences. For the same preference, it will always give me the same demand.

Exercise 7

a. The amount of good 1 the consumer purchases will weakly decrease.

   Proof. From the law of compensated demand, we have:

$$\begin{aligned}
\left(p' - p\right) \cdot \left(x\left(p', w'\right) - x(p,w)\right) &\leq 0 \\
\Rightarrow (t, 0, \ldots, 0) \cdot \left(x_1\left(p', w'\right) - x_1(p,w), \ldots, x_L\left(p', w'\right) - x_L(p,w)\right) &\leq 0 \\
\Rightarrow t\left(x_1\left(p', w'\right) - x_1(p,w)\right) &\leq 0 \\
\Rightarrow x_1\left(p', w'\right) \leq x_1(p,w)&
\end{aligned}$$

   Hence, the amount of good 1 the consumer purchases weakly decreases.
   Note that the equality holds if and only if $x(p', w') = x(p,w) = x^*$. □

b. If $R = 0$, we do not have a compensated price change, so we can no longer affirm what will happen with the amount of good 1 the consumer purchases, since it will depend on the type of good it is. For instance, if it is a Giffen good, an increase in its price would generate an increase in its consumed amount.

Exercise 8

Yes it is. To prove: $\forall x, y, z \in [0,1]$, $\forall\ \alpha \in [0,1]$, if $y \succeq z$ and $z \succeq x$, then $\alpha y + (1-\alpha)z \succeq x$, which means $U(\alpha y + (1-\alpha)z) \geq U(x)$

Proof.

$$
\begin{aligned}
& U(\alpha y + (1-\alpha)z) - U(x) \\
=& U(\alpha y + (1-\alpha)z) - (\alpha U(x) + (1-\alpha)U(x)) \\
=& \alpha y + (1-\alpha)z - (\alpha y + (1-\alpha)z)^2 - \alpha x + \alpha x^2 - (1-\alpha)x + (1-\alpha)x^2 \\
=& \alpha(U(y) - U(x)) + (1-\alpha)(U(z) - U(x)) + \alpha(1-\alpha)(y-z)^2 \geq 0
\end{aligned}
$$

Therefore, this consumer's preference relation is convex. □

Exercise 9

Assuming Walras Law, the choices are inconsistent with WARP if and only if: (1) $p^1 x^0 \leq w^1$ and (2) $p^0 x^1 \leq w^0$ and (3)$x^1 \neq x^0$.

1. Consider condition (1). $p^1 x^0 = 15 \times 50 + 9 \times 50 = 1200 < 1250 = w^1$. Condition (1) is satisfied.

2. Consider condition (2). We need

$$
\begin{aligned}
p_1^0 x_1^1 + p_2^0 x_2^1 = 10x_1^1 + 10x_2^1 \leq 1000 \quad (1) \\
x_1^1 \geq 0 \quad (2) \\
x_2^1 \geq 0 \quad (3) \\
p_1^1 x_1^1 + p_2^1 x_2^1 = 15x_1^1 + 9x_2^1 = 1250 \quad (4)
\end{aligned}
$$

   Solving the conditions, $x_2^1 \in [0, \frac{125}{3}]$.

3. Consider condition (3). Assume $(x_1^1, x_2^1) = (50, 50)$, the total expenditure is 1200 ¡ 1250, which violates Walras Law. Then $(x_1^1, x_2^1) \neq (x_1^0, x_2^0)$.

Combining these conditions, if the consumer's choices satisfy $x_2^1 \in [0, \frac{125}{3}]$, it's inconsistent with the weak axiom.

Exercise 10

a. Suppose two consumers have identical preferences which are rational and locally nonsatiated:

   1. $p^1 x^2 < w^1$ and $p^2 x^1 > w^2$: Since $x^1 \in C^*(B_{p^1 w^1}, \succeq)$ and $x^2 \in B_{p^1 w^1}$, we have $x^1 \succeq x^2$, so consumer is weakly better than consumer 2. Furthermore, since the preferences $\succeq$ are

locally nonsatiated, we have $x^1 \succ x^2$ and consumer 1 is strictly better than consumer 2.

(If not, suppose $x^1 \sim x^2$, since $p^1x^2 < w^1$, we can find a $\delta > 0$ such that $B_\delta(x^2) \cap \mathbb{R}^L_+ \subset B_{p^1w^1}$. Due to local non-satiation, $\exists\ x'' \in B_\delta(x^2) \cap \mathbb{R}^L_+$ such that $x'' \succ x^1$, contradicting x^1 is preference-maximizing bundle, so $x^1 \succ x^2$.)

2. $p^1x^2 > w^1$ and $p^2x^1 > w^2$: Since $x^1 \notin B_{p^2w^2}$ and $x^2 \notin B_{p^1w^1}$, we cannot tell which consumer is better off.
3. $p^1x^2 < w^1$ and $p^2x^1 < w^2$: Since $x^1 \in C^*(B_{p^1w^1}, \succeq)$ and $x^2 \in B_{p^1w^1}$, we have $x^1 \succeq x^2$. Symmetrically, we have $x^2 \succeq x^1$. Due to local nonsatiation, we have $x^1 \succ x^2$ and $x^2 \succ x^1$, contradicting the fact that $\succeq$ is a rational preference. The assumption that consumers hold common preferences which are rational and locally non-satiated is not valid. We also can't tell who is better off in this case.

b. The colleague's argument is mostly correct:

- First, we need to recognize that by choosing to live in different locations, each person is choosing a budget set $B_{p^1w^1}$ and $B_{p^2w^2}$ and then making an optimal choice from the chosen budget.
- Second, if they have same preferences, they must be indifferent between these two locations and they are equally better off. Otherwise, one consumer has incentive to move to the other location. (For example, in data 1, consumer 1 is better-off, if with common preferences, consumer 2 has incentive to move to location 1 , so it cannot be).
- Third, if they have different preferences, we cannot compare who is better off because there is not a uniform standard to compare welfare across individuals. We can only infer each person's preference orderings by choice data. (pick one example and explain, e.g. in data 1, we can say $x^1 \succeq_1 x^2$ and $x^2 \succeq_2 x^1$ given $\succeq_1$, $\succeq_2$ being rational, locally non-satiated.)