ECON 6170 Section 11

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Monotone Comparative Statics

Remark 1 (Notation for increasing differences)**.** For $f : Z \subseteq \mathbb{R}^d \to \mathbb{R}$,

- Increasing differences in (x, θ) means that if $x' \geq x$ and $\theta' \geq \theta$ then

$$f(x', \theta') - f(x, \theta') \geq f(x', \theta) - f(x, \theta)$$

- Increasing differences in $(z_i, z_j; z_{-ij})$ means that $f(\cdot, z_{-ij})$ has increasing differences in (z_i, z_j). That is, if $z_i' \geq z_i$ and $z_j' \geq z_j$ then

$$f(z_i', z_j'; z_{-ij}) - f(z_i, z_j'; z_{-ij}) \geq f(z_i', z_j; z_{-ij}) - f(z_i, z_j; z_{-ij})$$

- Increasing differences on Z means increasing differences in $(z_i, z_j; z_{-ij})$ for all i, j.

- Increasing differences in $((x, y), \theta)$ means increasing differences in (w, θ) where $w := (x, y)$.

Section Exercise 1. Prove the following lemma: f has increasing differences in (x, θ) if and only if f has increasing differences in $(x_i, \theta_j; x_{-i}, \theta_{-j})$ for all $i \in \{1, \ldots, d\}$ and all $j \in \{1, \ldots, m\}$.

Suppose f has increasing differences in (x, θ). In particular, suppose $x_i' \geq x_i$, $\theta_j' \geq \theta_j$, x' is x with x_i replaced by x_i', and θ' is θ with θ_j replaced by θ_j'. Then $x' \geq x$ and $\theta' \geq \theta$, so

$$f(x', \theta') - f(x, \theta') \geq f(x', \theta) - f(x, \theta)$$

or equivalently

$$f(x_i', \theta_j'; x_{-i}, \theta_{-j}) - f(x_i, \theta_j'; x_{-i}, \theta_{-j}) \geq f(x_i', \theta_j; x_{-i}, \theta_{-j}) - f(x_i, \theta_j; x_{-i}, \theta_{-j})$$

Conversely, suppose f has increasing differences in $(x_i, \theta_j; x_{-i}, \theta_{-j})$ for all $i \in \{1, \ldots, d\}$ and all $j \in \{1, \ldots, m\}$. Suppose $x' \geq x$ and $\theta' \geq \theta$. Then $x_i' \geq x_i$ for all i and $\theta_j' \geq \theta_j$ for all j. Let $i \in \{1, \ldots, d\}$ and $x^i := (x_1, \ldots, x_{i-1}, x_i', \ldots, x_d)$. Then

$$\begin{aligned}
&f(x^i, \theta') - f(x^{i+1}, \theta') \\
&\geq f(x^i, \theta_1, \theta_2', \ldots, \theta_m') - f(x^{i+1}, \theta_1, \theta_2', \ldots, \theta_m') \\
&\geq f(x^i, \theta_1, \theta_2, \theta_3', \ldots, \theta_m') - f(x^{i+1}, \theta_1, \theta_2, \theta_3', \ldots, \theta_m') \\
&\geq \ldots \\
&\geq f(x^i, \theta) - f(x^{i+1}, \theta)
\end{aligned}$$

Each step j follows from increasing differences in $(x_i, \theta_j; x_{-i}, \theta_{-j})$. We can rewrite:

$$f(x^i, \theta') - f(x^i, \theta) \geq f(x^{i+1}, \theta') - f(x^{i+1}, \theta)$$

Applying this iteratively to $i = 1, 2, \ldots, d$, we have

$$\begin{aligned} f(x', \theta') - f(x', \theta) &\geq f(x^2, \theta') - f(x^2, \theta) \\ &\geq f(x^3, \theta') - f(x^3, \theta) \\ &\geq \ldots \\ &\geq f(x, \theta') - f(x, \theta) \end{aligned}$$

Therefore, f has increasing differences in (x, θ).

Section Exercise 2. If f has increasing differences in $(x, \theta) \in X \times \Theta$, does f have increasing differences on $Z := X \times \Theta$?

Only if $x, \theta \in \mathbb{R}$. f does not necessarily have increasing differences in $(x_i, x_j; x_{-ij}, \theta)$ for example.

Section Exercise 3. Prove that $X \geq_S X$ if and only if X is a sublattice of $\mathbb{R}^d$.

X is a sublattice of $\mathbb{R}^d \iff x \vee x', x \wedge x' \in X$ for all $x, x' \in X \iff X \geq_S X$.

Theorem 4 (Milgrom and Shannon). *Let $(X, \geq)$ be a lattice and $(\Theta, \geq)$ be a partially ordered set. Suppose $f : X \times \Theta \to \mathbb{R}$ and define $X^* : \Theta \times 2^X \rightrightarrows X$ as*

$$X^*_\Gamma(\theta) := \arg\max_{x \in \Gamma} f(x, \theta)$$

Then $X^*_\Gamma(\theta)$ is nondecreasing in the strong set order if and only if (i) $f(\cdot, \theta)$ is quasi-supermodular in x for all $\theta \in \Theta$, and (ii) f has single-crossing differences in (x, θ).

Section Exercise 4 (2023 Midterm 3 Q4)**.** There are two firms: 1 and 2. Let $S \subseteq \mathbb{R}_{++}$ be a nondegenerate compact interval. Define each firm $i \in \{1, 2\}$'s C^2 profit function $\pi_i : S^2 \to \mathbb{R}$ as

$$\pi_i(p_i, p_{-i}) := (p_i - c_i) D_i(p_i, p_{-i})$$

where $p_i \in S$ is the price charged by firm i, p_{-i} is the price charged by the other firm, $c_i \in (0, \min S)$ is firm i's marginal cost and $D_i(p_i, p_{-i}) > 0$ is the demand for firm i given that firm i charges p_i and the other firm charges p_{-i}.

(i) Suppose that D_i is such that the own-price elasticity falls as p_{-i} increases. Prove that π_i satisfies single-crossing differences in (p_i, p_{-i}).

Hint: Firm i's own-price elasticity is $\varepsilon_i(p_i, p_{-i}) := -\frac{p_i}{D_i(p_i, p_{-i})} \frac{\partial D_i}{\partial p_i}(p_i, p_{-i})$.

Take logs of both sides of the expression for π_i:

$$\log \pi_i(p_i, p_{-i}) = \log(p_i - c_i) + \log D_i(p_i, p_{-i})$$

Then

$$\begin{aligned} \frac{\partial \log \pi_i}{\partial p_i}(p_i, p_{-i}) &= \frac{1}{p_i - c_i} + \frac{1}{D_i(p_i, p_{-i})} \frac{\partial D_i}{\partial p_i}(p_i, p_{-i}) \\ &= \frac{1}{p_i - c_i} - \frac{\varepsilon_i(p_i, p_{-i})}{p_i} \end{aligned}$$

and

$$\frac{\partial^2 \log \pi_i}{\partial p_{-i} \partial p_i}(p_i, p_{-i}) = -\frac{1}{p_i} \frac{\partial \varepsilon_i}{\partial p_{-i}}(p_i, p_{-i})$$

We know that $\partial \varepsilon_i / \partial p_{-i}$ is negative, so we have that

$$\frac{\partial^2 \log \pi_i}{\partial p_{-i} \partial p_i}(p_i, p_{-i}) \geq 0$$

It follows that $\log \pi_i$ has increasing differences in (p_i, p_{-i}) and therefore also has single-crossing differences in (p_i, p_{-i}). Therefore $\pi_i = \exp(\log \pi_i)$ also has single-crossing differences in (p_i, p_{-i}).

(ii) Show that $\pi_i(\cdot; p_{-i})$ is quasi-supermodular for each $p_{-i} \in S$.

Because p_i is a scalar $x \vee x' = \max\{x, x'\}$ and $x \wedge x' = \min\{x, x'\}$. Then

$$f(x) + f(x') = f(\max\{x, x'\}) + f(\min\{x, x'\}) = f(x \vee x') + f(x \wedge x')$$

It follows that $\pi_i(\cdot, p_{-i})$ is supermodular and therefore quasi-supermodular.

(iii) Let $B_i : S \rightrightarrows S$ denote firm i's best-response correspondence, i.e.,

$$B_i(p_{-i}) := \arg\max_{p_i \in S} \pi_i(p_i, p_{-i})$$

Suppose that B_i is nonempty- and compact-valued. Prove that $B_i(\cdot)$ is monotone in the strong set order.

S is a lattice and the conditions of the Theorem of Milgrom and Shannon are satisfied by (i) and (ii). Monotonicity follows.