ECON 6170 Module 5 and Problem Sets 7, 8, 9 Answers

Patrick Ferguson*

Exercise 1. False. Take any function with a "kink", e.g.,

$$|x| := \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

which is not differentiable at $x_0 := 0$.

Exercise 2.

(i)

$$\begin{aligned}
(f+g)'(x_0) &= \lim_{h\to 0} \frac{f(x_0+h)+g(x_0+h)-f(x_0)-g(x_0)}{h} \\
&= \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} + \lim_{h\to 0} \frac{g(x_0+h)-g(x_0)}{h} \\
&= f'(x_0)+g'(x_0)
\end{aligned}$$

(ii)

$$\begin{aligned}
(\alpha f)'(x_0) &= \lim_{h\to 0} \frac{\alpha f(x_0+h)-\alpha f(x_0)}{h} \\
&= \alpha \cdot \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} \\
&= \alpha f'(x_0)
\end{aligned}$$

(iii)

$$\begin{aligned}
(f\cdot g)'(x_0) &= \lim_{h\to 0} \frac{f(x_0+h)g(x_0+h)-f(x_0)g(x_0)}{h} \\
&= \lim_{h\to 0} \frac{f(x_0+h)g(x_0+h)-f(x_0)g(x_0+h)+f(x_0)g(x_0+h)-f(x_0)g(x_0)}{h} \\
&= \lim_{h\to 0} \frac{(f(x_0+h)-f(x_0))g(x_0+h)+f(x_0)(g(x_0+h)-g(x_0))}{h} \\
&= \lim_{h\to 0} \frac{(f(x_0+h)-f(x_0))}{h} g(x_0+h) + \lim_{h\to 0} f(x_0)\frac{g(x_0+h)-g(x_0)}{h} \\
&= \lim_{h\to 0} \frac{(f(x_0+h)-f(x_0))}{h} \lim_{h\to 0} g(x_0+h) + f(x_0)\lim_{h\to 0} \frac{g(x_0+h)-g(x_0)}{h} \\
&= f'(x_0)g(x_0)+f(x_0)g'(x_0)
\end{aligned}$$

*Based on Professor Takuma Habu's solutions.

(iv) It suffices to show that

$$\left(\frac{1}{g}\right)'(x_0) = -\frac{g'(x_0)}{g(x_0)^2}$$

from which the product rule gives the rest. Note that $g(x_0) > 0$ implies $g(x_0 + h) > 0$ for h sufficiently close to 0.

$$\begin{aligned}
\left(\frac{1}{g}\right)'(x_0) &= \lim_{h\to 0} \frac{\frac{1}{g(x_0+h)} - \frac{1}{g(x_0)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{g(x_0)-g(x_0+h)}{g(x_0+h)g(x_0)}}{h} \\
&= \lim_{h\to 0} \frac{g(x_0) - g(x_0+h)}{h} \frac{1}{g(x_0+h)g(x_0)} \\
&= \lim_{h\to 0} \frac{g(x_0) - g(x_0+h)}{h} \lim_{h\to 0} \frac{1}{g(x_0+h)g(x_0)} \\
&= \frac{g'(x_0)}{g(x_0)^2}
\end{aligned}$$

Exercise 3. First consider the case in which $f'(x_0) \neq 0$. Then $f(x) \neq f(x_0)$ for $x \neq x_0$ sufficiently close to x_0. For otherwise, we would have a sequence $x_n \to x_0$ with $f(x_n) = f(x_0)$ for all n and thus $\frac{f(x_n)-f(x_0)}{x_n - x_0} = 0 \to 0$, a contradiction.

Then we can write

$$\frac{(g\circ f)(x_n) - (g\circ f)(x_0)}{x_n - x_0} = \frac{(g\circ f)(x_n) - (g\circ f)(x_0)}{f(x_n) - f(x_0)} \cdot \frac{f(x_n) - f(x_0)}{x_n - x_0}$$

By continuity of f, $x_n \to x_0$ implies $f(x_n) \to f(x_0)$, so letting $x_n \to x_0$ on both sides, we obtain

$$(g\circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$$

Now suppose $f'(x_0) = 0$. We will use the following lemma: *If $f : X \subseteq \mathbb{R} \to \mathbb{R}$ is differentiable at x_0, then f is Lipschitz on some $(x_0 - \delta, x_0 + \delta)$.* To see this, note that differentiability of f implies that there exist $\delta > 0$ such that $\left|\frac{f(x)-f(x_0)}{x - x_0}\right| \leq L$ for $x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$. Rearranging, we have $|f(x) - f(x_0)| \leq L \cdot |x - x_0|$.

Continuity of f at x_0 implies that for all $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in (x_0 - \delta, x_0 + \delta)$ implies $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$. Therefore, differentiability of g combined with our lemma implies that there exists $\delta > 0$ such that if $x \in (x_0 - \delta, x_0 + \delta)$ then

$$|(g\circ f)(x) - (g\circ f)(x_0)| \leq L|f(x) - f(x_0)|$$

Dividing across by $|x - x_0|$, we get

$$\left|\frac{(g\circ f)(x) - (g\circ f)(x_0)}{x - x_0}\right| \leq L \left|\frac{f(x) - f(x_0)}{x - x_0}\right|$$

Taking $x \to x_0$ implies that the the term on the right-hand side converges to $L \cdot |f'(x_0)| = 0$. The term on the left-hand side is bounded below by the sequence $0, 0, 0, \ldots$, so it too converges to 0. Therefore, $(g\circ f)'(x_0) = 0 = f'(x_0)$.

Exercise 4. Since f is strictly increasing, $x \neq y$ implies $f(x) \neq f(y)$, so f^{-1} is well-defined on some subset of $\mathbb{R}$. By the intermediate value theorem and continuity of f, this subset will be an interval, call it (c, d). By the extreme value theorem, $f(S)$ is compact for any closed (implying compact) $S \subseteq (a, b)$. Equivalently, the preimage of any closed $S \subseteq (a, b)$ under f^{-1} is closed. By (a slight generalisation of) a result seen in section, this means that f^{-1} is continuous.

The derivative of f^{-1} at $y = f(x)$, if it exists, is given by

$$\left(f^{-1}\right)'(y) = \lim_{\delta \to 0} \frac{f^{-1}(y+\delta) - f^{-1}(y)}{\delta}$$

By the intermediate value theorem, for δ sufficiently small, we can find an $x + h \in (a, b)$ such that $h \neq 0$ and $f(x+h) = y + \delta$. We can then rewrite the quotient above as

$$\frac{f^{-1}(y+\delta) - f^{-1}(y)}{\delta} = \frac{f^{-1}(f(x+h)) - f^{-1}(f(x+h))}{f(x+h) - f(x)} = \frac{h}{f(x+h) - f(x)}$$

Note that, by construction, as $\delta \to 0$, $f(x+h) \to y = f(x)$. Since $x + h$ is the only element of (a, b) that gives $f(x+h)$, this means that $h \to 0$ as $\delta \to 0$. It follows that we can write

$$\left(f^{-1}\right)'(y) = \lim_{h \to 0} \frac{h}{f(x+h) - f(x)} = \frac{1}{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}} = \frac{1}{f'(x)}$$

using the fact that $f'(x) > 0$ for all x.

Exercise 5. Take any x and y such that $a \leq x < y \leq b$. By the mean value theorem, there exists $z \in (x, y)$ such that

$$f(y) - f(x) = f'(z)(y - x)$$

Since $y - x > 0$ and $z \in (x, y)$, it follows that $f(y) - f(x) = 0$. Since x and y are arbitrary, this holds for all $x, y \in (a, b)$.

Exercise 6. Since $f^{(k)}$ is continuous at x_0 and $f^{(k)}(x_0) > 0$, there exists $\delta > 0$ such that $f^{(k)}(x) > 0$ for all $x \in (x_0 - \delta, x_0 + \delta)$. By Taylor's theorem, and using the fact that the first $k - 1$ derivatives of f at x_0 are 0, we have that for any $x \in (x_0 - \delta, x_0 + \delta)$,

$$f(x) = f(x_0) + \frac{f^{(k)}(p)}{k!}(x - x_0)^k,$$

where p is some point between x and x_0. But then $f^{(k)}(p) > 0$, and, since k is even, $(x - x_0)^k > 0$. Therefore $f(x)$ is larger than $f(x_0)$, i.e., f has a local minimum at x_0.

Exercise (Additional Exercise on PS 7)**.** Note that[1]

$$|x \cdot y|^2 = (x_1 y_1 + \cdots + x_d y_d)^2 = \sum_{i,j} x_i y_j x_j y_i$$

and

$$(\|x\| \cdot \|y\|)^2 = (x_1^2 + \cdots + x_d^2) \cdot (y_1^2 + \cdots + y_d^2) = \sum_{i,j} x_i^2 y_j^2$$

[1] $\sum_{i,j} f(i,j) := \sum_{i=1}^d \sum_{j=1}^d f(i,j)$ and $\sum_{i<j} f(i,j) := \sum_{i=1}^d \sum_{j=i+1}^d f(i,j)$.

But

$$(x_i y_j - x_j y_i)^2 = x_i^2 y_j^2 - 2x_i y_j x_j y_i + x_j^2 y_i^2$$

so

$$\begin{aligned}
\sum_{i<j}(x_i y_j - x_j y_i)^2 &= \sum_{i<j}\left(x_i^2 y_j^2 - 2x_i y_j x_j y_i + x_j^2 y_i^2\right) \\
&= \sum_{i<j}\left(x_i^2 y_j^2 + x_j^2 y_i^2\right) - 2\sum_{i<j} x_i y_j x_j y_i \\
&= \sum_{i<j}\left(x_i^2 y_j^2 + x_j^2 y_i^2\right) - 2\sum_{i<j} x_i y_j x_j y_i + \sum_{i=1}^{d}\left(x_i^2 y_i^2 - x_i y_i x_i y_i\right) \\
&= \sum_{i,j} x_i^2 y_j^2 - \sum_{i,j} x_i y_j x_j y_i \\
&= (\|x\| \cdot \|y\|)^2 - |x \cdot y|^2
\end{aligned}$$

But $\sum_{i<j}(x_i y_j - x_j y_i)^2$ is nonnegative, so $(\|x\| \cdot \|y\|)^2 \geq |x \cdot y|^2$, implying $\|x\| \cdot \|y\| > |x \cdot y|$.

Exercise 7.

$$\begin{aligned}
0 &= \lim_{h\to 0} \frac{\|f(x_0 + h) - f(x_0) - Df(x_0)h\|}{\|h\|} \\
&= \lim_{h\to 0} \|h\| \lim_{h\to 0} \frac{\|f(x_0 + h) - f(x_0) - Df(x_0)h\|}{\|h\|} \\
&= \lim_{h\to 0} \|f(x_0 + h) - f(x_0) - Df(x_0)h\| \\
&\geq \lim_{h\to 0} \|f(x_0 + h) - f(x_0)\| - \lim_{h\to 0} \|Df(x_0)h\| \\
&= \lim_{h\to 0} \|f(x_0 + h) - f(x_0)\|
\end{aligned}$$

Exercise 8. Differentiability of $f : X \subseteq \mathbb{R}^d \to \mathbb{R}^m$ at x_0 means that

$$\frac{\|f(x_0 + \vec{h}) - f(x_0) - A\vec{h}\|_m}{\|\vec{h}\|_d} \to 0$$

as $\vec{h} \to 0$, for some $A \in \mathbb{R}^{m\times d}$. For $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, d\}$, we want to show that[2]

$$\lim_{h\to 0} \frac{f_i(x_0 + he_j) - f_i(x_0)}{h}$$

exists, and equals the (i, j)-th entry of A. To do so, it suffices to show that

$$\left|\frac{f_i(x_0 + he_j) - f_i(x_0)}{h} - a_{ij}\right|$$

[2]Note that $\vec{h}$ above is a d-vector, whereas here it is a scalar.

is bounded above by by some function that converges to zero with h. Letting $A_{i\bullet}$ be the i-th row of A as a vector in $\mathbb{R}^d$, we can rewrite this as

$$\begin{aligned}\left|\frac{f_i(x_0+he_j)-f_i(x_0)-A_{i\bullet}^\top he_j}{h}\right| &= \left|\frac{f_i(x_0+he_j)-f_i(x_0)-A_{i\bullet}^\top he_j}{\|he_j\|_d}\right| \\ &\le \frac{1}{\|he_j\|_d}\sqrt{\sum_{i=1}^m \left[f_i(x_0+he_j)-f_i(x_0)-A_{i\bullet}^\top he_j\right]^2} \\ &= \frac{\|f(x_0+he_j)-f(x_0)-Ahe_j\|_m}{\|he_j\|_d} \qquad (*)\end{aligned}$$

But he_j is a sequence of d-vectors converging to zero with h, so $(*)$ converges to zero as $h \to 0$.

Exercise 9.

$$\frac{1}{h}\left[\frac{(0+h)0}{(0+h)^2+0^2}-f(0,0)\right]=0$$

so $\left.\frac{\partial f}{\partial x}\right|_{(0,0)}=0$. Similarly, $\left.\frac{\partial f}{\partial y}\right|_{(0,0)}=0$.

To prove that f is not differentiable at $(0,0)$, it suffices, by Exercise 7, to show that f is not continuous at $(0,0)$. Observe that the sequence $z_n=(\frac{1}{n},\frac{1}{n})$ converges to $(0,0)$, but $f(z_n)=\frac{1}{2}$ for all n, so $f(z_n)\to\frac{1}{2}\neq 0=f(0,0)$.

Exercise 10. By Proposition 10 (the chain rule), we can write

$$\begin{aligned}D(g\circ f)(x_0) &= Dg(f(x_0))\cdot Df(x_0) \\ &= \begin{bmatrix}\frac{\partial g(f(x_0))}{\partial y_1} & \cdots & \frac{\partial g(f(x_0))}{\partial y_d}\end{bmatrix}\begin{bmatrix}\frac{\partial f_1(x_0)}{\partial x} \\ \vdots \\ \frac{\partial f_d(x_0)}{\partial x}\end{bmatrix} \\ &= \sum_{i=1}^d \frac{\partial g(f(x_0))}{\partial y_i}\cdot\frac{\partial f_i(x_0)}{\partial x}\end{aligned}$$

where $y=f(x)$.

Exercise 11. To simplify notation, I write $x:=(x_1,x_2)$ (i.e., drop the 0 subscript). Define

$$\begin{aligned}r(h_1,h_2) &:= f(x_1+h_1,x_2+h_2)-f(x_1+h_1,x_2) \\ t(h_1,h_2) &:= f(x_1+h_1,x_2+h_2)-f(x_1,x_2+h_2)\end{aligned}$$

Then

$$\begin{aligned}r(h_1,h_2)-r(0,h_2) &= f(x_1+h_1,x_2+h_2)-f(x_1+h_1,x_2)-f(x_1,x_2+h_2)+f(x_1,x_2) \\ &= t(h_1,h_2)-t(h_1,0)\end{aligned}$$

By the mean-value theorem applied to $r(\cdot,h_2)$ and $t(h_1,\cdot)$

$$\frac{\partial r(c_1,h_2)}{\partial x_1}h_1=\frac{\partial t(h_1,c_2)}{\partial x_2}h_2$$

for some $c_1 \in (0, h_1)$ and $c_2 \in (0, h_2)$. Rewriting in terms of f,

$$h_1 \left(\frac{\partial f(x_1 + c_1, x_2 + h_2)}{\partial x_1} - \frac{\partial f(x_1 + c_1, x_2)}{\partial x_1} \right) = h_2 \left(\frac{\partial f(x_1 + h_1, x_2 + c_2)}{\partial x_2} - \frac{\partial f(x_1, x_2 + c_2)}{\partial x_2} \right)$$

Applying the mean value theorem to $\frac{\partial f(x_1+c_1,\cdot)}{\partial x_1}$ and $\frac{\partial f(\cdot,x_2+c_2)}{\partial x_2}$,

$$h_1 h_2 \frac{\partial^2 f(x_1 + c_1, \gamma_2)}{\partial x_2 \partial x_1} = h_2 h_1 \frac{\partial^2 f(\gamma_1, x_2 + c_2)}{\partial x_1 \partial x_2}$$

for some $\gamma_1 \in (x_1, x_1 + h_1)$, $\gamma_2 \in (x_2, x_2 + h_2)$. We can divide both sides by $h_1 h_2$ to get

$$\frac{\partial^2 f(x_1 + c_1, \gamma_2)}{\partial x_2 \partial x_1} = \frac{\partial^2 f(\gamma_1, x_2 + c_2)}{\partial x_1 \partial x_2}$$

Note that as $h_1 \to 0$, $c_1 \to 0$ and $\gamma_1 \to x_1$; and as $h_2 \to 0$, $c_2 \to 0$ and $\gamma_2 \to x_2$. Taking the limit of both sides as $h_1, h_2 \to 0$ and using that $f \in C^2$,

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_2 \partial x_1} = \frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2}$$

Exercise 13. Write $x_n := \alpha x + (1 - \alpha) y_n$ with $\alpha \in (0, 1)$. Then concavity of f implies that

$$\begin{aligned}
f(x_n) &\geq \alpha f(x) + (1 - \alpha) f(y_n) \\
f(x_n) - f(x) &\geq (1 - \alpha)(f(y_n) - f(x)) \\
\frac{f(x_n) - f(x)}{(1 - \alpha)(y_n - x)} &\geq \frac{f(y_n) - f(x)}{y_n - x} \\
\frac{f(x_n) - f(x)}{x_n - x} &\geq \frac{f(y_n) - f(x)}{y_n - x} \\
\frac{f(x) - f(x_n)}{x - x_n} &\geq \frac{f(x) - f(y_n)}{x - y_n}
\end{aligned}$$

and

$$\begin{aligned}
f(x_n) - f(y_n) &\geq \alpha(f(x) - f(y_n)) \\
\frac{f(x_n) - f(y_n)}{\alpha(x - y_n)} &\leq \frac{f(x) - f(y_n)}{x - y_n} \\
\frac{f(x_n) - f(y_n)}{x_n - y_n} &\leq \frac{f(x) - f(y_n)}{x - y_n}
\end{aligned}$$

Combining, we have

$$\frac{f(x) - f(x_n)}{x - x_n} \geq \frac{f(x_n) - f(y_n)}{x_n - y_n}$$

An analogous argument gives the second inequality,

$$\frac{f(x_n) - f(y_n)}{x_n - y_n} \geq \frac{f(y_n) - f(y)}{y_n - y}$$

Exercise 14. Suppose first that f is concave on X. Fix any $x, v \in \mathbb{R}^d$ with $v \neq 0$. For any $t, t' \in S_{x,v}$ and any $\alpha \in [0,1]$,

$$\begin{aligned}
g_{x,v}\left(\alpha t + (1-\alpha)\, t'\right) &= f\left(x + \left(\alpha t + (1-\alpha)\, t'\right) v\right) \\
&= f\left(\alpha\,(x + tv) + (1-\alpha)\left(x + t'v\right)\right) \\
&\geq \alpha f\,(x + tv) + (1-\alpha)\, f\left(x + t'v\right) \\
&= \alpha g_{x,v}\,(t) + (1-\alpha)\, g_{x,v}\left(t'\right)
\end{aligned}$$

Hence, $g_{x,v}(\cdot)$ is concave. Conversely, suppose that for any $x, v \in \mathbb{R}^d$ with $v \neq 0, g_{x,v}(\cdot)$ is concave. Pick any $z_1, z_2 \in X$ and any $\alpha \in [0,1]$. Letting $x = z_1$ and $v = z_2 - z_1$, observe that $g_{x,v}(0) = f(z_1)$, $g_{x,v}(1) = f(z_2)$, and

$$g_{x,v}\,(\alpha) = f\,(z_1 + \alpha\,(z_2 - z_1)) = f\,((1-\alpha)\, z_1 + \alpha z_2)$$

Since $g_{x,v}(\cdot)$ is concave, for any $\alpha \in [0,1]$,

$$\begin{aligned}
f\,((1-\alpha)\, z_1 + \alpha z_2) &= g_{x,v}\,(\alpha) \\
&= g_{x,v}\,((1-\alpha)\cdot 0 + \alpha \cdot 1) \\
&\geq (1-\alpha)\, g_{x,v}\,(0) + \alpha g_{x,v}\,(1) \\
&= (1-\alpha)\, f\,(z_1) + \alpha f\,(z_2)
\end{aligned}$$

i.e., f is concave. The proof for the case of strict concavity is analogous.

Exercise 16. By Proposition 15 and Remark 16, $f''(x) < 0$ is a sufficient condition for strict concavity, but not a necessary one. Note that in this case $f''(x) = -12x^2 < 0$ for all $x \in \mathbb{R} \setminus \{0\}$. Therefore, f is strictly concave on $\mathbb{R} \setminus \{0\}$. We therefore only need to prove strict concavity at $\alpha x + (1-\alpha)y = 0$ and for $x = 0$ or $y = 0$ (every other convex combination is not strict or is in the restricted domain $\mathbb{R} \setminus \{0\}$). Suppose that neither x nor y is 0 but $\alpha x + (1 - \alpha y) = 0$. Then

$$\begin{aligned}
\alpha f(x) + (1-\alpha) f(y) &= -\left(\alpha x^4 + (1-\alpha) y^4\right) \\
&< 0 \\
&= -0^4 \\
&= f(\alpha x + (1-\alpha) y)
\end{aligned}$$

Suppose $y = 0$ and $x \neq 0$. Then

$$\begin{aligned}
\alpha f(x) + (1-\alpha) f(y) &= -\alpha x^4 \\
&< -(\alpha x)^4 \\
&= f(\alpha x + (1-\alpha) y)
\end{aligned}$$

Exercise 17. First we calculate the Hessian. The first derivatives are

$$\frac{\partial f}{\partial x} = \alpha x^{\alpha-1} y^{\beta} \quad \frac{\partial f}{\partial y} = \beta x^{\alpha} y^{\beta-1}$$

The Hessian is then

$$H = \begin{bmatrix} \alpha(\alpha-1) x^{\alpha-2} y^{\beta} & \alpha\beta x^{\alpha-1} y^{\beta-1} \\ \alpha\beta x^{\alpha-1} y^{\beta-1} & \beta(\beta-1) x^{\alpha} \beta^{\beta-2} \end{bmatrix}$$

(i) Sylvester's criterion says that H is negative definite if $h_{11} < 0$ and $\det H > 0$. The former is true if $\alpha - 1 < 0$, or $\alpha < 1$. The latter is true if

$$\begin{aligned} 0 &< \alpha(\alpha-1)x^{\alpha-2}y^{\beta} \cdot \beta(\beta-1)x^{\alpha}\beta^{\beta-2} - (\alpha\beta x^{\alpha-1}y^{\beta-1})^2 \\ &= \alpha\beta x^{2\alpha-2}y^{2\beta-2}[(\alpha-1)(\beta-1) - \alpha\beta] \end{aligned}$$

which is true if $\alpha\beta - \alpha - \beta + 1 - \alpha\beta > 0$, or $\alpha + \beta < 1$. This condition is then sufficient for strict concavity.

(ii) I claim that $\alpha + \beta = 0$ is a sufficient (and necessary) condition. Sylvester's criterion implies that H is negative semidefinite if $h_{11}, h_{22} \leq 0$ and $\det H \geq 0$. Given $\alpha > 0$ and $\alpha + \beta = 1$, we know that $\beta < 1$, so $h_{22} < 0$. By the same reasoning as in (i), $\alpha + \beta = 1$ implies $\det H = 0$. Therefore, $\alpha + \beta = 1$ is sufficient for concavity. We will require $\alpha + \beta \geq 1$ to avoid strict concavity (negating (i)). However, this alone is insufficient, as it only says the Hessian is not negative definite, which does not disprove strict concavity. We need to find some other way of showing that f with $\alpha + \beta = 1$ will violate strict concavity. If $\alpha + \beta = 1$, we can write $\beta = 1 - \alpha$. Therefore, we can write $f(x, y) = x^{\alpha}y^{1-\alpha}$. Consider the distinct points $(1, 1)$ and $(3, 3)$. Then

$$\begin{gathered} f\left(\frac{1}{2}(1,1) + \frac{1}{2}(3,3)\right) = f(2,2) = 2^{\alpha}2^{1-\alpha} = 2 \\ = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 3 = \frac{1}{2}(1^{\alpha}1^{1-\alpha}) + \frac{1}{2}(3^{\alpha}3^{1-\alpha}) = \frac{1}{2}f(1,1) + \frac{1}{2}f(3,3) \end{gathered}$$

But if f were strictly concave, this would not hold with equality. Thus, f is not strictly concave.

(iii) For f to be neither concave nor convex, Sylvester's criterion implies that it suffices that $\det H < 0$ or $h_{11} > 0$ or $h_{22} \geq 0$. This will be the case iff $\alpha + \beta > 1$.

Exercise (Additional Exercise on PS 8)**.** Because f is hod k, we can write

$$f(\lambda x) = \lambda^k f(x)$$

Differentiating both sides with respect to λ, we have

$$\nabla f(\lambda x) \cdot x = k\lambda^{k-1} f(x)$$

which evaluated at $\lambda = 1$ gives

$$\nabla f(x) \cdot x = kf(x)$$

Exercise 18. We can rewrite the equation defining our implicit function as

$$F(x^*, h(x^*)) = 0$$

The implicit function theorem then says that at in some neighbourhood of (x^*, y^*), $h : \mathbb{R}^2 \to \mathbb{R}^2$ is well-defined and has derivative

$$Dh(x) = -(D_y F(x, h(x)))^{-1} \cdot D_x F(x, h(x))$$

which we can write as

$$\begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} \end{bmatrix} = - \begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} \end{bmatrix}^{-1} \cdot \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{bmatrix}$$

where $y = h(x)$. Using the formula for inverting a 2×2 matrix, we get

$$\begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} \end{bmatrix}^{-1} = \frac{1}{\frac{\partial F_1}{\partial y_1} \cdot \frac{\partial F_2}{\partial y_2} - \frac{\partial F_1}{\partial y_2} \cdot \frac{\partial F_2}{\partial y_1}} \begin{bmatrix} \frac{\partial F_2}{\partial y_2} & -\frac{\partial F_1}{\partial y_2} \\ -\frac{\partial F_2}{\partial y_1} & \frac{\partial F_1}{\partial y_1} \end{bmatrix}$$

Inserting this into the previous formula, we have

$$\begin{aligned}
\frac{\partial h_1}{\partial x_1} &= -\frac{\frac{\partial F_2}{\partial y_2} \cdot \frac{\partial F_1}{\partial x_1} - \frac{\partial F_1}{\partial y_2} \cdot \frac{\partial F_2}{\partial x_1}}{\frac{\partial F_1}{\partial y_1} \cdot \frac{\partial F_2}{\partial y_2} - \frac{\partial F_1}{\partial y_2} \cdot \frac{\partial F_2}{\partial y_1}} = \frac{\frac{\partial F_1}{\partial y_2} \cdot \frac{\partial F_2}{\partial x_1} - \frac{\partial F_2}{\partial y_2} \cdot \frac{\partial F_1}{\partial x_1}}{\frac{\partial F_1}{\partial y_1} \cdot \frac{\partial F_2}{\partial y_2} - \frac{\partial F_1}{\partial y_2} \cdot \frac{\partial F_2}{\partial y_1}} \\
\frac{\partial h_1}{\partial x_2} &= -\frac{\frac{\partial F_2}{\partial y_2} \cdot \frac{\partial F_1}{\partial x_2} - \frac{\partial F_1}{\partial y_2} \cdot \frac{\partial F_2}{\partial x_2}}{\frac{\partial F_1}{\partial y_1} \cdot \frac{\partial F_2}{\partial y_2} - \frac{\partial F_1}{\partial y_2} \cdot \frac{\partial F_2}{\partial y_1}} = \frac{\frac{\partial F_1}{\partial y_2} \cdot \frac{\partial F_2}{\partial x_2} - \frac{\partial F_2}{\partial y_2} \cdot \frac{\partial F_1}{\partial x_2}}{\frac{\partial F_1}{\partial y_1} \cdot \frac{\partial F_2}{\partial y_2} - \frac{\partial F_1}{\partial y_2} \cdot \frac{\partial F_2}{\partial y_1}} \\
\frac{\partial h_2}{\partial x_1} &= -\frac{-\frac{\partial F_2}{\partial y_1} \cdot \frac{\partial F_1}{\partial x_1} + \frac{\partial F_1}{\partial y_1} \cdot \frac{\partial F_2}{\partial x_1}}{\frac{\partial F_1}{\partial y_1} \cdot \frac{\partial F_2}{\partial y_2} - \frac{\partial F_1}{\partial y_2} \cdot \frac{\partial F_2}{\partial y_1}} = \frac{\frac{\partial F_2}{\partial y_1} \cdot \frac{\partial F_1}{\partial x_1} - \frac{\partial F_1}{\partial y_1} \cdot \frac{\partial F_2}{\partial x_1}}{\frac{\partial F_1}{\partial y_1} \cdot \frac{\partial F_2}{\partial y_2} - \frac{\partial F_1}{\partial y_2} \cdot \frac{\partial F_2}{\partial y_1}} \\
\frac{\partial h_2}{\partial x_2} &= -\frac{-\frac{\partial F_2}{\partial y_1} \cdot \frac{\partial F_1}{\partial x_2} + \frac{\partial F_1}{\partial y_1} \cdot \frac{\partial F_2}{\partial x_2}}{\frac{\partial F_1}{\partial y_1} \cdot \frac{\partial F_2}{\partial y_2} - \frac{\partial F_1}{\partial y_2} \cdot \frac{\partial F_2}{\partial y_1}} = \frac{\frac{\partial F_2}{\partial y_1} \cdot \frac{\partial F_1}{\partial x_2} - \frac{\partial F_1}{\partial y_1} \cdot \frac{\partial F_2}{\partial x_2}}{\frac{\partial F_1}{\partial y_1} \cdot \frac{\partial F_2}{\partial y_2} - \frac{\partial F_1}{\partial y_2} \cdot \frac{\partial F_2}{\partial y_1}}
\end{aligned}$$

Exercise 19. Define

$$F(y, x) = y - f(x)$$

for all $(y, x) \in Y \times X$ Note that $x_0 \in \text{int} X$ by assumption. It is WLOG to assume that Y is open,[3] implying $y_0 := f(x_0) \in \text{int}\, Y$. Note that this is implies $(x_0, y_0) \in \text{int}(X \times Y)$.

We also have $F(y_0, x_0) = 0$. That f is C^1 at x_0 implies that F is C^1 at x_0. Moreover, $D_x F(y_0, x_0) = -Df(x_0)$ is invertible. Thus, applying the implicit function theorem on F gives us that x is implicitly defined as a function g of y on an open ball $B_Y(y_0)$ such that $F(y, g(y)) = 0$ for all $y \in B_{\varepsilon_Y}(y_0)$. Furthermore, g is differentiable and

$$\begin{aligned}
Dg(y) &= -\left[D_x F(y, g(y))\right]^{-1} D_y F(y, g(y)) \\
&= -\left[-Df(g(y))\right]^{-1} I \\
&= Df(g(y))^{-1}.
\end{aligned}$$

[3]Because every potential codomain can be extended to an open superset.