Module 2 answer key

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1. Prove Proposition 4: If (x_n) is a Cauchy sequence and there is a subsequence (x_{n_k}) that converges to $x \in \mathbb{R}$, then (x_n) converges to x as well.
Solution: Assume that there exists a subsequence (x_{n_k}) of (x_n) such that $(x_{n_k}) \to x$. Take $\varepsilon > 0$ and choose N big enough so that $|x_{n_k} - x| < \varepsilon$ and $|x_n - x_m| < \varepsilon$ for $n_k, n, m > N$. Now, take $n > N$ and $n_k > n > N$ and get $|x_n - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| < 2\varepsilon$, hence, $(x_n) \to x$.

2. Verify that open intervals (a, b) are indeed open according to this definition. Do the same for $(a, +\infty)$.
Solution: Pick arbitrary $x \in (a, b)$. Let $\epsilon = \min\{(x-a), (b-x)\}$. Then for any $y \in B_\epsilon(x)$, we have $y > a$ and $y < b$. For (a, ∞), pick $\epsilon = x - a$.

3. The (arbitrary) union of open sets is open. The intersection of finitely many open sets is open. Prove this. What about arbitrary intersections of open sets?

   Solution: (a) Consider a union $S = \bigcup_{i=1}^{n} s_i$, where n may equal ∞. If $x \in S$, then $x \in s_i$ for at least some particular i. Then, there exists an epsilon ball around x that is in the same s_i (because s_i is open) and consequently also in S.
(b) Consider an intersection $S = \bigcap_{i=1}^{m} s_i$, where $m \neq \infty$. Take $x \in S$. It's true that $\forall i, x \in s_i$. Since s_i is open, there exists ϵ_i such that $B_{\epsilon_i}(x)$ is in s_i. Take $\epsilon = \min_i\{\epsilon_1, \epsilon_2, ..., \epsilon_m\}$, it exists because we looking the interection of a finite number of open sets. Notice that $B_\epsilon(x) \subseteq B\epsilon_i(x) \subseteq s_i$ for all i. Therefore, $B_\epsilon(x)$ lies in the intersection S. Hence, S is open.
(c) This does not hold for the intersection of an infinite number of open sets. Consider $\bigcap_{i=1}^{\infty}(-\frac{1}{i}, 1) = [0, 1)$.

4. Prove that the closed interval $[a, b]$ is indeed closed. (Feel free to use Exercise 2.)
Solution: We know that the complement, $(-\infty, a) \cup (b, \infty)$ is the union of two open sets and therefore open.

5. The arbitrary intersection of closed sets is closed. The union of finitely many closed sets is closed. Prove this.[1] What about arbitrary unions of closed sets?

   Solution: By DeMorgan's laws, this follows from exercise 4.
(a) Take the arbitrary intersection of closed sets s_i indexed by i. Denote $\bar{s_i}$ to be a complement of s_i. Because s_i is closed, $\bar{s_i}$ is open. Take $S = \bigcap_i s_i$.

[1] *Hint:* Recall De Morgan's laws.

Its complement by De Morgan's law is $\bar{S} = \bigcup_i \bar{s_i}$ is open because it is the union of open sets. Therefore, S is closed by definition of a closed set.
(b) Take the union of finitely many closed sets: $S = \bigcup_i^m s_i$. Its complement: $\bar{S} = \bigcap_i^m \bar{s_i}$. The intersection of finitely many open sets is open. Hence, S is closed.
(c) Consider $\bigcup_i \left[\frac{1}{i}, +\infty\right) = (0, +\infty)$

6. Give an example of open cover that does not admit a finite subcover for $(0, 1] \subset \mathbb{R}$ (which is not closed) and for $[0, +\infty)$ (which is not bounded).
Solution: Consider the examples $(0, 1] \subseteq \bigcup_{i=1}^{\infty} (\frac{1}{i}, 2) = (0, 2)$ and $[0, +\infty) \subseteq \bigcup_{i=1}^{\infty} (-1, i) = (-1, \infty)$.

7. Suppose S has an isolated point—for example, $S = \{1\} \cup [2, 3]$. What functions are continuous at 1?
Solution: Every function is continuous at 1. Consider any sequence that converges to $x_0 = 1$ in the domain S. Picking $0 < \epsilon < 1$, by definition there exists N such that for all $n > N$ $|x_n - 1| < \epsilon < 1$. Hence, $x_n = 1$ for all $n > N$, since x_n has to belong to S and be no further away from $x_0 = 1$ than $\epsilon < 1$. Hence, for $n > N$ $f(x_n) = f(1)$ and, therefore, converges to 1.

8. Prove Proposition **??** using ε-δ definition of continuity.
Solution: Since g is continuous at $f(x_0)$, then $\forall \epsilon > 0$ there exists $\delta_1 > 0$ such that:
$$|f(x) - f(x_0)| < \delta_1 \Rightarrow |g(f(x)) - g(f(x_0))| < \epsilon$$
Since f is continuous at x_0, then for $\delta_1 > 0$ there exists $\delta > 0$ such that:
$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \delta_1$$
Together the above two lines imply that $\forall \epsilon > 0$ there exists $\delta > 0$ such that:
$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \delta_1 \Rightarrow |g(f(x)) - g(f(x_0))| < \epsilon$$

9. Let f and g be continuous at x_0. Prove or disprove: $\max(f, g)$ is continuous at x_0.[2]
Solution: True; $\max\{f(x), g(x)\} = \frac{1}{2}(f(x) + g(x)) + \frac{1}{2}|f(x) - g(x)|$. Note that these are all continues functions, and the sum of continuous functions is continuous.

[2] *Hint:* $\max(f, g) = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|$

10. Prove or disprove: $f : S \to \mathbb{R}$ is continuous at x_0 if and only if for every *monotonic* sequence (z_n) in S converging to x_0, $\lim_n f(z_n) = f(x_0)$.[3]
    Solution:
    (a) Take any sequence (z_n) that converges to x_0, then take any subsequence of (z_n), let's call it (z_{n_k}), then (z_{n_k}) has a monotone subsequence $(z_{n_{k_l}})$ such that $f(z_{n_{k_l}})$ converges to $f(x_0)$. Consider $f(z_n)$ for any sequence z_n converging to x_0, for any subsequence $f(z_{n_k})$, it has a sub-subsequence $f(z_{n_{k_l}})$ that converges to $f(x_0)$. Now we can use the lemma in the hint which gives us that $f(z_n)$ converges to $f(x_0)$.
    (b) If $f : S \to \mathbb{R}$ is continuous at x_0, then for every monotonic sequence z_n in S, $\lim f(z_n) = f(x_0)$. This holds trivially because in fact $\lim f(z_n) = f(x_0)$ for every sequence z_n in S, monotonic or not.

11. Prove or disprove: The extreme value theorem still holds if f is defined on (a, b).
    Solution: False. Take, for example, any function f such that $\lim_{x \to b-} f(x) = +\infty$

12. Prove or disprove: The extreme value theorem still holds if f is defined on (a, b) and we add the assumption that f is bounded.
    Solution: False. Take any continuous and strictly increasing function, the maximum of it does not exist on (a,b).

[3] *Hint:* the following Lemma could be useful: (x_n) converges to x if and only if for every subsequence x_{n_k} there exists sub-subsequence $x_{n_{k_l}}$ that converges to x. Proving this Lemma is a good exercise in itself.