ECON 6130: Endowment Economy with Complete Markets

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Outline

We will formalize the main ideas explored in our previous examples. The main reference for this part of the course is LS chapter 8.

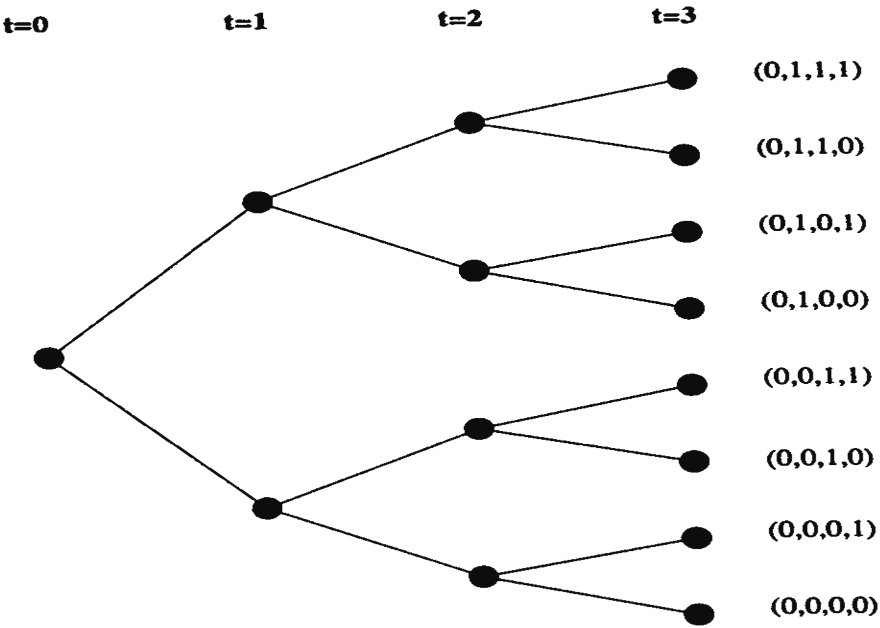
Outline

1. Refresher on probability theory
2. Arrow-Debreu equilibrium
3. Sequential trade equilibrium
4. Social planner and Pareto efficiency
5. Welfare theorems

Probability theory refresher

A stochastic world:

- In each period $t \geq 0$, a stochastic event $s_t \in S$ is realized.
- Denote $s^t = [s_0, s_1, \ldots s_t]$ a history up and until time t.
- The *unconditional* probability of observing s^t is given by the measure $\pi_t(s^t)$
- The *conditional* probability of observing s^t given that s^τ happened is $\pi_t(s^t|s^\tau)$
- Assume that a given s_0 happened before trading starts


t=0
t=1
t=2
t=3
(0,1,1,1)
(0,1,1,0)
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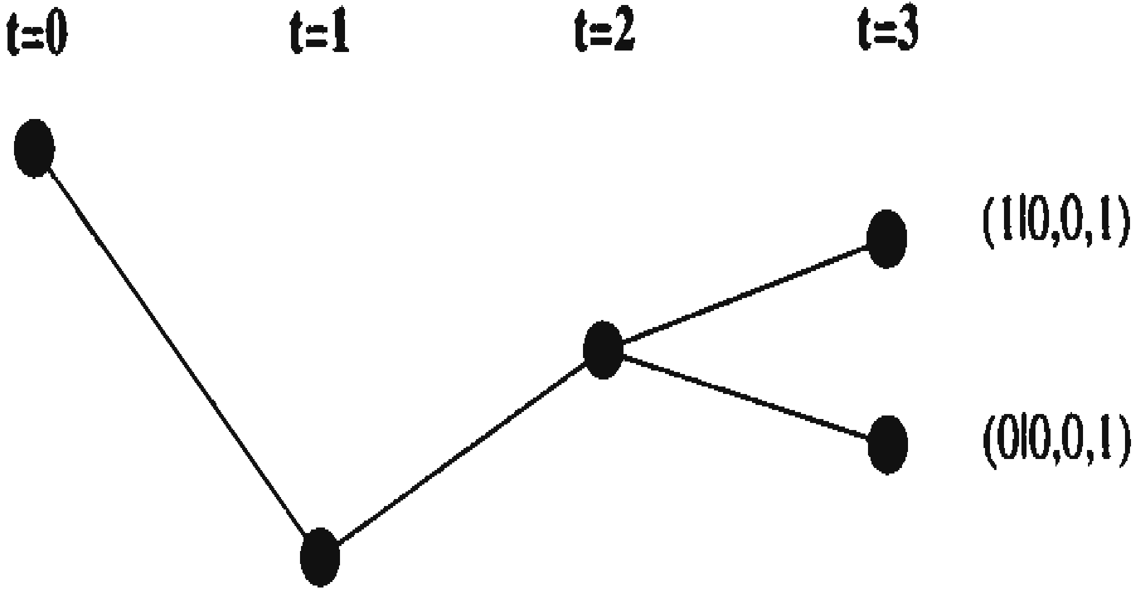
t=0
t=1
t=2
t=3
(1|0,0,1)
(0|0,0,1)

The economy

Environment:

- There are I agents indexed by $i = 1, \ldots, I$. Agent i owns a stochastic endowment of goods $y_t^i(s^t)$.
- Household i values a history-dependent consumption plan $c^i = \{c_t^i(s^t)\}_{t=0}^{\infty}$ according to

$$U(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u[c_t^i(s^t)] \pi_s(s^t)$$

- where $u' > 0$, $u'' < 0$, $\lim_{c \to 0} u'(c) = +\infty$.

Definition 1 (Feasible allocation)

A feasible allocation satisfies

$$\sum_i c_t^i(s^t) \leq \sum_i y_t^i(s^t)$$

for all t and all s^t.

Trading arrangements

Suppose that each household evolves in autarky:

- What's their consumption $c_t(s^t)$?
- Does it depend on s^t?

We will study two types of trading arrangements:

1. Arrow-Debreu securities: At $t = 0$ households trade claims to consumption at all time $t > 0$ contingent on all possible histories up to time t, s^t. There is no trade at time $t > 0$.
2. Sequential markets: trade occurs at each $t \geq 0$. Trades for history s^{t+1}-contingent $t + 1$ goods occur only at node s^t.

Efficient allocation

Definition 2 ((Pareto) Efficient allocation)

An allocation $\{c^i\}_{i\in\{1,I\}}$ *is efficient if there is no feasible allocation* $\{\tilde{c}^i\}_{i\in\{1,I\}}$ *such that*

$$U(\tilde{c}^i) \geq U(c^i) \text{ for all } i$$
$$U(\tilde{c}^i) > U(c^i) \text{ for at least one } i$$

Proposition 1

An allocation is efficient if and only if it solves the social planner's problem

$$\max_{\{c^i\}_i} \sum_{i=1}^{I} \lambda_i U(c^i), \text{ s.t. } \{c^i\}_i \text{ being feasible}$$

for some non-negative λ_i *for all* i*. The* λ*'s are the Pareto weights.*

Lagrangian ($\theta_t(s^t) \geq 0$ are the Lagrange multipliers):

$$L = \sum_{t=0}^{\infty} \sum_{s^t} \left(\sum_{i=1}^{I} \lambda_i \beta^t u(c_t^i(s^t)) \pi_t(s^t) + \theta_t(s^t) \sum_{i=1}^{I} \left[y_t^i(s^t) - c_t^i(s^t) \right] \right)$$

FOC:

$$\lambda_i \beta^t u'(c_t^i(s^t)) \pi_t(s^t) = \theta_t(s^t)$$

Therefore:

$$c_t^i(s^t) = u'^{-1}(\lambda_i^{-1} \lambda_1 u'(c_t^1(s^t)))$$

and

$$\sum_i u'^{-1}(\lambda_i^{-1} \lambda_1 u'(c_t^1(s^t))) = \sum_i y_t^i(s^t)$$

- How does $c_t^1(s^t)$ depend on the endowments? Insurance?
- How does $c_t^1(s^t)$ depend on the Pareto weights?

Arrow-Debreu equilibrium

At time $t = 0$, and only then, agents trade claims to consumption at time t contingent on history s^t at price $q_t^0(s^t)$.

Definition 3 (ADE)

An Arrow-Debreu equilibrium is a sequence of allocations $\{c_t^i(s^t)\}_{t=0}^{\infty}$ for all agents i and prices $\{q_t^0(s^t)\}_{t=0}^{\infty}$ such that:

1. *Given prices, household's i allocation solves it maximization problem:*

$$\max_{\{c_t^i(s^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u[c_t^i(s^t)] \pi_t(s^t)$$

$$s.t. \ \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) y_t^i(s^t)$$

2. *The allocation is feasible (markets clear)*

Solving the equilibrium

Each agent's FOC is

$$\beta^t u'[c_t^i(s^t)]\pi_t(s^t) = \mu_i q_t^0(s^t)$$

where μ_i is the Lagrange multiplier on the budget constraint. Therefore,

$$c_t^i(s^t) = u'^{-1}\left(u'(c_t^1(s^t))\frac{\mu_i}{\mu_1}\right)$$

and

$$\sum_i u'^{-1}\left(u'(c_t^1(s^t))\frac{\mu_i}{\mu_1}\right) = \sum_i y_t^i(s^t).$$

- How does $c_t^1(s^t)$ depend on the endowments? Insurance?
- Have we seen a similar equation before?
- At the ADE allocation, the shadow prices $\theta_t(s^t)$ are equal to $q_t^0(s^t)$.

Efficiency of ADE

Remember: efficient allocation solves a social planner's problem.

Theorem 1 (First welfare theorem)

Any Arrow-Debreu equilibrium allocation is efficient.

Idea of the proof: Just set $\lambda_i = \mu_i^{-1}$ and normalize the weights. Need to check the RC. Also, the *shadow prices* $\theta_t(s^t) = q_t^0(s^t)$.

Theorem 2 (Second welfare theorem)

Let $\{c_t^i(s^t, \lambda\}_{t=0}^{\infty}$ be an efficient allocation for some Pareto weights $\{\lambda^i\}_{i=1}^{\infty}$. Then there exist transfers $\{\tau^i\}_{i=1}^{I}$ such that the allocation is an Arrow-Debreu equilibrium. Intuition?

See Mas-Colell, Whinston and Green (1995) for a proof.

Negishi's method

The first welfare theorems gives us a way to easily find the set of Arrow-Debreu equilibria (Negishi's (1960) method):

1. Compute all efficient allocations. (SP problem with arbitrary weights)
2. The first welfare theorem tells us that all competitive allocation are efficient. By solving for all efficient allocation we therefore have solved for the competitive ones.
3. Isolate the efficient allocation that are also competitive allocations.

Example of Negishi's method

Remember our 2-agent economy with varying endowments 2,0.
With Pareto weight $\alpha \in [0,1]$, the SP problem is

$$\max_{c^1,c^2} \sum_{t=0}^{\infty} \beta^t [\alpha \log(c_t^1) + (1-\alpha)\log(c_t^2)]$$

$$c_t^i \geq 0, \forall i, \forall t$$

$$c_t^1 + c_t^2 = e_t^1 + e_t^2 \equiv 2, \forall t$$

Attach multipliers $\theta_t/2$ to to the resource constraints. The FOCs are

$$\frac{\alpha\beta^t}{c_t^1} = \frac{\theta_t}{2}$$

$$\frac{(1-\alpha)\beta^t}{c_t^2} = \frac{\theta_t}{2}$$

and therefore

$$c_t^1 = \frac{\alpha}{1-\alpha} c_t^2$$

Example of Negishi's method

Combining with the resource constraints, we get

$$c_t^1(\alpha) = 2\alpha$$
$$c_t^2(\alpha) = 2(1-\alpha)$$
$$\theta_t = \beta^t$$

So there seems to be a continuum of efficient allocations... But we had a unique solution when we solved that economy earlier. There must be an extra condition on CE that will help us select from the set of efficient allocation.

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Budget constraints

$$t^i(\alpha) = \sum_t \theta_t [c_t^i(\alpha) - e_t^i]$$

We look for α^* such that $t^1(\alpha) = t^2(\alpha) = 0$.

Example of Negishi's method

$$t^1(\alpha) = \sum_t \theta_t [c_t^1(\alpha) - e_t^1] = \sum_t \beta^t [2\alpha - e_t^1] = \frac{2\alpha}{1-\beta} - \frac{2}{1-\beta^2}$$

$$t^2(\alpha) = \sum_t \theta_t [c_t^2(\alpha) - e_t^2] = \sum_t \beta^t [2(1-\alpha) - e_t^2] = \frac{2(1-\alpha)}{1-\beta} - \frac{2\beta}{1-\beta^2}$$

Our solution is $\alpha^* = \frac{1}{1+\beta}$ and, for that α, the consumptions are

$$c_t^1 = \frac{2}{1+\beta}$$

$$c^2{}_t = \frac{2\beta}{1+\beta}$$

which is what we got when we solved the ADE.

Solving the equilibrium with no aggregate uncertainty

Now we go back to ADE. Suppose that there is no aggregate uncertainty and that $I = 2$. Let the stochastic events $s_t \sim U([0,1])$ be independent across time. Suppose that the endowments are $y_t^1(s^t) = s_t$ and $y_t^2(s^t) = 1 - s_t$.

- How do $c_t^i(s^t)$ vary across time?
- From the FOC we have

$$q_t^0(s^t) = \beta^t \pi_t(s^t) \frac{u'(c^i)}{\mu_i}$$

- We can use the household budget constraint to write:

$$c^i = (1-\beta) \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t) y_t^i(s^t)$$

- What is the interpretation?
- What is $c^1 + c^2$ equal to?

Asset pricing with AD securities

Suppose that we have an asset that provides dividends $\{d_t(s^t)\}_{t=0}^{\infty}$, what should the price of this asset be?

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What's the price of an asset that pays 1 at each t regardless of s^t?

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What's the price of an asset that pays 1 at period τ only regardless of s^τ?

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What's the price of an asset that pays 1 at period τ only regardless of s^τ?

$$\sum_{s^\tau} q_\tau^0(s^\tau)$$

Asset pricing with AD securities

What is the time 0 price of an asset that entitles the owner to dividend stream $\{d_t(s^t)\}_{t \geq \tau}$ if history s^τ is realized?

Asset pricing with AD securities

What is the time 0 price of an asset that entitles the owner to dividend stream $\{d_t(s^t)\}_{t\geq\tau}$ if history s^τ is realized?

$$p_\tau^0(s^\tau) = \sum_{t\geq\tau}\sum_{s^t|s^\tau} q_t^0(s^t)d_t(s^t)$$

The units of the price are time 0 goods: $q_0^0(s_0) = 1$. To convert the price into units of time τ, history s^τ consumption goods, we must divide by $q_\tau^0(s^\tau)$:

$$p_\tau^\tau(s^\tau) = \frac{p_\tau^0(s^\tau)}{q_\tau^0(s^\tau)} = \sum_{t\geq\tau}\sum_{s^t|s^\tau} \frac{q_t^0(s^t)}{q_\tau^0(s^\tau)}d_t(s^t)$$

Asset pricing with AD securities

Notice that (using the FOCs) ($q_t^\tau(s^t)$ is the price of one unit of s^t goods in terms of s^τ goods)

$$q_t^\tau(s^t) \equiv \frac{q_t^0(s^t)}{q_\tau^0(s^\tau)} = \frac{\beta^t u'(c_t^i(s^t))\pi_t(s^t)}{\beta^\tau u'(c_\tau^i(s^\tau))\pi_\tau(s^\tau)} = \beta^{t-\tau}\frac{u'(c_t^i(s^t))}{u'(c_\tau^i(s^\tau))}\pi_t(s^t|s^\tau)$$

Remember that by Bayes law:

$$\pi_t(s^t|s^\tau) \times \pi_\tau(s^\tau) = \pi_t(s^t, s^\tau) = \pi_t(s^t)$$

So we can write:

$$p_\tau^\tau(s^\tau) = \sum_{t\geq\tau}\sum_{s^t|s^\tau} q_t^\tau(s^t)d_t(s^t)$$

Why did we go to all this trouble?

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$$p_\tau^\tau(s^\tau) = \sum_{t \geq \tau} \sum_{s^t|s^\tau} q_t^\tau(s^t) d_t(s^t)$$

Why did we go to all this trouble? Price of equity at time τ in state s^τ.

Asset pricing with AD securities

We have:

$$q^{\tau}_{\tau+1}(s^{\tau+1}) = \beta \frac{u'(c^i_{\tau+1}(s^{\tau+1}))}{u'(c^i_{\tau}(s^{\tau}))} \pi_{\tau+1}(s^{\tau+1}|s^{\tau})$$

Intuitively, what is this quantity and why is it useful?

Asset pricing with AD securities

We have:

$$q_{\tau+1}^{\tau}(s^{\tau+1}) = \beta \frac{u'(c_{\tau+1}^{i}(s^{\tau+1}))}{u'(c_{\tau}^{i}(s^{\tau}))} \pi_{\tau+1}(s^{\tau+1}|s^{\tau})$$

Intuitively, what is this quantity and why is it useful? Pricing kernel.

We can write the price at time τ in history s^{τ} of a claim to a random payoff $\omega(s_{\tau+1})$ as

$$p_{\tau}^{\tau}(s^{\tau}) = \sum_{s_{\tau+1}} q_{\tau+1}^{\tau}(s^{\tau+1})\omega(s_{\tau+1}) = E_{\tau}\left(\beta \frac{u'(c_{\tau+1})}{u'(c_{\tau})}\omega(s_{\tau+1})\right)$$

Defining the gross return $R_{\tau+1} \equiv \omega(s_{\tau+1})/p_{\tau}^{\tau}(s^{\tau})$, we can write

$$1 = E_{\tau}\left(\beta \frac{u'(c_{\tau+1})}{u'(c_{\tau})} R_{\tau+1}\right) \equiv E_{\tau}(m_{\tau+1}R_{\tau+1})$$

The term $m_{\tau+1}$ is called the stochastic discount factor.

Sequential trading

So far we've looked at Arrow-Debreu equilibrium. We've seen that the allocation is equivalent to an efficient allocation and we've seen how to price assets. We now move to a different market structure in which assets are traded each period.

Arrow securities: At each date $t \geq 0$, trade occurs in a set of claims to one-period-ahead state-contingent consumption.

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Are markets complete? yes, they are sequentially complete...

Sequential trading

Define the *natural debt limit* (q_τ^t are the AD prices):

$$A_t^i(s^t) = \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} q_\tau^t(s^\tau) y_\tau^i(s^\tau)$$

Intuition:

Sequential trading

Define the *natural debt limit* (q_τ^t are the AD prices):

$$A_t^i(s^t) = \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} q_\tau^t(s^\tau) y_\tau^i(s^\tau)$$

Intuition: Household i at time $t-1$ cannot promise to pay more than $A_t^i(s^t)$ at time t in state s^t, otherwise their consumption would be negative.

Denote by $\tilde{a}_t^i(s^t)$ the claims to time t consumption, on top of its endowment, that agent i get in period t in state s^t.

Denote by $\tilde{Q}_t(s_{t+1}|s^t)$ the price of a claim to one unit of consumption at time $t+1$ in state s^{t+1} when the current history is s^t.

Sequential trading

The objective function of households is unchanged. Using our new notation, the budget constraint is

$$\tilde{c}_t^i(s^t) + \sum_{s_{t+1}} \tilde{a}_{t+1}^i(s_{t+1}, s^t)\tilde{Q}_t(s_{t+1}|s^t) \leq y_t^i(s^t) + \tilde{a}_t^i(s^t)$$

To rule out Ponzi schemes, we impose the condition

$$-\tilde{a}_{t+1}^i(s^{t+1}) \leq A_{t+1}^i(s^{t+1})$$

This is not the only condition that would work.

Sequential trading

Definition 4 (Sequential trading equilibrium)

A sequential trading competitive equilibrium is a distribution of assets $\tilde{a}^i_{t+1}$ for all i and t, an allocation $\{\tilde{c}^i\}$ for all i, and pricing kernels $\tilde{Q}_t(s_{t+1}|s^t)$ such that

1. *For all i, $\tilde{c}^i$ solves household i's problem.*
2. *For all $\{s^t\}_{t=0}^{\infty}$, we have $\sum_i \tilde{c}^i_t(s^t) = \sum_i y^i_t(s^t)$ and $\sum_i \tilde{a}^i_{t+1}(s_{t+1}, s^t) = 0$.*

Solving the sequential trading equilibrium

The Lagrangien is

$$
\begin{aligned}
L^i = \sum_{t=0}^{\infty} \sum_{s^t} & \Big(\beta^t u[\tilde{c}_t^i(s^t)]\pi_t(s^t) \\
& + \eta_t^i(s^t)\left(y_t^i(s^t) + \tilde{a}_t^i(s^t) - \tilde{c}_t^i(s^t) - \sum_{s_{t+1}} \tilde{a}_{t+1}^i(s_{t+1}, s^t)\tilde{Q}_t(s_{t+1}|s^t)\right) \\
& + \sum_{s^{t+1}} \nu_t^i(s^t, s_{t+1})(A_{t+1}^i(s^{t+1}) + \tilde{a}_{t+1}^i(s^{t+1}))\Big)
\end{aligned}
$$

The FOC's are:

Solving the sequential trading equilibrium

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& + \sum_{s^{t+1}} \nu_t^i(s^t, s_{t+1})(A_{t+1}^i(s^{t+1}) + \tilde{a}_{t+1}^i(s^{t+1}))\Big)
\end{aligned}
$$

The FOC's are:

$$
\begin{aligned}
\beta^t u'[\tilde{c}_t^i(s^t)]\pi_t(s^t) - \eta_t^i(s^t) &= 0 \\
-\eta_t^i(s^t)\tilde{Q}_t(s_{t+1}|s^t) + \nu_t^i(s^t, s_{t+1}) + \eta_{t+1}^i(s_{t+1}, s^t) &= 0
\end{aligned}
$$

We can set all the $\nu_t^i(s^t, s_{t+1})$ equal to 0, why?

Solving the sequential trading equilibrium

After playing with the FOC's, we get:

$$\tilde{Q}_t(s_{t+1}|s^t) = \beta \frac{u'(\tilde{c}^i_{t+1}(s^{t+1}))}{u'(\tilde{c}^i_t(s^t))} \pi(s^{t+1}|s^t)$$

- What is the intuition here?
- Does this pricing kernel look like something we've seen already?

Solving the sequential trading equilibrium

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- What is the intuition here?
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Remember from the AD equilibrium:

$$q^t_{t+1}(s^{t+1}) = \beta \frac{u'(c^i_{t+1}(s^{t+1}))}{u'(c^i_t(s^t))}\pi(s^{t+1}|s^t)$$

Equivalence of ADE and STE

Proposition 2 (Equivalence of ADE and STE)

Let $\{c_t^i(s^t)\}_{t=0}^{\infty}$ be an Arrow-Debreu Equilibrium allocation with associated prices $\{q_t^0(s^t)\}_{t=0}^{\infty}$. Then, the pricing kernel given by $q_{t+1}^0(s^{t+1}) = \tilde{Q}_t(s_{t+1}|s^t)q_t^0(s^t)$, the consumption $\tilde{c}_t^i(s^t) = c_t^i(s^t)$ and associated assets holdings form a Sequential Trading Equilibrium.

Proof: See LS chapter 8.

The converse is also true.

Intuitively, both market structure allow agents to move resources across all histories.

What have we learnt so far?

- The set of equilibria is the same under Arrow-Debreu and sequential trading.
- Competitive allocations are solutions to a social planner problem (they are Pareto efficient).
- We can decentralize any Pareto efficient with a set of lump sum transfers.
- Pricing kernel allows us to price any securities.