ECON 6170

Problem Set 7

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Worked with Fenglin Ye on Exercise 5, Exercise 6, and additional exercises Exercise 1.

Exercise 1. False. Consider the example of $f(x) = |x|$ on $X = [-10, 10]$. This function is not differentiable at $x_0 = 0$, since $\lim_{x \searrow x_0} \frac{f(x)-f(x_0)}{x-x_0} = 1$, and $\lim_{x \nearrow x_0} \frac{f(x)-f(x_0)}{x-x_0} = -1$. Since $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ DNE, f is not differentiable at x_0. It is, however, continuous at x_0.

Exercise 3. Prove the chain rule.

Proof. Suppose that $f : X \subseteq \mathbb{R} \to \mathbb{R}$ is differentiable at $x_0 \in \text{int}(X)$ and that $g : Y \to \mathbb{R}$ (where $f(X) \subseteq Y$) is differentiable at $f(x_0)$. This means that $f'(x_0)$ exists and $g'(f(x_0))$ exist. Consider the limit:

$$\lim_{x \to x_0} \frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \frac{f(x) - f(x_0)}{x - x_0}$$

and since f is differentiable and therefore continuous, $x \to x_0 \implies f(x) \to f(x_0)$, and

$$= \lim_{f(x) \to f(x_0)} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = g'(f(x_0))f'(x_0) = (g' \circ f)(x_0)f'(x_0)$$

Since both of the above derivatives exist, their limits are finite, so this limit exists and is finite. Thus, we have that $(g \circ f)'(x_0) = (g' \circ f)(x_0)f'(x_0)$ $\square$

Exercise 4. Prove: Suppose $f : (a, b) \subseteq \mathbb{R} \to \mathbb{R}$, and f is strictly increasing and differentiable on (a, b). Then

$$\left(f^{-1}\right)'(f(x)) = \frac{1}{f'(x)} \; \forall\, x \in (a, b)$$

Proof. Fix some $x_0 \in (a, b)$. Consider the limit:

$$\lim_{f(x) \to f(x_0)} \frac{f^{-1}(f(x)) - f^{-1}(f(x_0))}{f(x) - f(x_0)} = \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{f'(x_0)}$$

Which follows from the fact that $f^{-1}(f(x)) = x$ and the fact that f is differentiable (and therefore continuous) and strictly increasing implies that as $x \to x_0$, $f(x) \to f(x_0)$. $\square$

Exercise 5. Prove: Let $[a, b]$ be a compact interval in $\mathbb{R}$ and suppose $f : [a, b] \to \mathbb{R}$ is continuous and differentiable on (a, b). If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant.

Proof. By the Mean Value Theorem, we have that $f(b) - f(a) = f'(c)(b - a)$ for some $c \in (a, b)$. Since $f'(x) = 0 \; \forall\, x \in (a, b)$, $f(b) - f(a) = 0 \implies f(a) = f(b)$. Say that $f(a) = f(b) = y$ for some $y \in \mathbb{R}$. Then fix some $x_0 \in (a, b)$, and consider $f(x_0) - f(a)$. By the Mean Value Theorem, we have that $f(x_0) - f(a) = f'(c)(x_0 - a)$ for some $c \in (a, x_0)$. Since $f'(x) = 0 \; \forall\, x \in (a, x_0) \subseteq (a, b)$, $f(x_0) - f(a) = 0 \Rightarrow f(x_0) = f(a) = y$. Since this holds for arbitrary $x_0 \in (a, b)$, it must be the case that $f(x) = y \; \forall\, x \in [a, b]$, and f is constant. $\square$

Exercise 6. Prove: Suppose $f : (a, b) \subseteq \mathbb{R} \to \mathbb{R}$, $f \in C^k$, and that $f'(x_0) = f''(x_0) = \cdots = f^{(k-1)}(x_0) = 0$ and $f^{(k)}(x_0) \neq 0$. Then if k is even and $f^{(k)}(x_0) > 0$, f has a local minimum at x_0.

Proof. We have that $f^{(k)}(x_0) > 0$ at x_0, meaning that since $f^{(k)}$ is continuous, there exists $\varepsilon > 0$ such that $f^{(k)}(y) > 0 \ \forall \ y \in B_\varepsilon(x_0)$. Take some $y \in B_\varepsilon(x_0)$ such that $y > x_0$. Then by Taylor's Theorem, there exists x' between x_0 and y such that

$$f'(y) = P_{k-1}(x_0) + \frac{f^{(k)}(x')}{(k-1)!}(y - x_0)^{k-1}$$

Since $f''(x_0) = \cdots = f^{(k-1)}(x_0) = 0$, $P_{k-1}(x_0) = 0$, and since $(y - x_0) > 0$ and $x' \in B_\varepsilon(x_0)$, $\frac{f^{(k)}(x')}{(k-1)!}(y - x_0)^{k-1} = f'(y) > 0$, meaning that since $f'(x_0) = 0$, $f(x) > f(x_0)$ for all $x \in (x_0, y)$.

Next, take some $y' < x_0$. Since $y' - x_0 < 0$, we have that $f'(y') < 0$ from Taylor's Theorem. Then we have $f'(x) < 0$ for all $x \in (y', x_0)$, so thus $f(x) > f(x_0)$.

Since $f(x) > f(x_0)$ for all $x \in B_\varepsilon(x_0)$, x_0 is a local minimum. □

Exercise 1. Prove the following:
Theorem 1. ***Cauchy-Schwartz Inequality***. *For any* $x, y \in \mathbb{R}^d$,

$$|x \cdot y| \leq \|x\| \cdot \|y\|$$

Proof. Assume that $\| \cdot \|$ is the induced norm of the d-dimensional Euclidean space, the Euclidean norm, *i.e.*,

$$\|x\| = \sqrt{\sum_{i=1}^{d} x_i^2}$$

We have that

$$|x \cdot y| = \left| \sum_{i=1}^{d} x_i y_i \right| \leq \sqrt{\sum_{i=1}^{d} (x_i y_i)^2}$$

from the Triangle Inequality. Then this is equal to

$$\sqrt{\sum_{i=1}^{d} (x_i)^2 (y_i)^2} = \sqrt{\sum_{i=1}^{d} x_i^2 \sum_{i=1}^{d} y_i^2} = \sqrt{\sum_{i=1}^{d} x_i^2 \sum_{i=1}^{d} y_i^2} = \|x\| \cdot \|y\|$$

Thus, $|x \cdot y| \leq \|x\| \cdot \|y\|$. □