Midterm 1

ECON 6170

September 21, 2021

Instructions: You have the full class time to complete the following problems. You are to work alone. This test is not open book. In your answers, you are free to cite results that you can recall from class or previous homeworks *unless explicitly stated otherwise*. The exam is out of 20 points, and there is one extra credit question. The highest possible score is 22/20.

1. (5pts) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$.

   Let $h = f * g$, i.e., $h(x) = f(x) * g(x)$ for all $x \in \mathbb{R}$.

   If f and g are continuous, prove that h is continuous.

   Note: **Prove it directly using the definition of continuity**. You **may not** simply cite the proposition in the notes that states this result.

2. (5pts) Let $X \subset \mathbb{R}^k$ be compact and suppose that $f : X \to \mathbb{R}$ is continuous. Prove that $f(X)$ is a compact set in $\mathbb{R}$.
   Note: Recall that $f(X) = \{f(x) | x \in X\}$.

3. (5pts) Suppose that $f : \mathbb{R}^k \to \mathbb{R}$ is convex.

   Let $\alpha_1, \ldots, \alpha_n \in [0, 1]$ be such that $\sum_{i=1}^n \alpha_i = 1$.

   Let $x_1, \ldots, x_n \in \mathbb{R}^k$.

   Prove that $f(\sum_{i=1}^n \alpha_i x_i) \leq \sum_{i=1}^n \alpha_i f(x_i)$.

   Note: This is known as Jensen's Inequality.

   Friendly reminder: f is convex if $f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)$ for all $\alpha \in [0, 1]$ and $x, y \in \mathbb{R}^k$.

4. (5pts) Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are convex. Let $h \equiv \max\{f, g\}$, i.e., $h(x) = \max\{f(x), g(x)\}$ for all $x \in \mathbb{R}$. Prove that h is convex.

5. (Extra Credit: 2 pts) Prove that $f(x) = x^{0.5}$ is continuous on $[0, \infty)$.

   Hint 1: When showing continuity at $x_0 \in [0, \infty)$, treat the cases where $x_0 = 0$ and $x_0 > 0$ separately.

   Hint 2: Note that $x - x_0 = (x^{0.5} - x_0^{0.5})(x^{0.5} + x_0^{0.5})$.