ECON 6170 Section 2

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Section Exercise 1. Let $A \subseteq \mathbb{R}$ be nonempty. Then there exists a sequence of elements of A, (x_n) such that $x_n \to \sup A$.

Suppose A is bounded above. We know that, for all $\epsilon > 0$, there exists $x \in A$ such that $\sup A > x > \sup A - \epsilon$. Then choose $x_n \in A$ such that $\sup A > x_n > \sup A - 1/n$. This defines a sequence converging to $\sup A$.

Suppose A is unbounded above. Then $\sup A = \infty$ and for all $n \in \mathbb{N}$, we can find an $x_n \in A$ such that $x_n \geq n$. This defines a sequence diverging to $\infty = \sup A$. $\square$

Remark 1. Because Exercise 14 appears before Definition 8 (infinite limits) in the lecture notes, I assume that the limits in the exercise are real numbers. Analogous results do hold with infinite limits, so long as everything is well-defined (no $\infty - \infty$ or $\infty \cdot 0$ expressions).

Exercise 14 (i). Prove or disprove: If $x_n \to x$ and $y_n \to y$, then $(x_n + y_n)_n$ converges to $x + y$.

Solution: True. For any ϵ, for sufficiently large n, we have $|x_n - x| < \frac{\epsilon}{2}$ and $|y_n - y| < \frac{\epsilon}{2}$. This gives us $|x_n - x| + |y_n - y| < \epsilon$. Using the triangle inequality, we have $|(x_n + y_n) - (x + y)| = |x_n - x + y_n - y| \leq |x_n - x| + |y_n - y| < \epsilon$.

Exercise 14 (ii). Show that if $x_n \to x$ and $y_n \to y$, then $x_n y_n \to xy$.

Fix $\epsilon > 0$. Taking $N \in \mathbb{N}$ sufficiently large we know that $n \geq N$ implies $|x_n - x| < \epsilon$ and $|y_n - y| < \epsilon$. The sequence (y_n) converges, so it is bounded. We can thus say $0 \leq |y_n| < m$ for some $m > 0$. Then

$$|x_n y_n - xy| = |x_n y_n - x y_n + x y_n - xy| = |(x_n - x) y_n + x(y_n - y)| \leq |x_n - x| \cdot |y_n| + |x| \cdot |y_n - y|$$

$$< \epsilon m + |x| \epsilon = (m + |x|)\epsilon$$

which is just a positive constant times ϵ.

Exercise 14 (iv). Show that if $x_n \to x \neq 0$ with $x_n \neq 0$ for all n, then $\frac{1}{x_n} \to \frac{1}{x}$.

Fix $\epsilon > 0$ and choose $N \in \mathbb{N}$ sufficiently large that $n \geq N$ implies $|x_n - x| < \epsilon$. Note that $\epsilon > |x - x_n| \geq |x| - |x_n|$. Rearranging, we have $|x_n| > |x| - \epsilon$. Without loss of generality, take $\epsilon < \frac{1}{2}|x|$, so that $|x| - \epsilon > \frac{1}{2}|x|$. Taking reciprocals, $\frac{1}{|x_n|} < \frac{1}{|x| - \epsilon} < \frac{1}{\frac{1}{2}|x|}$.

Then
$$\left|\frac{1}{x_n} - \frac{1}{x}\right| = \left|\frac{x - x_n}{x_n x}\right| = \frac{|x - x_n|}{|x_n| \cdot |x|} < \frac{\epsilon}{\frac{1}{2}|x| \cdot |x|}$$

Section Exercise 2. Prove or disprove: (x_n) has a subsequence converging to $x \in \mathbb{R}$ iff for all $\epsilon > 0$ infinitely many terms of (x_n) lie in $(x - \epsilon, x + \epsilon)$.

By Exercise 16, if a subsequence $x_{n_k} \to x$, then for all $\epsilon > 0$ all but finitely many terms of x_{n_k} are contained in $(x - \epsilon, x + \epsilon)$. It follows that infinitely many terms of the original sequence (x_n) are contained in the same interval. Conversely, if for all positive ϵ, infinitely many terms of (x_n) lie in $(x - \epsilon, x + \epsilon)$, then we can define a subsequence converging to x as follows: let $x_{n_1} \in (x - 1, x + 1)$, and for all $k \geq 2$, let $x_{n_k} \in (x - \frac{1}{k}, x + \frac{1}{k})$ and $n_k > n_{k-1}$.

Section Exercise 3. Prove: A sequence x_n converges to $x \in \mathbb{R}$ if and only if every subsequence (x_{n_k}) contains a subsubsequence $(x_{n_{k_i}})$ that converges to x.

"Only if" is straightforward—every subsequence of a convergent sequence converges itself. "If" is more challenging. Let (x_n) be a sequence such that every subsequence, (x_{n_k}), contains its own subsubsequence, $(x_{n_{k_i}})$, converging to x. Choose an arbitrary $\epsilon > 0$. We want to show that, for sufficiently large N, $n \geq N$ implies $|x_n - x| < \epsilon$. Suppose not. Then for all N, there exists $k \geq N$ such that $|x_k - x| \geq \epsilon$. This defines a subsequence, (x_k), which clearly has no subsubsequence converging to x. This contradicts our hypothesis, so we must have $x_n \to x$.

Exercise 33. Consider the following non-theorem: *Let $x_n \to x \geq 0$ and (y_n) be any sequence. Then* $\limsup x_n y_n = x \limsup y_n$. Disprove this, then identify a tiny change to the assumptions that makes it true (but don't prove it).

Solution: A counterexample would be $x_n = 1/n$ and $y_n = n$. Another would be $x_n = 1/n$ and $y_n = -n$. Note that in these cases the right-hand-side would be undefined. Either the assumption that $x > 0$ or the assumption that (y_n) is bounded would make the statement true.

Section Exercise 4. Show that
$$\limsup_{n \to \infty} x_n = \sup\{x \mid x_{n_k} \to x \text{ for some subsequence of } (x_n), (x_{n_k})\}$$

First we consider the case $-\infty < \limsup_n x_n < \infty$.

Let $x^* = \limsup_n x_n$. Suppose some subsequence $x_{n_k} \to x^* + \epsilon$, for some $\epsilon > 0$. Then for all $K \in \mathbb{N}$, there exists $k \geq K$ with $x_{n_k} \geq x^* + \frac{\epsilon}{2}$. It follows that for all $N \in \mathbb{N}$, there exists $m \geq N$ with $x_m \geq x^* + \frac{\epsilon}{2}$. Thus, $\sup\{x_n, x_{n+1}, \dots\} \geq x^* + \frac{\epsilon}{2}$ for all n. This implies $\limsup_n x_n \geq x^* + \frac{\epsilon}{2} = \limsup_n x_n + \frac{\epsilon}{2}$, which is a contradiction. So x^* is an upper bound on the set of subsequential limits.

Alternatively, suppose that the limits of all convergent subsequences are weakly less than $x^* - \epsilon$, for some $\epsilon > 0$. By Section Exercise 3, there exists some positive $\delta < \epsilon$ such that at most finitely many x_n lie in $(x^* - \delta, x^*]$. Denote these terms by $x_{n_1}, \dots, x_{n_K}$. Then $n \geq n_K + 1$ implies $\sup\{x_n, x_{n+1}, \dots\} \leq x^* - \delta$. It follows that $\limsup_n x_n \leq x^* - \delta = \limsup_n x_n - \delta$, which is another contradiction. So x^* is the *least* upper bound on the set of subsequential limits.

Now we consider the case $\limsup_n x_n = \infty$. This implies that $\sup_n\{x_n, x_{n+1}, x_{n+2}, \dots\} = \infty$ for all $n \in \mathbb{N}$. That is, for all $M \in \mathbb{R}$ and all $n \in \mathbb{N}$, there exists $k \geq n$ such that $x_k \geq M$. In fact, for each M, there exists infinitely many such x_k. Applying this with $M = 1, 2, 3, \dots$, we obtain a subsequence (x_k) that diverges to ∞. Thus $\sup\{x \mid x_{n_k} \to x \text{ for some subsequence of } (x_n), (x_{n_k})\} = \infty$.

Finally, we consider the case $\limsup_n x_n = -\infty$. This implies $x_n \to -\infty$, and thus every subsequence $x_{n_k} \to -\infty$.