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I solemnly declare that the submitted work is my own only, and I have not collaborated with anyone else in any form, and I have not plagiarized the work of anyone else. By signing this statement, I explicitly declare that I am aware of the consequences if I was found to have compromised CORNELL CODE of ACADEMIC INTEGRITY and Cornell's Code of Campus Conduct.

Name:
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Econ 6190 Second Exam

Submit via Canvas by 6 pm, Nov 10

Instructions

This exam consists of three questions, not of equal length or difficulty. Answer all questions. Remember to always explain your answer. Good luck!

1. **[35 pts]** Let X be a random variable following a normal distribution with mean μ and variance $\sigma^2 > 0$. We draw a random sample $\{X_1, X_2, \dots X_n\}$ from X and construct a sample mean statistic $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

   (a) **[5 pts]** Fix $\delta > 0$. Find an upper bound of $P\{\left|\bar{X} - \mu\right| > \delta\}$ by using Markov inequality with $r = 2$.

   (b) **[5 pts]** Repeat the exercise (a) but using Markov inequality with $r = 4$.

   (c) **[5 pts]** Compare the two bounds in (a) and (b) above when $\delta = \sigma$ and when n is at least 2. Which one of them gives you a tighter bound of $P\{\left|\bar{X} - \mu\right| > \delta\}$?

   (d) **[5 pts]** Since we know X is normal, find the exact value of $P\{\left|\bar{X} - \mu\right| > \delta\}$.

   (e) **[10 pts]** From (d), we see that the tail probability of a normal sample mean is much thinner than what Markov inequality predicts. In fact, we can show that if $Z \sim N(\mu, \sigma^2)$, then

   $$P\{|Z - \mu| > \delta\} \leq 2\exp\left(-\frac{\delta^2}{2\sigma^2}\right). \tag{1}$$

   Given (1), find a constant c such that

   $$P\{\left|\bar{X} - \mu\right| \leq c\} > 0.95.$$

   That is, we can predict that with a probability of at least 0.95, sample average is within $c-$distance of its true mean. What is the prediction of c if you only use Chebyshev's inequality?

   (f) **[5 pts]** Given your answer to (e), how much more data do we have to collect if we want the prediction of c based on Chebyshev's inequality to be the same as that based on (1)?

2. **[40 pts]** Delta method.

(a) **[10 pts]** Assume

$$\sqrt{n}\begin{pmatrix} \hat{\theta}_1 - \theta_1 \\ \hat{\theta}_2 - \theta_2 \end{pmatrix} \xrightarrow{d} N(0, \Sigma),$$

where Σ is the 2×2 variance-covariance matrix. Use the delta method to find the asymptotic distribution of $\hat{\theta}_1^3 + \hat{\theta}_1\hat{\theta}_2^2$. Clearly demonstrate your derivations.

(b) **[30 pts]** Suppose I use $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2$, where $\bar{X}$ is the sample mean, to estimate $\sigma^2 = \text{var}(X)$. By imposing suitable assumptions and by using delta method, find the asymptotic distribution of $\hat{\sigma}^2$ (i.e., show $\sqrt{n}(\hat{\sigma}^2 - \sigma^2)$ converges in distribution to some normal distribution). Clearly demonstrate your derivations.

3. **[25 pts]** Consider a sample of data $\{X_1, \dots X_n\}$, where

$$X_i = \mu + \sigma_i e_i, i = 1 \dots n,$$

where $\{e_i\}_{i=1}^n$ are iid and $\mathbb{E}[e_i] = 0$, $\text{var}(e_i) = 1$, $\{\sigma_i\}_{i=1}^n$ are n finite and positive constants, and $\mu \in \mathbb{R}$ is the parameter of interest.

(a) **[5 pts]** Let

$$\hat{\mu}_1 = \frac{1}{n}\sum_{i=1}^n X_i$$

be the sample mean estimator. Under what condition is $\hat{\mu}_1$ a consistent estimator of μ? Under what condition is $\hat{\mu}_1 - \mu = O_p(\frac{1}{\sqrt{n}})$?

(b) **[10 pts]** Let

$$\hat{\mu}_2 = \frac{\frac{1}{n}\sum_{i=1}^n \frac{X_i}{\sigma_i^2}}{\frac{1}{n}\sum_{i=1}^n \frac{1}{\sigma_i^2}}$$

be an alternative estimator of μ. Under what condition is $\hat{\mu}_2$ a consistent estimator of μ? Under what condition is $\hat{\mu}_2 - \mu = O_p(\frac{1}{\sqrt{n}})$?

(c) **[10 pts]** Compare the MSE of $\hat{\mu}_1$ and $\hat{\mu}_2$. Which one is more efficient and why?