ECON 6130: Macroeconomics 1

Problem Set 2

1 Problem I

1. In each period, the aggregate endowment in the economy is the total of each individual's endowment. That is, $e_t^1 + e_t^2 + e_t^3 = 3$. This is because at a given time period t, only one of the individuals receives an endowment of 3.

2. In the Arrow-Debreu equilibrium, markets open at t=0. Agents trade claims to consumption at all time periods t. There is no trade at time $t > 0$.

   A competitive ADE is a set of prices $\{\hat{p}_t\}_{t=0}^{\infty}$ and allocations $\{\hat{c}_t\}_{t=0}^{\infty}$ such that -

   - Given prices, the allocation is the solution of the following maximisation problem -

   $$\begin{aligned} \max_{\{c_t^i\}} \quad & \sum_t \beta^t log(c_t^i) \\ \text{s.t.} \quad & \sum_t \hat{p}_t c_t^i \leq \sum_t \hat{p}_t e_t^i \\ & c_t^i \geq 0 \forall i \ \ t \end{aligned} \tag{1}$$

   - The market clears at all time periods -

   $$\sum_i \hat{c}_t^i = \sum_i e_t^i = 3 \quad \forall t \tag{2}$$

3. In this case, market opens at every period t. Each agent trade claims to consumption one-period-ahead. That is, in period t, each agent trades claims for period $t+1$. Markets close at period t and then open again at period $t+1$ and so on. The natural debt limit for each individual is as follows -

   $$A_t^i = \sum_{\tau=t}^{\infty} p_\tau^t e_\tau^i \tag{3}$$

   In the above equation, τ is every period after t. The debt limit signifies the maximum amount the agent can borrow at period t, if they do not consume anything after period τ, at prices p_τ.

   The price of a claim in period t to one unit of consumption in time $t+1$ is given by $\tilde{Q}_t$. An agent i's consumption claim in period is given by $\tilde{a}_t^i$.

   A sequential trading equilibrium is a distribution of assets $\{\tilde{a}_{t+1}^i\} \quad \forall \quad i, t$, and allocation of $\{\tilde{c}_t^i\} \ \forall \ i, t$, and pricing kernels $\tilde{Q}_t$, such that

– Each consumer maximises utility in period t, -

$$\begin{aligned} \max_{\{a^i_{t+1}\},\{c^i_t\}} \quad & \sum_t \beta^t log(c^i_t) \\ \text{s.t.} \quad & c^i_t + a^i_{t+1}\tilde{Q}_t \leq e^i_t + a^i_t \forall i \ \ t \\ & -a^i_{t+1} \leq A^i_{t+1} \\ & c^i_t \geq 0 \forall i \ \ t \end{aligned} \tag{4}$$

– Markets clear for all t

$$\sum_i \tilde{c}^i_t = \sum e^i_t \tag{5}$$

$$\sum_i \tilde{a}^i_{t+1} = 0 \tag{6}$$

4. If $\{\hat{c}^i_t\}$ is the ADE allocation under prices $\{\hat{p}_t\}^{\infty}_{t=0}$, then there exists a pricing kernel $\tilde{Q}_t$ such that $\hat{p}_{t+1} = \hat{p}_t\tilde{Q}_t$ such that $\tilde{c}^i_t = \hat{c}^i_t \ \ \forall i, t$ and the associated assets hold.

5. The lagrangian for the utility maximisation problem given that markets clear at all time periods is given as follows -

$$L(c^i_t, \lambda) = \sum_t \beta^t log(c^i_t) - \lambda[\sum_t \hat{p}_t c^i_t - \sum_t \hat{p}_t e^i_t] \tag{7}$$

The budget constraint is binding given a concave utility function and $c^i_t > 0$ since the INADA conditions hold. Taking the derivative with respect to c^i_t gives us for time period t

$$\frac{\beta^t}{c^i_t} = \lambda \hat{p}_t \implies \beta^t = \lambda \hat{p}_t c^i_t \tag{8}$$

and for time period $t+1$,

$$\beta^{t+1} = \lambda \hat{p_{t+1}} c^i_{t+1} \tag{9}$$

Dividing (9) by (8), we get

$$\beta = \frac{\hat{p_{t+1}} c^i_{t+1}}{\hat{p}_t c^i_t} \implies \beta \hat{p}_t c^i_t = \hat{p}_{t+1} c^i_{t+1} \tag{10}$$

Using the market clearing equality ($\sum_i c^i_t = \sum e^i_t$ for every time period), we get

$$\beta \hat{p}_t \sum_i c^i_t = \hat{p}_{t+1} \sum_i c^i_{t+1} \implies \beta \hat{p}_t \sum_i e^i_t = \hat{p}_{t+1} \sum_i e^i_{t+1} \tag{11}$$

$$\implies \beta \hat{p}_t = \hat{p}_{t+1} \tag{12}$$

Since we are given that $\hat{p}_0 = 1$, $\hat{p}_1 = \beta$, $\hat{p}_2 = \beta^2$ and so on, or $\hat{p}_t = \beta^t$. Now, revisiting the budge constraint, we have -

$$\sum_t \beta^t c_t^i = \sum_t \beta^t e_t^i \tag{13}$$

Note that c_t^i is a constant for each i from (9). Therefore, for agent 1 we have,

$$c^1 \sum_t \beta^t = \beta^0.3 + \beta^1.0 + \beta^2.0 + \beta^3.3... \tag{14}$$

$$\implies \frac{c^1}{1-\beta} = \frac{3}{1-\beta^3} \implies \hat{c}^1 = \frac{3(1-\beta)}{1-\beta^3} \tag{15}$$

could simplify further

Following a similar logic, we have for person 2

$$\frac{c^2}{1-\beta} = \frac{3\beta}{1-\beta^3} \implies \hat{c}^2 = \frac{3\beta(1-\beta)}{1-\beta^3} \tag{16}$$

and for person 3,

$$\frac{c^3}{1-\beta} = \frac{3\beta^2}{1-\beta^3} \implies \hat{c}^3 = \frac{3\beta^2(1-\beta)}{1-\beta^3} \tag{17}$$

6. All agents consumer a positive amount in each time period which makes them better off compared to scenario of no trade since utility would to tend to negative infinity in time periods with no endowment. Note that person 1 is better off then person 2 and so one because person 1 gets endowed in period 1 and therefore the present value of the future endowments is the highest for person 1.

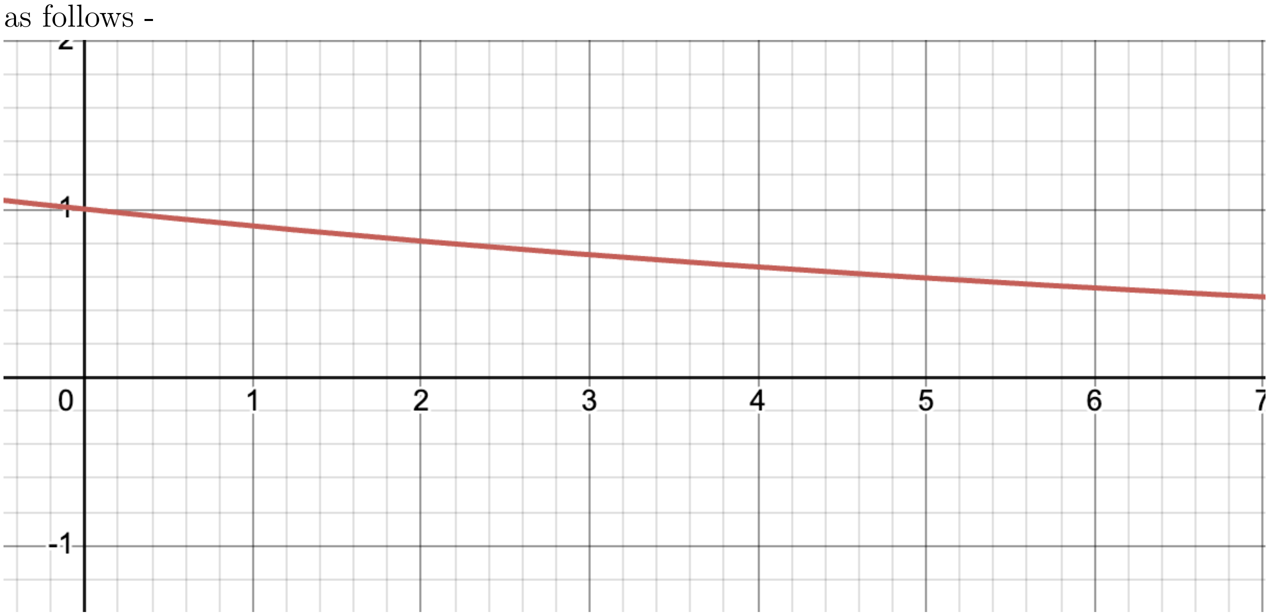
7. From (10) in the previous answer we have,

$$\beta \hat{p}_t c_t^i = \hat{p}_{t+1} c_{t+1}^i \tag{18}$$

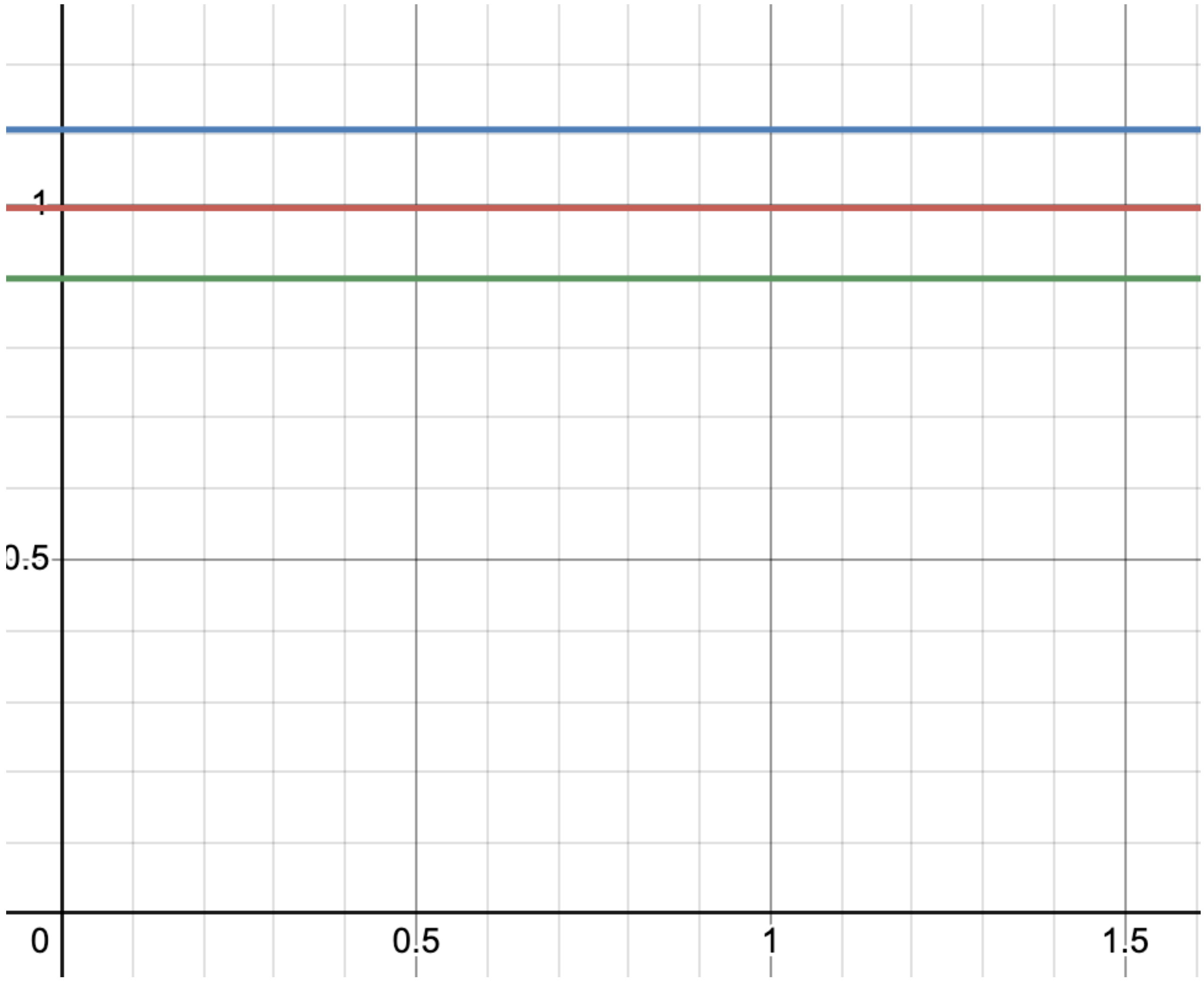
Since $\hat{p}_t = \beta^t$,

$$\beta^{t+1} c_t^i = \beta^{t+1} c_{t+1}^i \implies c_t^i = c_{t+1}^i \tag{19}$$

Note that consumption is constant is over time and given by equations (15), (16), (17) before. For a value of $\beta = 0.9$, the graph of prices (with price on the y-axis and time on the x-axis) is given as follows -



The consumption graphs with time on the x-axis is as follows -



The blue line is the consumption path for agent 1, with red for agent 2 and green for agent 3. As explained previously, the constant consumption paths are different for each of the agents because agent 1 was the first to be endowed.

8. Suppose the price of an asset at period 0 is p_0^0. The price of one unit of consumption delivered in period t is $\hat{p}_t^0$. So,

$$p_0^0 = \sum_{t=0}^{\infty} \hat{p}_t^0 dt = \sum_{t=0}^{\infty} \beta^t 0.05 = \frac{0.05}{1-\beta} \tag{20}$$

9. The optimisation problem of the social planner who maximises total welfare is given as follows -

$$\begin{aligned} \max_{\{c_t^i\}} \quad & \sum_t [\lambda_1 \beta^t log(c_t^1) + \lambda_2 \beta^t log(c_t^2) + (1-\lambda_1-\lambda_2)\beta^t log(c_t^3)] \\ \text{s.t.} \quad & \sum_i c_t^i \leq \sum_i e_t^i \forall \quad t \\ & c_t^i \geq 0 \forall i \quad t \end{aligned} \tag{21}$$

where λ_1, λ_2, λ_3 are the pareto weights.

10. The lagrangian for the social planner's optimisation is given as -

$$L = \sum_t [\lambda_1 \beta^t log(c_t^1) + \lambda_2 \beta^t log(c_t^2) + (1 - \lambda_1 - \lambda_2)\beta^t log(c_t^3)] - \mu_t [\sum_i e_t^i - \sum_i c_t^i] \tag{22}$$

Taking the derivative, we get -

$$c_t^1 : \frac{\lambda_1 \beta^t}{c_t^1} = \mu_t \implies c_t^1 = \frac{\lambda_1 \beta^t}{\mu_t} \tag{23}$$

$$c_t^2 : \frac{\lambda_2 \beta^t}{c_t^1} = \mu_t = \frac{\lambda_2 \beta^t}{\mu_t} \tag{24}$$

$$c_t^1 : \frac{(1 - \lambda_1 - \lambda_2)\beta^t}{c_t^1} = \mu_t = \frac{(1 - \lambda_1 - \lambda_2)\beta^t}{\mu_t} \tag{25}$$

Given the market clearing condition, in a given time period we have,

$$\frac{\beta^t}{\mu_t} = 3 \implies \mu_t = \frac{\beta^t}{3} \tag{26}$$

Substituting this in (23)-(25), we get,

$$c_t^1 = 3\lambda_1 \tag{27}$$

$$c_t^2 = 3\lambda_2 \tag{28}$$

$$c_t^1 = 3(1 - \lambda_1 - \lambda_2) \tag{29}$$

Given the ADE prices, we can substitute the pareto efficient bundles in each consumer's budge constraint to get -

$$\sum_t \beta^t c_t^1 = \sum_t \beta^t e_t^1 \text{ since } p_t = \beta^t \tag{30}$$

$$3\lambda_1 \sum_t \beta^t = \frac{3}{1 - \beta^3} \tag{31}$$

$$\implies \lambda_1 = \frac{(1 - \beta)}{1 - \beta^3} \tag{32}$$

Doing the same for the rest, we get

$$\lambda_2 = \frac{\beta(1 - \beta)}{1 - \beta^3} \tag{33}$$

$$\lambda_3 = \frac{\beta^2(1 - \beta)}{1 - \beta^3} \tag{34}$$

For the CE allocations to be the same as allocations for the social planner, we need the above restrictions on the pareto weights, $p_t = \beta^t$ and that each consumer's budget constraint is satisfied.

11. The consumption sequence will not remain constant since the endowments change according to the time period. In this case, the total endowment is 3 for periods $t = 0, 3, 6..$ and 4 for the rest of the periods. Therefore, for time periods $t = 0, 3, 6..$, using the market clearing conditions we get $\frac{\beta^t}{\mu_t} = 3$ and $\frac{\beta^t}{\mu_t} = 4$ for the other time periods. Therefore, we have $c_t^i = 3\lambda_i \ \ \forall\ i$ in periods $t = 0, 3, 6..$ and $c_t^i = 4\lambda_i \ \ \forall\ i$ in the other periods.

2 Problem 2

1. For $v_\theta(c)$ to be concave, we need that
$$\lambda v_\theta(x) + (1-\lambda)v_\theta(y) \leq v_\theta(\lambda x + (1-\lambda y)) \tag{35}$$
for two pareto optimal solutions (x_1, x_2) and (y_1, y_2). We have,
$$\begin{aligned}
v_\theta(x) &= \theta u(x_1) + (1-\theta)w(x_2) && (36)\\
v_\theta(y) &= \theta u(y_1) + (1-\theta)w(y_2) && (37)\\
\lambda v_\theta(x) + (1-\lambda)v_\theta(y) &= \lambda\theta u(x_1) + \lambda(1-\theta)w(x_2) + (1-\lambda)\theta w(y_1) + (1-\lambda)(1-\theta)w(y_2) && (38)\\
&= \theta[\lambda u(x_1) + (1-\lambda)u(y_1)] + (1-\theta)[\lambda w(x_2) + (1-\lambda)w(y_2)] && (39)\\
&\leq \theta u(\lambda(x_1) + (1-\lambda)(y_1)) + (1-\theta)w(\lambda x_2 + (1-\lambda)(y_2)) \text{ since u,w are concave} && (40)\\
&\implies \lambda v_\theta(x) + (1-\lambda)v_\theta(y) \leq v_\theta[\lambda x + (1-\lambda y] && (41)
\end{aligned}$$

2. The Lagrangian of the optimisation problem is as follows -
$$L = \theta u(c^1) + (1-\theta)w(c^2) - \lambda[c^1 + c^2 - c] \tag{42}$$
The FOCs are as follows -
$$\theta u'(c^1) = \lambda = (1-\theta)w'(c^2) \tag{43}$$
Given that u, c are concave, the budget constraint must be binding that is $c^1 + c^2 = c$ that is $\frac{\partial L}{\partial c} = \lambda$.
$$\implies \frac{\partial L}{\partial c} = v'_\theta(c) = \theta u'(c^1) = (1-\theta)w'(c^2) \tag{44}$$

3 Problem 3

1. For i =1, $u(\tilde{c^1}) > u(\hat{c^1})$. Since $u(.)$ is increasing, it must be that $\tilde{c^1} > \hat{c^1} \implies \hat{p}_t\tilde{c^1} > \hat{p}_t\hat{c^1}$. Which in turn implies that, $\sum_{t=0}^\infty \hat{p}_t\tilde{c^1} > \sum_{t=0}^\infty \hat{p}_t\hat{c^1}$ since all prices are strictly positive.

2. Following the same logic as above, since $u(.)$ is increasing, for ay i, $u(\tilde{c^i}) > u(\hat{c^i}) \implies \tilde{c^i} > \hat{c^i}$. And we have, $\hat{p}_t\tilde{c^i} > \hat{p}_t\hat{c^i} \implies \sum_{t=0}^\infty \hat{p}_t\tilde{c^i} > \sum_{t=0}^\infty \hat{p}_t\hat{c^i}$.

3. Let e_t^i be each consumer i's endowment and since the Budget constraint binds,
$$\begin{aligned}
\sum_t\sum_i \hat{p}_t e_t^i &= \sum_t\sum_i \hat{p}_t\hat{c^i} && (45)\\
&= \sum_t \hat{p}_t\hat{c^1} + \sum_t\sum_{i=2} \hat{p}_t\hat{c^i} && (46)\\
\sum_t \widetilde{\hat{p}_t c}^1 + \sum_t\sum_{i=2} \widetilde{\hat{p}_t c}^i &= \sum_t \widetilde{\hat{p}_t c}^i && (47)\\
\implies \sum_t\sum_i \hat{p}_t e_t^i &< \sum_t \widetilde{\hat{p}_t c}^i && (48)
\end{aligned}$$
(48) violates the feasibility condition and therefore $\{\hat{c}_t^i\}_{t=0}^\infty$ is pareto optimal.