Final

ECON 6170

December 9, 2022

Instructions: You have the full final exam time to complete the following problems. You are to work alone. This test is not open book. In your answers, you are free to cite results that you can recall from class or previous homeworks *unless explicitly stated otherwise*. The exam is out of 20 points, and there are extra credit questions. The highest possible score is 24/20.

1. (5pts) Let $f, g : [0, 1] \to \mathbb{R}$. Let $h = \max\{f, g\}$. Let $k = f + g$.

   (a) Define quasiconvexity and quasiconcavity.

   (b) Prove or disprove: If f and g are quasiconcave, h is quasiconcave

   (c) Prove or disprove: If f and g are quasiconvex, h is quasiconvex

   (d) Prove or disprove: If f and g are quasiconvex, k is quasiconvex

2. (5pts)

   (a) State the intermediate value theorem.

   (b) State the mean value theorem.

   (c) State Taylor's theorem.

   (d) Let $f : (0,1) \to \mathbb{R}$ be differentiable with $f'(x) > 0$ for all $x \in (0,1)$. Prove that f is strictly increasing.

3. (5pts) Implicit function theorem.

(a) Let $F(x_1, x_2, y) = x_1 + x_2 + y - e^{x_1 x_2 y}$ and $(x_1^0, x_2^0, y^0) = (0, 0.5, 0.5)$. Show that the set of (x_1, x_2, y) that solve $F(x_1, x_2, y) = 0$ near (x_1^0, x_2^0, y^0) is the graph of some function $y = h(x_1, x_2)$. Compute Dh.
Note: If $g(x) = e^x$, $\frac{d}{dx}g(x) = g(x)$.

(b) Let $F : \mathbb{R}^4 \to \mathbb{R}^2$ be defined as

$$F(x_1, x_2, y_1, y_2) = (x_1^2 - x_2^2 - y_1^3 + y_2^2 + 4,\ \ 2x_1 x_2 + x_2^2 - 2y_1^2 + 3y_2^4 + 8).$$

Let $(x_1^0, x_2^0, y_1^0, y_2^0) = (2, -1, 2, 1)$. Show that the set of (x_1, x_2, y_1, y_2) that solve $F(x_1, x_2, y_1, y_2) = 0$ near $(x_1^0, x_2^0, y_1^0, y_2^0)$ is the graph of some function $(y_1, y_2) = h(x_1, x_2)$. Compute Dh.

4. (5pts) Suppose that $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is continuously differentiable. Then f has increasing differences in $(x, t) \in \mathbb{R}^n \times \mathbb{R}^m$ if and only if $\frac{\partial f}{\partial x_i \partial t_j} \geq 0$ for all $i \in \{1, \ldots n\}, j \in \{1, \ldots m\}$.

   Prove only one direction—that f having increasing differences implies $\frac{\partial f}{\partial x_i \partial t_j} \geq 0$ for all $i \in \{1, \ldots n\}, j \in \{1, \ldots m\}$.

   Note: The other direction is also simple to prove and involves using the fundamental theorem of calculus. But you need not prove it for this problem.

5. (Extra Credit: 4 pts)

   (a) Just as in the last question, state a characterization for the *supermodularity* of f in $x \in \mathbb{R}^n$ in terms of conditions on the partials of f.

   Note: State this for the case of general m and n. Be careful about which partials you place conditions on.

   (b) State the KKT theorem, precisely defining the constraint qualification, first-order conditions and complementary-slackness conditions.

   (c) A function $f : E \subset \mathbb{R} \to \mathbb{R}$ is *uniformly continuous* if for all $\epsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in E$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. Give an example of a function which is continuous but not uniformly continuous.

   (d) Prove your example in (c) works.