ECON 6130 PROBLEM SET 5

Problem 1. Consider a neoclassical growth model with two sectors, one producing consumption goods and one producing investment goods. Consumption is given by $C_t = F(K_{C,t}, L_{C,t})$ and investment is given by $I_t = G(K_{I,t}, L_{I,t})$, where $K_{j,t}$ is the amount of capital in sector j at the beginning of period t and $L_{j,t}$ is the amount of labor used in sector j in period t. The total amount of labor in each period is equal to L (leisure is not valued by the household). Labor can be freely allocated in each period between the two sectors: $L = L_{C,t} + L_{I,t}$. Capital, by contrast, is sector-specific. Investment goods, however, can be used to augment the capital stock in either sector. In particular, the capital stocks in the two sectors evolve according to

$$K_{j,t+1} = (1-\delta) K_{j,t} + I_{j,t},$$

where $I_t = I_{C,t} + I_{I,t}$.

The social planner seeks to maximize $\sum_{t=0}^{\infty} \beta^t u(C_t)$, given $K_{C,0}$ and $K_{I,0}$, subject to the constraints on technology. Note that although leisure is not valued in the utility function, the planner must nonetheless decide in each period how to allocate L across the two sectors.

1. Formulate the planner's optimization problem as a dynamic programming problem. What are the state variables? What are the choice variables? (Hint: You should have two of each.)

The planner's problem is given by

$$\begin{aligned}
V(K_C, K_I) = \max_{I_C, L_C} \; & u(C) + \beta V\left(K_C', K_I'\right) \\
& \text{subject to} \\
& C = F(K_C, L_C), \\
& K_C' = (1-\delta) K_C + I_C, \\
& K_I' = (1-\delta) K_I + G(K_I, L - L_C) - I_C, \\
& L_C \in [0, L], I_C \in [0, G(K_I, L - L_C)].
\end{aligned}$$

The state variables are the capital levels K_C and K_I, and after appropriate substitution, the control variables are consumption investment I_C and consumption labor L_C. The first constraint defines the current consumption level C, the second constraint defines the law of motion for consumption capital K_C', and the third constraint defines the law of motion for investment capital K_I'. The final constraint places bounds on consumption labor L_C and consumption investment I_C.

2. Find a set of first order conditions and envelope conditions (using Benveniste-Scheinkman) that an optimal solution of the planning problem must satisfy.

Date: October 10, 2018.

Assuming that the bounds on L_C and I_C are non-binding, the first order conditions are

$$V_1\left(K_C', K_I'\right) = V_2\left(K_C', K_I'\right),$$
$$u'(C) F_2(K_C, L_C) = \beta V_2\left(K_C', K_I'\right) G_2(K_I, L_I).$$

The envelope conditions are

$$V_1(K_C, K_I) = u'(C) F_1(K_C, L_C) + \beta(1-\delta) V_1\left(K_C', K_I'\right),$$
$$V_2(K_C, K_I) = \beta V_2\left(K_C', K_I'\right)\left[(1-\delta) + G_1(K_I, L_I)\right].$$

3. Use your answer to part 2 to find a set of equations that determine the steady-state values of capital and labor (in each sector) in this economy.

In the steady state, we must have $K_C = K_C'$ and $K_I = K_I'$. With these conditions, the first order and envelope conditions are

$$\begin{aligned}
V_1(K_C, K_I) &= V_2(K_C, K_I), \\
u'(C) F_2(K_C, L_C) &= \beta V_2(K_C, K_I) G_2(K_I, L_I), \\
V_1(K_C, K_I) &= u'(C) F_1(K_C, L_C) + \beta(1-\delta) V_1(K_C, K_I), \\
V_2(K_C, K_I) &= \beta V_2(K_C, K_I)\left[(1-\delta) + G_1(K_I, L_I)\right].
\end{aligned}$$

Simplifying, we find

$$F_2(K_C, L_C)\left[1 - \beta(1-\delta)\right] = \beta F_1(K_C, L_C) G_2(K_I, L_I), \tag{1}$$
$$1 = \beta\left[(1-\delta) + G_1(K_I, L_I)\right]. \tag{2}$$

We also have the dependencies

$$L_C + L_I = L, \tag{3}$$
$$\delta(K_C + K_I) = G(K_I, L_I). \tag{4}$$

Equations (1)-(4) suffice to determine the steady-state values of capital and labor.

4. Suppose that $F(K_{C,t}, L_{C,t}) = K_{C,t}^{\alpha} L_{C,t}^{1-\alpha}$ and $G(K_{I,t}, L_{I,t}) = K_{I,t}^{\gamma} L_{I,t}^{1-\gamma}$. Express the steady-state as a function of the structural parameters of the model.

The steady-state equations (1)-(4) are

$$\begin{aligned}
(1-\alpha)\left(\frac{K_C}{L_C}\right)^{\alpha}\left[1-\beta(1-\delta)\right] &= \beta\alpha\left(\frac{K_C}{L_C}\right)^{\alpha-1}(1-\gamma)\left(\frac{K_I}{L_I}\right)^{\gamma}, \\
1 &= \beta\left[1 - \delta + \gamma\left(\frac{K_I}{L_I}\right)^{\gamma-1}\right], \\
L_C + L_I &= L, \\
\delta(K_C + K_I) &= K_I^{\gamma} L_I^{1-\gamma}.
\end{aligned}$$

The second equation implies

$$\frac{K_I}{L_I} = \left(\frac{1-\beta(1-\delta)}{\beta\gamma}\right)^{\frac{1}{\gamma-1}}.$$

We can then rearrange the first equation to find

$$\frac{K_C}{L_C} = \frac{\beta\alpha\left(1-\gamma\right)\left(\frac{1-\beta(1-\delta)}{\beta\gamma}\right)^{\frac{\gamma}{\gamma-1}}}{\left(1-\alpha\right)\left[1-\beta\left(1-\delta\right)\right]}.$$

Hence we can express the steady-state as a function of the structural parameters with the equations

$$\begin{aligned}
\frac{K_C}{L_C} &= \frac{\beta\alpha\left(1-\gamma\right)\left(\frac{1-\beta(1-\delta)}{\beta\gamma}\right)^{\frac{\gamma}{\gamma-1}}}{\left(1-\alpha\right)\left[1-\beta\left(1-\delta\right)\right]} \\
\frac{K_I}{L_I} &= \left(\frac{1-\beta\left(1-\delta\right)}{\beta\gamma}\right)^{\frac{1}{\gamma-1}}, \\
L_C + L_I &= L, \\
\delta\left(K_C + K_I\right) &= K_I^{\gamma} L_I^{1-\gamma}.
\end{aligned}$$

Problem 2. Consider the following problem:

$$\begin{aligned}
&\max_{\{c_t\}_{t=0}^{\infty}} \mathbb{E}\left[\sum_{t=0}^{\infty}\beta^t u\left(c_t\right)\right] \\
&\text{subject to} \\
&c_t + k_{t+1} = e^{y_t}k_t^{\alpha} + \left(1-\delta\right)k_t \quad t \geq 0,
\end{aligned}$$

where y_t is a random process.

1. Write this problem recursively (Bellman equation) assuming that y_t is a Markov chain. State conditions to guarantee the value function is continuous, monotone, and concave. What are the state variables? What are the control variables?

The recursive problem is

$$\begin{aligned}
V\left(k,y\right) = \max_{k'} & \, u\left(c\right) + \beta\mathbb{E}\left[\left.V\left(k',y'\right)\right| y\right] \\
&\text{subject to} \\
&c + k' = e^{y}k^{\alpha} + \left(1-\delta\right)k, \\
&k' \in \left[0, e^{y}k^{\alpha} + \left(1-\delta\right)k\right].
\end{aligned}$$

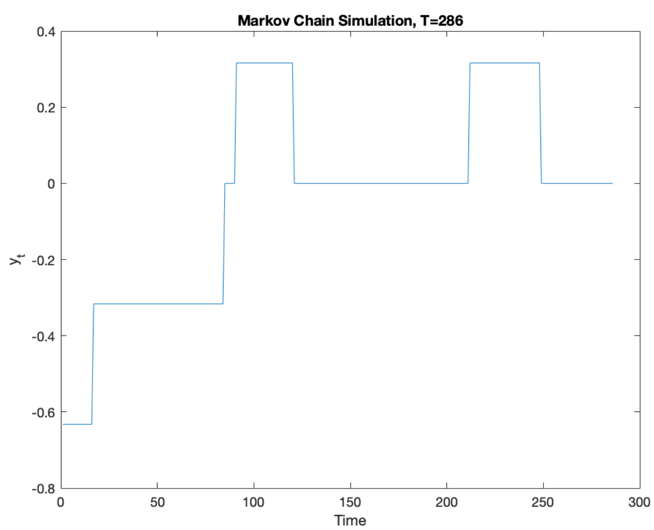
The state variables are the current capital level k and the current productivity shock y. The control variable is next-period capital k'. The first constraint is a budget constraint that defines current consumption c, and the second constraint is a non-negativity constraint on current consumption c and next-period capital k'. By Theorem 9.6 in Stokey, Lucas, and Prescott (1989), V is continuous if the Markov chain $\{y_t\}_{t=0}^{\infty}$ has an at most countable state space, u is bounded and continuous, $\beta \in [0,1)$, and $\delta \in [0,1]$. If u is additionally strictly increasing, then Theorem 9.7 implies that $V\left(\cdot,y\right)$ is also strictly increasing for each y. Finally, if u is strictly concave and $\alpha \in [0,1]$, then Theorem 9.8 implies that $V\left(\cdot,y\right)$ is strictly concave for each y. Note that any hypotheses in these Theorems not stated here are satisfied immediately because of the specification of the problem.

2. Assume the following process for income:

$$y_{t+1} = 0.98y_t + \epsilon_t,$$

where ϵ_t is an IID normal shock with mean 0 and variance such that the long-run variance of y_t is 0.1. Construct a 7-point Markov chain approximation to this process. To do so, you can use any commonly used technique. I recommend Tauchen (1986). Space approximation nodes between -3 and 3 standard deviations from the long-run mean. Simulate your Markov chain. Compute its long-run mean, its serial correlation, and its volatility.

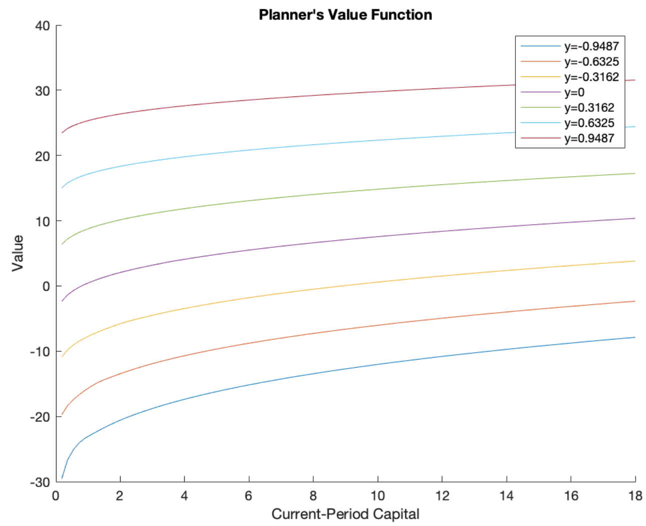
Since $0.98 < 1$ and each shock ϵ_t is IID with mean 0, the long-run mean of y_t is 0, and we are given that the long-run variance of y_t is 0.1. These facts were used to approximate the process using the method found in Tauchen (1986). The initial income y_0 is drawn at random from the stationary distribution of the Markov chain, so the symmetry of the stationary distribution about 0 and Birkhoff's Ergodic Theorem imply that the long-run mean of the Markov chain is 0. Given that the chain is stationary, the serial correlation is time-independent, and it was calculated to be 0.9962. The chain's volatility is given by the standard deviation of ϵ_t, which is necessarily .0629 to ensure that the long-run variance of y_t is 0.1. The following graph depicts one simulation of the chain over 286 periods:

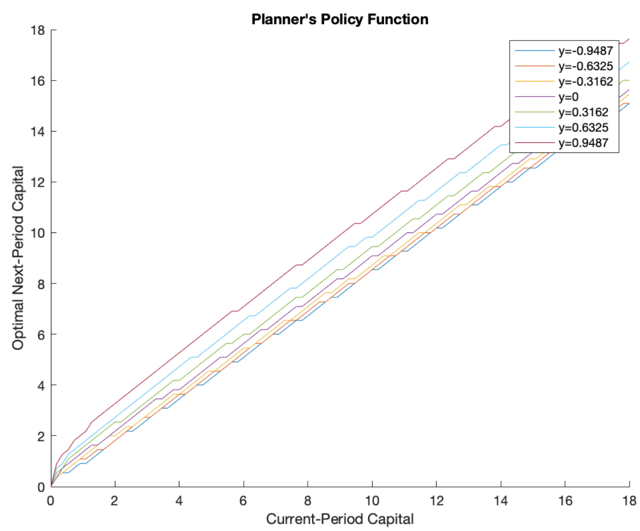


3. Suppose preferences are log. Assume that $\beta = 0.95$, $\delta = 0.1$, and $\alpha = 0.35$. Compute the value function for this problem assuming a discrete grid of 100 equally spaced points for k. You need to pick appropriate bounds for this grid. Plot the value function as a function of k for all the values of the random process. Do the same for the policy function.

Value function:

Policy function:

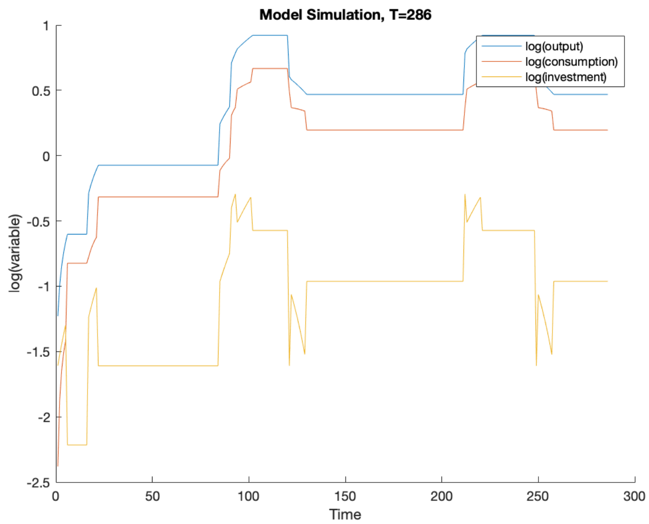


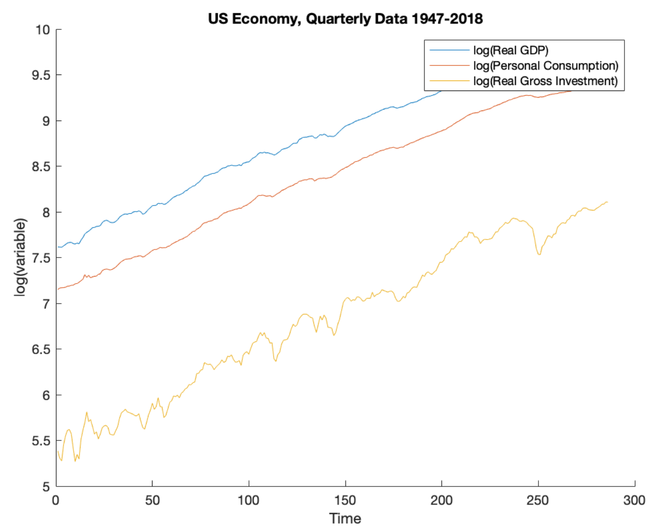


4. Simulate the model for a large number of periods and compute the standard deviations of (the log of) output, consumption, and investment, and the correlations between output, consumption, and investment. How does that compare to the data? (Hint: You should detrend the data using a method of your choice. All the data you need is on Fred. Make sure to use the "real," i.e. adjusted for inflation, time series and to take the log.)

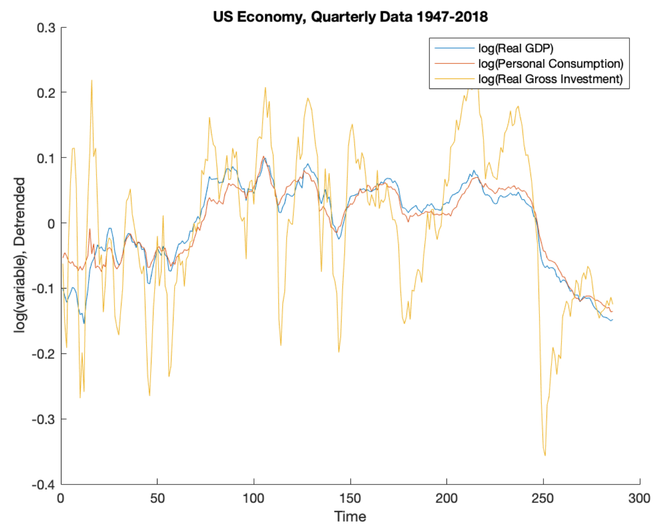
Simulation:

US economy:





The standard deviations of the logs of simulated output, consumption, and investment are 0.4316, 0.4680, and 0.4494, respectively. These values are substantially higher than the standard deviations of the detrended logs of US output, consumption, and investment (0.0661, 0.0581, and 0.1221, respectively). The correlations between the logs of simulated output and consumption, output and investment, and consumption and investment are 0.9803, 0.8986, and 0.8011, respectively. The correlation between output and consumption closely matches that in the detrended US data (0.9561), while the correlations between output and investment and consumption and investment are substantially higher than those in the US data (0.7185 and 0.7044, respectively). Quantitatively, the model appears to show a poor fit for the US economy, though



the relative levels of the logs of output, consumption, and investment appear similar in both the simulation and in the US data.

Problem 3. We consider the neoclassical growth model with an externality. A representative firm produces output according to the production function

$$Y_t = F(K_t, N_t).$$

Capital depreciates at a constant rate $\delta \geq 0$. The aggregate resource constraint is

$$Y_t = C_t + K_{t+1} - (1-\delta) K_t,$$

where C_t is aggregate consumption.

There is a large number of identical households with total mass equal to 1. Each household is endowed with $k_0 = K_0$ units of capital and one unit of time in every period. The household has preferences over identical consumption streams $\{c_t\}_{t=0}^{\infty}$ representable by the lifetime utility function

$$\sum_{t=0}^{\infty} \beta^t U(c_t, C_t).$$

Note that the household chooses $\{c_t\}_{t=0}^{\infty}$ but takes aggregate consumption $\{C_t\}_{t=0}^{\infty}$ as given.

1. State the social planner's problem recursively. Clearly identify the state and control variables.

The social planner's problem is

$$\begin{aligned} V(k) = \max_{k'} u(c,c) + \beta V(k') \\ &\text{subject to} \\ &c = F(k,1) + (1-\delta)k - k', \\ &k' \in [0, F(k,1) + (1-\delta)k]. \end{aligned}$$

The state variable is current-period capital k, and the control variable is next-period capital k'. The first constraint is a budget constraint that defines current-period consumption c, and the second constraint is a non-negativity constraint on next-period capital k' and current-period consumption c. In principle, the planner could allocate different amounts of consumption to each household for a given amount of aggregate consumption, but this is suboptimal assuming that u is concave in its first argument. Similarly, we assume that F displays constant returns to scale, so the planner allocates the same amount of labor n and capital k' to each household. For simplicity, we also assume that full employment is optimal ($n = 1$), but this may not be the case if u is decreasing in its second argument since the planner can internalize the externality from aggregate consumption.

2. Use the first order condition and the envelope condition to derive the Euler equation of the social planner's problem.

Assuming that the optimal choice of k' is interior, the first order condition is

$$u_1(c,c) + u_2(c,c) = \beta V'(k').$$

The envelope condition is

$$V'(k) = (F_1(k,1) + 1 - \delta)\left[u_1(c,c) + u_2(c,c)\right].$$

Combining the two conditions, we have the Euler equation

$$u_1(c,c) + u_2(c,c) = \beta\left(F_1(k',1) + 1 - \delta\right)\left[u_1(c',c') + u_2(c',c')\right].$$

3. Define a recursive competitive equilibrium.

A recursive competitive equilibrium is a value function $V : \mathbb{R}_+^2 \to \mathbb{R}$, policy functions for consumption and capital $C, G : \mathbb{R}_+^2 \to \mathbb{R}_+$, aggregate laws of motion for consumption and capital $J, H : \mathbb{R}_+ \to \mathbb{R}_+$, and pricing functions for wages and rent $w, r : \mathbb{R}_+ \to \mathbb{R}$ such that

(1) Given w, r, H, and J, V solves the households' Bellman equation, and C, G are the optimal policy functions:

$$\begin{aligned} V(k,K) = &\max_{c,k'} u(c, J(K)) + \beta V(k', H(K)) \\ &\text{subject to} \\ &c = w(K) + (1 - \delta + r(K))k - k', \\ &k' \in [0, w(K) + (1 - \delta + r(K))k]. \end{aligned}$$

Thus the household takes current-period aggregate capital K, aggregate consumption $C = J(K)$, next-period aggregate capital $K' = H(K)$, the wage rate $w(K)$, and the rental rate $r(K)$ as given when choosing current-period consumption c and next-period capital k'. The first constraint is simply a budget constraint, and the second constraint ensures non-negative current-period consumption c and non-negative next-period capital k'.

(2) The pricing functions satisfy the firm's first order conditions:

$$\begin{aligned} r(K) &= F_1(K,1), \\ w(K) &= F_2(K,1). \end{aligned}$$

(3) The policy functions and aggregate laws of motion are consistent:

$$J(K) = C(K,K),$$
$$H(K) = G(K,K).$$

(4) The goods market clears: $\forall K \in \mathbb{R}_+$,

$$C(K,K) + G(K,K) = F(K,1) + (1-\delta)K.$$

4. Use the first order condition and the envelope condition to derive the Euler equation that the representative household faces.

Letting $C = J(K)$ and $K' = H(K)$, the first order condition is

$$u_1(c,C) = \beta V_1(k',K').$$

The envelope condition is

$$V_1(k,K) = u_1(c,C)(1-\delta+F_1(K,1)).$$

Combining the two conditions, we have the Euler equation

$$u_1(c,C) = \beta(1-\delta+F_1(K',1))\,u_1(c',C').$$

5. For the purpose of this question, you can assume that there exists a unique competitive equilibrium. Is this equilibrium Pareto efficient? You don't need to provide a formal proof, but use the answers to questions 2 and 4 to explain your answer. What is the intuition?

If there is a unique competitive equilibrium and

$$u_2(c,c) \neq \beta(F_1(k',1)+1-\delta)\,u_2(c',c'), \tag{5}$$

where the consumption and capital levels are the optimal ones under the planner's problem, the competitive equilibrium is not Pareto efficient. Since all households have the same initial capital, if the competitive equilibrium were Pareto optimal, it would correspond to the solution of the planner's problem in which all households are weighted equally. This problem is considered in question 2, but the Euler equation derived in question 2 is not the same as the Euler equation derived in question 4 for the competitive equilibrium, given (5). Intuitively, the competitive equilibrium fails to be Pareto optimal because households take aggregate consumption as given and so do not internalize the impact of their own consumption on other households' utilities. The planner is able to internalize this externality, so she can strictly improve upon the competitive equilibrium allocation.