Macroeconomics II ECON6140

My suggested solutions - Final Exam 2023

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Problem 1

(a)

Endogenous jump variables: C_t, I_t, A_t

Endogenous state variables: K_t

Exogenous state variables: $\emptyset$

Remark. From the perspective of the central planner, all variables are endogenous!

(b)

The household does not internalize the externality and thus takes A_t as given. She only optimizes subject to constraints (1) and (2). Her optimization problem thus writes...

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \log(C_t) \quad \text{subject to (1) and (2)}$$

The Lagrangian of the household thus writes :

$$\mathcal{L}(C_t, I_t, K_{t+1}, \lambda_{1,t}, \lambda_{2,t}) = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\log(C_t) + \lambda_{1,t}((1-\delta)K_t + I_t - K_{t+1}) + \lambda_{2,t}(A_t K_t^{\alpha} - I_t - C_t)\right]$$

The FOC w.r.t C_t, I_t and K_{t+1} yields :

$$\frac{\partial \mathcal{L}}{\partial C_t}(\cdot) = 0 \iff \frac{1}{C_t} = \lambda_{2,t} \tag{4}$$

$$\frac{\partial \mathcal{L}}{\partial I_t}(\cdot) = 0 \iff \lambda_{1,t} = \lambda_{2,t} \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}}(\cdot) = 0 \iff -\lambda_{1,t} + \beta \mathbb{E}_t \left[\lambda_{1,t+1}(1-\delta) + \lambda_{2,t+1} \alpha A_{t+1} K_{t+1}^{\alpha-1} \right] = 0 \tag{6}$$

Combining (4), (5) and (6) yields the following:

$$-\frac{1}{C_t} + \beta \mathbb{E}_t \left[\frac{1-\delta+\alpha A_{t+1} K_{t+1}^{\alpha-1}}{C_{t+1}} \right] = 0 \iff 1 = \beta \mathbb{E}_t \left[\frac{C_t}{C_{t+1}} \left(1-\delta+\alpha A_{t+1} K_{t+1}^{\alpha-1}\right) \right]$$

The above Euler equation describes the necessary conditions for an optimal intertemporal decision of the household

(c)

The central planner is omnipotent and omnipresent and thus internalizes the externality. Unlike the household, she takes constraint (3) into account. Also, she knows that K_t and $\bar{K}_t$ are equal in equilibrium: she thus solves the following problem :

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \log(C_t) \quad \text{subject to (1), (2) and (3)}$$

The Lagrangian of the planner writes :

$$\mathcal{L}(C_t, I_t, K_{t+1}, A_{t+1}, \theta_{1,t}, \theta_{2,t}, \theta_{3,t}) = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (\log(C_t) + \theta_{1,t}((1-\delta)K_t + I_t - K_{t+1}) + \theta_{2,t}(A_t K_t^{\alpha} - I_t - C_t) +$$
$$\theta_{3,t}(a_0 \log(K_t) - \log(A_t)))$$

We find it more convenient to rexpress the problem as follows :

$$\mathcal{L}(C_t, I_t, K_{t+1}, \theta_{1,t}, \theta_{2,t}, \theta_{3,t}) = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\log(C_t) + \theta_{1,t}((1-\delta)K_t + I_t - K_{t+1}) + \theta_{2,t}(K_t^{\alpha+a_0} - I_t - C_t) \right]$$

The FOC w.r.t C_t, I_t and K_{t+1} yields :

$$\frac{\partial \mathcal{L}}{\partial C_t}(\cdot) = 0 \iff \frac{1}{C_t} = \theta_{2,t} \tag{7}$$

$$\frac{\partial \mathcal{L}}{\partial I_t}(\cdot) = 0 \iff \theta_{1,t} = \theta_{2,t} \tag{8}$$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}}(\cdot) = 0 \iff -\theta_{1,t} + \beta \mathbb{E}_t \left[\theta_{1,t+1}(1-\delta) + \theta_{2,t+1}(\alpha + a_0) K_{1+1}^{\alpha+a_0-1} \right] = 0 \tag{9}$$

Plugging (7) and (8) into (9) we get :

$$1 = \beta \mathbb{E} \left[\frac{C_t}{C_{t+1}} (1-\delta+(\alpha+a_0) K_{t+1}^{\alpha+a_0-1}) \right]$$

(d)

We know express the planner's problem as that of solving the Bellman equation. One can rewrite the above problem as :

$$V(K_t, A_t) = \max_{K_{t+1}, A_{t+1}} \log(C_t) + \beta E_t \left[V(K_{t+1}, A_{t+1}) \right] \tag{10}$$

Subject to (1), (2), (3). For convenience, we rewrite C_t as $C_t = K_t^{\alpha+a_0} - K_{t+1} + (1-\delta)K_t$. And (10) thus rewrites:

$$V(K_t) = \max_{K_{t+1}} \log(C_t) + \beta E_t \left[V(K_{t+1})\right] \tag{11}$$

Taking the first order condition of (11) with respect to K_{t+1} yields:

$$-\frac{1}{C_t} + \beta \mathbb{E}_t V'(K_{t+1}) = 0 \iff \beta \mathbb{E}_t V'(K_{t+1}) = \frac{1}{C_t} \tag{12}$$

Using the Benveniste-Scheinkman theorem, we have that

$$V'(K_t) = \frac{(\alpha + a_0)K_t^{\alpha+a_0-1} + (1-\delta)}{C_t}$$

It follows that :

$$\mathbb{E}_t V'(K_{t+1}) = \mathbb{E}_t \left[\frac{(\alpha + a_0)K_{t+1}^{\alpha+a_0-1} + (1-\delta)}{C_{t+1}}\right] \tag{13}$$

Combining (12) and (13) yields the desired optimality condition :

$$\beta \mathbb{E}_t \left[\frac{(\alpha + a_0)K_{t+1}^{\alpha+a_0-1} + (1-\delta)}{C_{t+1}}\right] = \frac{1}{C_t}$$

This is exactly the same optimality condition as in question (c).

(e)

Let us consider the household's optimality equation at the non-stochastic steady state (NSSS) (setting $C_t = C_{t+1}$). We thus get :

$$\frac{1}{\alpha}\left[\frac{1}{\beta} - 1 + \delta\right] = K^{\alpha+a_0-1} \Rightarrow K_h^{ss} = \left[\frac{1}{\beta\alpha}\right]^{\frac{1}{\alpha+a_0-1}}$$

Now let us consider the planner's optimal decision at the NSSS :

$$K_p^{ss} = \left[\frac{1}{\alpha + a_0}\left(\frac{1}{\beta} - 1 + \delta\right)\right]^{\frac{1}{\alpha+a_0-1}} = \left[\frac{1}{\beta(\alpha + a_0)}\right]^{\frac{1}{\alpha+a_0-1}}$$

Clearly, since by assumption we have $\alpha + a_0 < 1$, it must be the case that

$$\frac{\partial \left(\frac{1}{\beta x}\right)^{\frac{1}{\alpha+a_0-1}}}{\partial x} > 0$$

Hence $K_h^{ss} \leq K_p^{ss}$ as $a_0 \geq 0$

The household does not internalize the externality: she does not perceive the fact that an investment decision has not only a first-order effect on the level of output (through an increase in K_t) but also a second-order effect (through a rise in productivity A_t). Therefore, she underestimates the actual intertemporal return of saving (investing) and consequently does not invest sufficient amounts of output in capital accumulation so that the steady-state level of capital K_h^{ss} is lower than the level that would be achieved by a central and benevolent planner.

Problem 2

(a)

In the previous problem we had the following constraints for the household :

$$K_{t+1} = (1-\delta)K_t + I_t \tag{1}$$

$$C_t = A_t K_t^{\alpha} - I_t \tag{2}$$

We also derived the following optimality condition (Euler equation) :

$$\begin{aligned} 1 &= \beta \mathbb{E}_t \left[\frac{C_t}{C_{t+1}} \left(1 - \delta + \alpha A_{t+1} K_{t+1}^{\alpha-1}\right) \right] \\ 1 &= \beta \mathbb{E}_t \left[\frac{C_t}{C_{t+1}} \left(1 - \delta + \alpha K_{t+1}^{\alpha+a_0-1}\right) \right] \end{aligned} \tag{3}$$

Combining (1) and (2) we get that

$$\begin{aligned} C_t &= A_t K_t^{\alpha} - K_{t+1} + (1-\delta)K_t \\ C_t &= K_t^{\alpha+a_0} - K_{t+1} + (1-\delta)K_t \end{aligned} \tag{4}$$

(b)

At the NSSS we have the following :

$$1 = \beta(1 - \delta + \alpha K^{\alpha+a_0-1}) \iff K = \left[\frac{1}{\alpha}\left(\frac{1}{\beta} - 1 + \delta\right)\right]^{\frac{1}{\alpha+a_0-1}}$$

$$C = K\left(K^{\alpha+a_0-1} - \delta\right)$$

(c)

We now log-linearize the synthesized equilibrium equations (3) & (4) around the NSSS. Remember Ryan's convention: $x_t = \ln(X_t) - \ln(X)$

Following the method seen in class we get that the log-linearized version of (4) around the NSSS writes

$$Cc_t = \left((\alpha + a_0)K^{\alpha+a_0-1} + 1 - \delta)\right) Kk_t - Kk_{t+1}$$

$$\iff Cc_t - \left((\alpha + a_0)K^{\alpha+a_0-1} + 1 - \delta)\right) Kk_t + Kk_{t+1} = 0$$

Now log-linearizing the Euler equation (3), we get :

$$\begin{aligned} \frac{1}{C_t} &= \beta \mathbb{E}_t \left[\frac{1}{C_{t+1}}\left(1 - \delta + \alpha K_{t+1}^{\alpha+a_0-1}\right)\right] \\ \Rightarrow -\frac{1}{C}c_t &= \beta \mathbb{E}_t \left[-\frac{1}{C}\left(1 - \delta + \alpha K^{\alpha+a_0-1}\right) c_{t+1} + \alpha(\alpha + a_0 - 1)\frac{K^{\alpha+a_0-1}}{C} k_{t+1}\right] \\ \iff c_t &= \beta \mathbb{E}_t \left[\underbrace{\left(1 - \delta + \alpha K^{\alpha+a_0-1}\right)}_{=1/\beta} c_{t+1} - \alpha(\alpha + a_0 - 1)K^{\alpha+a_0-1} k_{t+1}\right] \\ \iff c_t &- \mathbb{E}_t\left[c_{t+1}\right] + \beta\alpha(\alpha + a_0 - 1)K^{\alpha+a_0-1} k_{t+1} = 0 \end{aligned}$$

(d)

We now compute the Jacobians of the above system. Our convention is that $X_t = [k_t]$, hence $X_{t+1} = [k_{t+1}]$. What's more, $Y_t = [c_t]$, hence $Y_{t+1} = [c_{t+1}]$

$$F_x = Jac_x = \begin{bmatrix} \left((\alpha + a_0)K^{\alpha + a_0 - 1} + 1 - \delta\right) K \\ 0 \end{bmatrix}$$

$$F_y = Jac_y = \begin{bmatrix} C \\ 1 \end{bmatrix}$$

$$F_{xp} = Jac_{xp} = \begin{bmatrix} K \\ \beta\alpha(\alpha + a_0 - 1)K^{\alpha + a_0 - 1} \end{bmatrix}$$

$$F_{yp} = Jac_{yp} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

```
clear
close all
%% Deep parameters of the model
Parameter
a_0 = 0.2;
%% Preliminary parametrization
ks = ((1/bet-1+delta)*1/(alph))^(1/(alph+a_0 -1));
nbk = 1000;
kmin = 0.8*ks;
kmax = 1.2*ks;
kgrid = linspace(kmin,kmax,nbk)';
vf = 0*kgrid;
%% Iteration
tt=0
while crit>1e-9 && tt < maxiter;
for i=1:nbk
%
% consumption and utility
%
c = kgrid(i)^(alph+a_0)+(1-delta)*kgrid(i)-kgrid(1:nbk);
util = log(c);
%
% find value function
%
[tv(i),dr(i)] = max(util+bet*vf(1:nbk));
end
crit = max(abs(tv'-vf))
vf = tv';
tt=tt+1;
disp(tt)
end
%% Final solution
kp = kgrid(dr);
```

Problem 4. Heterogeneous agents

First of all, that is not an easy question at all!!

I think the trick to "solve" this question is to feed a guess into an algorithm to get a new (provisional) steady-state level of capital feed it again to the algorithm and so on.

I think if you start by guessing a steady state level of capital (sth that makes sense) and then you solve the problem by **projection**, you can have a mapping of what to do in terms of capital choice, depending on your epsilon type. Now if you draw thousands of individuals and look at what they optimally choose in terms of capital stock, you can reconstruct an average capital stock (the average of those choices for instance), update the center capital stock, and solve the problem again by projection... and so on.