Econ 6170: Final

9 December 2023

You have the full allocated time (2.5 hours) to complete the following problems. You are to work alone. This test is not open book. Please write out your answer neatly below each question, and use a new sheet of paper if you need more space than provided. When using extra sheets, make sure to write out your name and the relevant question number. In your answers, you are free to cite results that you can recall from class or previous problem sets unless explicitly stated otherwise. There are seven questions and each question is worth five points, and so the exam is out of 35 point. There are no bonus points.

Question 1 Suppose $X \subseteq \mathbb{R}^d$ is nonempty and let $f : X \to \mathbb{R}$.

(i) Using either the sequential characterisation or the ϵ-δ criterion of continuity of f, prove that: if f is continuous at $x \in X$, then, for all open sets $O \subseteq \mathbb{R}$ such that $f(x) \in O$, there is an open ball with radius $\epsilon > 0$ centred at x, $B_\epsilon(x)$, such that $f(z) \in O$ for all $z \in B_\epsilon(x) \cap X$.

(ii) Explicitly prove that: If X is compact and f is continuous, then $f(X) := \{f(x) : x \in X\}$ is bounded.

(iii) Give an example in which $f(X)$ is not bounded but X is bounded and f is continuous.

(iv) Give an example in which $f(X)$ is not bounded but X is closed and f is continuous.

Hint: You will be given partial credit for writing down definitions of what you are being asked to prove (this applies to all other questions too!).

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(i) Suppose f is continuous at $x \in X$ so that for any sequence $(x_n)_n$ in X such that $x_n \to x$, we have $f(x_n) \to f(x)$. Fix $f(x) \in O \subseteq \mathbb{R}$. By way of contradiction, suppose that, for any $\epsilon > 0$, there exists $z_\epsilon \in B_\epsilon(x) \cap X$ such that $f(z_\epsilon) \notin O$. In particular, such a z must exist for any open ball centred at x with radius $\frac{1}{n}$ for all $n \in \mathbb{N}$. Let $z_n \in B_{n^{-1}}(x) \cap X$ be such that $f(z_n) \notin O$ for all $n \in \mathbb{N}$. Then the sequence $(z_n)_n$ converges to x. However, $f(z_n) \notin O$ for any $n \in \mathbb{N}$ and since O is open, O^c is closed and so $f(z_n) \to f(x) \in O^c$. This contradicts that $f(x) \in O$.

(ii) Here's a direct proof but you can also show that $f(X)$ is compact and thus bounded. Let $f : S \to \mathbb{R}^\ell$, where S is a nonempty, compact subset of $\mathbb{R}$. We will show that f is bounded on S; i.e., $\exists M \in \mathbb{R}$, $|f(x)| \le M \; \forall x \in S$. Fix any $x \in S$. Letting $\epsilon = 1$, since f is continuous, there exists $\delta > 0$ such that for any $x' \in S \backslash \{x\}$ such that $|x' - x| < \delta$, we have $|f(x) - f(x')| < 1$. Define $I_x := (x - \delta, x + \delta)$, then since $|x' - x| < \delta$ for any $x' \in I_x \cap S$, we have

$$1 > \left|f\left(x'\right) - f\left(x\right)\right|.$$

By the reverse triangle inequality, it follows that, for any $x' \in I_x \cap S$,

$$\begin{aligned} 1 &> \left|f\left(x'\right) - f\left(x\right)\right| \ge \left|\left|f\left(x'\right)\right| - \left|f\left(x\right)\right|\right| \ge \left|f\left(x'\right)\right| - \left|f\left(x\right)\right| \\ \Rightarrow 1 + \left|f\left(x\right)\right| &> \left|f\left(x'\right)\right|. \end{aligned}$$

Since S is compact and $\{I_x\}_{x \in S}$ is an open cover of S, there exists a finite number of points $\{x_1, \ldots, x_n\} \subseteq S$ such that $\{I_{x_i}\}_{i=1}^n$ is also a cover of S. Define

$$M := 1 + \max\left\{\left|f\left(x_i\right)\right| : i \in \{1, \ldots, n\}\right\}.$$

Then, M is a bound on $|f(x)|$ for any $x \in S$. To see this, for any $x' \in S$, since $S \subseteq \bigcup_{i=1}^n I_{x_i}$, there is an index $i \in \{1, \ldots, n\}$ such that $x' \in I_{x_i} \cap S$ so that

$$\left|f\left(x'\right)\right| < 1 + \left|f\left(x\right)\right| \le M.$$

Since $x' \in S$ was chosen arbitrarily, it follows that f is bounded on S by M.

(iii) $f : (0, 1) \to \mathbb{R}$, $f(x) := 1/x$ is a continuous function on an bounded set that is not bounded.

(iv) $g : \mathbb{R} \to \mathbb{R}$, $g(x) := x$ is a continuous function on a closed set (recall $\mathbb{R}$ is both open and closed) that is not bounded.

Question 2

(i) Define quasiconvexity and quasiconcavity.

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be a monotone function. Prove or disprove that: f is quasiconcave and/or quasiconvex.

(iii) Prove or disprove: If $f : \mathbb{R} \to \mathbb{R}$ is quasiconcave and quasiconvex, then f is monotone.

(iv) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is both strictly quasiconcave and strictly quasiconvex.

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(i) See notes.

(ii) Recall f is quasiconcave (resp. quasiconvex) if $f(\alpha x + (1-\alpha)y) \geq \min\{f(x), f(y)\}$ (resp. $\leq \max\{f(x), f(y)\}$) for any $x, y \in \mathbb{R}$ and $\alpha \in [0,1]$. Fix $x, y \in \mathbb{R}$ such that $x \geq y$ and suppose f is nondecreasing. Then, $f(x) \geq f(y)$ and

$$\max\{f(x), f(y)\} = f(x) \geq f(\alpha x + (1-\alpha)y) \geq f(y) = \min\{f(x), f(y)\}$$

for all $\alpha \in [0,1]$, where the inequalities follows from that fact that f is nondecreasing. Hence, f is both quasiconcave and quasiconvex. Argument for when f is nonincreasing is analogous.

(iii) Suppose f is both quasiconcave and quasiconvex. Toward a contradiction, suppose that f is not monotone; i.e., there exist $x, y, z \in \mathbb{R}$ such that $x < y < z$ and $f(x) < f(y) > f(z)$ (or $f(y) < f(x), f(z)$). Then, there exists an $\alpha \in (0,1)$ such that $y = \alpha x + (1-\alpha)z$ and

$$f(y) = f(\alpha x + (1-\alpha)z) > \max\{f(x), f(z)\},$$

which contradicts the fact that f is quasiconvex. Similarly, if $f(y) < f(x), f(z)$, then

$$f(y) = f(\alpha x + (1-\alpha)z) < \min\{f(x), f(z)\},$$

which contradicts the fact that f is quasiconcave..

(iv) Any strictly increasing function is both strictly quasiconcave and strictly quasiconvex.

Question 3 Let $f : \mathbb{R}^d_+ \to \mathbb{R}_+$ be a continuous function and define a correspondence $\Gamma : \mathbb{R}^d_+ \rightrightarrows \mathbb{R}_+$ by $\Gamma(\theta) := [0, f(\theta)]$. Show that Γ is continuous.

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See "4.1 More on Correspondences" Exercise 2/3.

Question 4 Let $X \subseteq \mathbb{R}^d$ and $Y \subseteq \mathbb{R}^m$ be two open sets. Let $f : X \to Y$.

(i) Give a definition of the directional derivative of f in the direction $\mathbf{v} \in \mathbb{R}^d \backslash \{\mathbf{0}\}$ at $\mathbf{x}_0 \in X$.

(ii) How is the directional derivative of f related to the partial derivative of f? Be precise.

Suppose $d = m = 1$ and $X := (a, b)$ and that f is convex. We say that $s \in \mathbb{R}$ is a *subgradient* of f at $x_0 \in X$ if

$$f(y) \geq f(x_0) + s(y - x_0) \ \ \forall y \in X.$$

(iii) Suppose f is differentiable. What is the subgradient of f at any $x \in X$?

(iv) Let $y_0, z_0 \in (a, b)$ with $z_0 > y_0$. Show that

$$f'\left(y_0^-\right) \leq f'\left(y_0^+\right) \leq \frac{f(z_0) - f(y_0)}{z_0 - y_0} \leq f'\left(z_0^-\right) \leq f'\left(z_0^+\right),$$

where $f'(x^-) := \lim_{h \nearrow 0} \frac{f(x+h)-f(x)}{h}$ and $f'(x^+) := \lim_{h \searrow 0} \frac{f(x+h)-f(x)}{h}$ are the left- and right-derivatives of f at $x \in X$, respectively. **Hint:** Recall the Chordal Slope lemma: For any convex $f : (a, b) \to \mathbb{R}$ and any $y < x < z$,

$$\frac{f(x) - f(y)}{x - y} \leq \frac{f(z) - f(y)}{z - y} \leq \frac{f(z) - f(x)}{z - x}.$$

Use this to first prove that $f'(y_0^-) \leq f'(y_0^+)$ and $f'(z_0^-) \leq f'(z_0^+)$. Then use the lemma again to prove the rest of the inequalities.

(v) Suppose f has a kink at $x_0 \in X$. Use the result above to show that any $s \in [f'(x_0^-), f'(x_0^+)]$ is a subgradient. **Hint:** Drawing helps!

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(i) See notes.

(ii) Partial derivative with respect to x_i is obtained by setting $\mathbf{v} = \mathbf{e}_i$.

(iii) When f is differentiable at $x \in X$, the subgradient is given by $f'(x)$.

(iv) For any $h > 0$, note that $x_0 - h < x_0 < x_0 + h$ so that by the hint:

$$\frac{f(x_0) - f(x_0 - h)}{h} \leq \frac{f(x_0 + h) - f(x_0 - h)}{2h} \leq \frac{f(x_0 + h) - f(x_0)}{h} \ \forall h > 0$$

so that taking limits, we have $f'(x_0^-) \leq f'(x_0^+)$, where $f'(x_0^-)$ is the left-derivative and $f'(x_0^+)$ is the right-derivative of f at $x_0 \in X$. We therefore have that the interval $[f'(x_0^-), f'(x_0^+)]$ is nonempty. Now take $y_0, z_0 \in (a, b)$ with $z_0 > y_0$, by the lemma again

$$f'\left(y_0^-\right) \leq f'\left(y_0^+\right) \leq \frac{f(z_0) - f(y_0)}{z_0 - y_0} \leq f'\left(z_0^-\right) \leq f'\left(z_0^+\right).$$

(v) Take any $s \in [f'(x_0^-), f'(x_0^+)]$, by above, For any $x \in (x_0, b)$, letting $z_0 = x > x_0 = y_0$ gives

$$s \leq f'\left(x_0^+\right) \leq \frac{f(x) - f(x_0)}{x - x_0}$$

and for any $x \in (a, x_0)$, letting $z_0 = x_0 > x = y_0$ gives

$$\frac{f(x_0) - f(x)}{x_0 - x} \leq f'\left(x_0^-\right) \leq s.$$

Hence,

$$f(x) \geq s(x - x_0) + f(x_0) \;\; \forall x \in (a, b).$$

Question 5 Let $f : \mathbb{R}^d \to \mathbb{R}$ and $h : \mathbb{R}^d \to \mathbb{R}^K$, where h_k is $\mathbf{C}^1$ for each $k \in \{1, \ldots, K\}$. Suppose $\mathbf{x}^*$ is a local maximum or minimum of f on the constraint set

$$\Gamma := \left\{ \mathbf{x} \in \mathbb{R}^d : h(\mathbf{x}) = \mathbf{0} \right\}.$$

(i) State the (rest of the) Theorem of Lagrange.

(ii) Suppose $K = 1 < d = 2$. Prove the Theorem of Lagrange. **Hint:** Write out the two first-order conditions and apply the implicit function theorem to the appropriate one to obtain the Lagrange multiplier. Then, use the fact that $\mathbf{x}^*$ is a local maximum to show that the constructed Lagrange multiplier also satisfies the other first-order condition.

(iii) What does the Theorem of Lagrange say when (a) $K = d$ and (b) $K > d$?

(iv) Prove or disprove that the converse of the Theorem of Lagrange is true.

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(i) See class notes.

(ii) See class notes.

(iii) If $K > d$, constraint qualification cannot be satisfied. If $K = d$, we have a simultaneous system of equations and so we don't need the Lagrange theorem to solve the problem.

(iv) See class notes.

Question 6 Fix some $Y \subseteq \mathbb{R}^d$ that is nonempty and has a nonempty interior. We say that a (production) vector $\mathbf{y} \in Y$ is *efficient* if there is no $\mathbf{y}' \in Y$ such that $\mathbf{y}' \geq \mathbf{y}$ and $\mathbf{y}' \neq \mathbf{y}$. A production vector $\mathbf{y} \in Y$ is *profit maximising for some* $\mathbf{p} \in \mathbb{R}^d_{++}$ *if*

$$\mathbf{p} \cdot \mathbf{y} \geq \mathbf{p} \cdot \mathbf{y}' \; \forall \mathbf{y}' \in Y.$$

(i) Prove or disprove: (a) If $\mathbf{y} \in Y$ is efficient, then $\mathbf{y}$ is a boundary point of Y; (b) If $\mathbf{y} \in Y$ is a boundary point of Y, then $\mathbf{y}$ is efficient.

(ii) Prove that: If $\mathbf{y} \in Y$ is profit maximising for some $\mathbf{p} \in \mathbb{R}^d_{++}$, then $\mathbf{y}$ is efficient.

(iii) State a separating hyperplane theorem.

(iv) Suppose that Y is convex. Prove that every efficient production vector $\mathbf{y} \in Y$ is a profit-maximising production vector for some $\mathbf{p} \in \mathbb{R}^d_{+}$ (i.e., $\mathbf{p} \neq \mathbf{0}$ and $\mathbf{p} \geq \mathbf{0}$). **Hint:** Apply the separating hyperplane theorem to the set Y and $P_{\mathbf{y}} := \{\mathbf{y}' \in Y : \mathbf{y}' \gg \mathbf{y}\}$, where $(y_i')_{i=1}^d = \mathbf{y}' \gg \mathbf{y} = (y_i)_{i=1}^d$ means that $y_i' > y_i$ for all $i = 1, \ldots, d$. Try drawing the case of $d = 2$.

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(i) (a) If $\mathbf{y} \in \text{int}(Y)$ is efficient, then there exists $\mathbf{y}' \in Y \backslash \text{int}(Y)$ with $\mathbf{y}' \geq \mathbf{y}$ and $\mathbf{y}' \neq \mathbf{y}$. (b) Suppose $Y = [0,1]^2$. Then, $(0,0)$ is a boundary point that is not efficient because $(0,1) \geq (0,0)$.

(ii) Suppose not; i.e., there exists $\mathbf{y}' \in Y$ such that $\mathbf{y}' \geq \mathbf{y}$ and $\mathbf{y}' \neq \mathbf{y}$. Then, because $\mathbf{p} \in \mathbb{R}^d_{++}$, we must have $\mathbf{p} \cdot (\mathbf{y} - \mathbf{y}') > 0$; i.e., $\mathbf{y}$ is not profit maximising.

(iii) See notes.

(iv) Suppose that $\mathbf{y} \in Y$ is efficient and define $P_{\mathbf{y}} := \{\mathbf{y}' \in Y : \mathbf{y}' \gg \mathbf{y}\}$. The set $P_{\mathbf{y}}$ is convex and because y is efficient, we must have $Y \cap P_{\mathbf{y}} = \varnothing$. By the separating hyperplane theorem, there exists $\mathbf{p} \in \mathbb{R}^d_{++} \backslash \{\mathbf{0}\}$ such that $\mathbf{p} \cdot \mathbf{y}' \geq \mathbf{p} \cdot \mathbf{y}''$ for all $\mathbf{y}' \in P_{\mathbf{y}}$ and $\mathbf{y}'' \in Y$. Hence, $\mathbf{p} \cdot \mathbf{y}' \geq \mathbf{p} \cdot \mathbf{y}$ for all $\mathbf{y}' \gg \mathbf{y}$. Hence, we must have $\mathbf{p} \geq 0$.

Question 7 Suppose $(X, \geq)$ is a nonempty convex sublattice of $(\mathbb{R}^d, \geq)$. We say that $f : X \to \mathbb{R}$ is *C-supermodular* if

$$f(x \vee y - tv) - f(y) \geq f(x) - f(x \wedge y + tv) \ \forall t \in [0,1],$$

where $v = y - x \wedge y = x \vee y - x$ for all $x, y \in X$.

(i) Suppose $d = 2$ and draw the points x, y, $x \wedge y$, and $x \vee y$ for the case in which x and y are not ordered. Draw in the points $b = x \vee y - tv$ and $a = x \wedge y + tv$ for some $t \in (0,1)$. Interpret the condition for C-supermodularity in this diagram.

(ii) Argue that C-supermodularity is a stronger property than supermodularity.

(iii) Prove that: If f is supermodular and $f(x_i, x_{-i})$ is a concave function of x_{-i} for all $i \in \{1, \ldots d\}$, then f is C-supermodular.

(iv) Given an example of f that is not concave but $f(x_i, x_{-i})$ is a concave function of x_{-i} for all $i \in \{1, \ldots d\}$. **Hint:** It suffices to consider $d = 2$.

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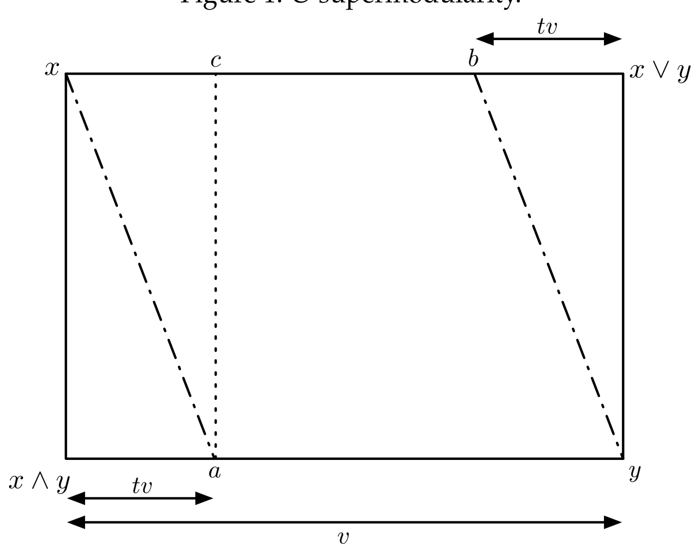
(i) For all possible (backward-bending) parallelogram, we need the line between x' and a to be at least as "long" as the line between b and y.

Figure 1: C-supermodularity.



(ii) Supermodularity is when $t = 0$.

(iii) Adding and subtracting $f(x \vee y)$ to $f(b) - f(y)$, we obtain that

$$f(b) - f(y) = f(b) - f(x \vee y) + f(x \vee y) - f(y).$$

Then, by i-concavity,

$$f(x \vee y) - f(b) \leq f(c) - f(x).$$
$$\Leftrightarrow f(b) - f(x \vee y) \geq f(x) - f(c)$$

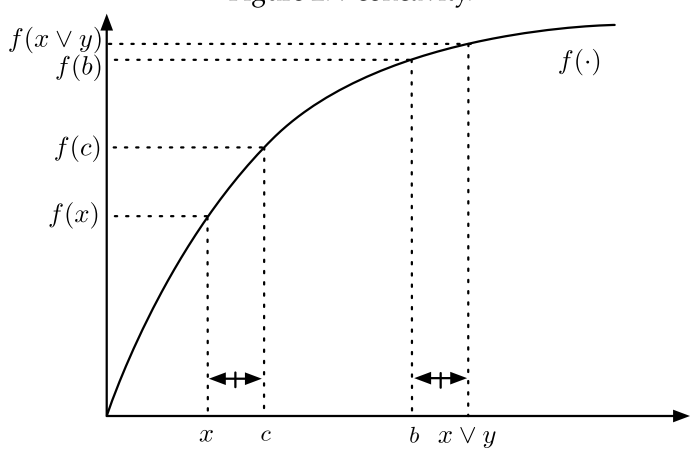
where $c = x + tv$. This is clear from Figure 2 (drawn when the function is increasing, the result also holds for decreasing concave function). Since F is supermodular, then

$$f(x \vee y) - f(y) \geq f(x) - f(x \vee y) = f(c) - f(a).$$

Together, these imply that

$$f(b) - f(y) \geq f(x) - f(c) + f(c) - f(a) = f(x) - f(a).$$

Figure 2: i-concavity.



(iv) Take $f(x_1, x_2) :- x_1^{3/4} x_2^{3/4}$. This is not itself concave (since the powers sum to more than 1) however, it does satisfy i-concavity. Since

$$\frac{\partial F(x_1, x_2)}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\frac{3}{4} x_i^{-\frac{1}{4}} x_j^{\frac{3}{4}}\right) = -\frac{3}{16} x_i^{-\frac{5}{4}} x_j^{\frac{3}{4}} \leq 0.$$