Econ 6170: Mid-Term 1

3 October 2024

You have the full class time to complete the following problems. You are to work alone. This test is not open book. Please write out your answer neatly below each question, and use a new sheet of paper if you need more space than provided. When using extra sheets, make sure to write out your name and the relevant question number. In your answers, you are free to cite results that you can recall from class or previous problem sets unless explicitly stated otherwise. The exam is out of 25 points.

Question 1 (5 points) Let $f : S \to \mathbb{R}$ be a function on a nonempty set $S \subseteq \mathbb{R}$.

(i) Suppose f is continuous. Give an example in which S is closed but not bounded and $f(S)$ is not bounded.

(ii) Suppose f is continuous. Given an example in which S is bounded but not closed and $f(S)$ is not bounded.

(iii) Suppose S is compact. Given an example in f is not continuous and $f(S)$ is not bounded

Make sure to argue/prove that $f(S)$ is not bounded in each of your example.

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Solution 1. A set is bounded if it is bounded from above and below. A set $S \subseteq \mathbb{R}$ is bounded from above (resp. below) if there exists $s^* \in \mathbb{R}$ such $s \leq s^*$ (resp. $s \geq s^*$) for all $s \in S$.

(i) $f(x) = x$ and $S = \mathbb{R}_+$. Note that $\lim_{x\to\infty} f(x) = \infty$.

(ii) $f(x) = \frac{1}{x}$ and $(0, 1]$. Note that f is continuous on $(0, 1]$ but $f(S)$ is not bounded above.

(iii) $f(x) = \begin{cases} x^{-1} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ and $S = [-1, 1]$. $f(S) = \mathbb{R}$.

Question 2 (5 points) A set $U \subseteq V \subseteq \mathbb{R}^d$ is *open relative to* V if and only if $U = O_u \cap V$ for some open set $O_u \subseteq \mathbb{R}^d$.

(i) Show that, if $\{U_i : i \in I\}$ is a collection of sets that are open relative to V, then $\bigcup_{i \in I} U_i$ is also open relative to V.

(ii) Show that, if $\{U_i : i = 1, 2, \ldots, N\}$ is a finite collection of sets that are open relative to V, then $\bigcap_{i=1}^N U_i$ is also open relative to V.

(iii) Show that, if U is open relative to V, then, for all $x \in U$, there exists $\epsilon > 0$ such that $B_\epsilon(x) \cap V \subseteq U$.

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Solution 2. (i) Since each U_i is open relative to V, there exists an $O_i \subseteq \mathbb{R}^d$ that is open such that $U_i = O_i \cap V$. Then,

$$\bigcup_{i \in I} U_i = \bigcup_{i \in I} (O_i \cap V) = \left(\bigcup_{i \in I} O_i \right) \cap V.$$

Since arbitrary unions of open sets are open, $\bigcup_{i \in I} O_i \subseteq \mathbb{R}^d$ is open and hence $\bigcup_{i \in I} U_i$ is also relative to V.

(ii) Since each U_i is open relative to V, there exists an $O_i \subseteq \mathbb{R}^d$ that is open such that $U_i = O_i \cap V$. Then,

$$\bigcap_{i=1}^{N} U_i = \bigcap_{i=1}^{N} (O_i \cap V) = \left(\bigcap_{i=1}^{N} O_i \right) \cap V.$$

Since finite intersections of open sets are open, $\bigcap_{i=1}^N O_i \subseteq \mathbb{R}^d$ is open and hence $\bigcap_{i=1}^N O_i$ is also relative to V.

(iii) Since each U is open relative to V, there exists an $O \subseteq \mathbb{R}^d$ that is open such that $U = O \cap V$. Fix $x \in U$. Since $x \in O$, there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq O$. Then, $B_\epsilon(x) \cap V \subseteq O \cap V = U$.

Question 3 (5 points) A set $S \subseteq \mathbb{R}^d$ is *totally bounded* if and only if, for any $\epsilon > 0$, there exists $\{s_1, s_2, \ldots, s_n\} \subseteq S$ for some $n \in \mathbb{N}$ such that

$$S \subseteq \bigcup_{s=1}^{n} B_\epsilon(s).$$

(i) Describe what it means for a set S to be totally bounded in words. How does it relate to compactness?

(ii) Show that if S is totally bounded, then S is bounded.

(iii) Show that if S is sequentially compact, then S is totally bounded. **Hint:** Prove by contradiction; i.e., construct a sequence that violates S being sequentially compact.

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Solution 3. (i) It's like compactness but the open sets that cover S must have a constant radius.

(ii) Suppose S is totally bounded and let $\{s_1, s_2, \ldots, s_n\} \subseteq S$ be a finite set such that, for $\epsilon > 0$, $\bigcup_{i=1}^n B_\epsilon(s_i)$ covers S. Fix some $i^* \in \{1, 2, \ldots, n\}$ and define

$$M := \max\{|s_i - s_{i^*}| : i \in \{1, 2, \ldots, n\}\} + 1.$$

Observe that $S \subseteq B_M(s_i^*)$ and so S is bounded.

(iii) Suppose S is sequentially compact but S is not totally bounded; i.e., for some $\epsilon > 0$. S, $\bigcup_{s \in T} B_\epsilon(s)$ does not cover S for any finite subset T of S. To obtain a contradiction, we will construct a sequence in S with no convergent subsequence. Fix $s_1 \in S$ and ϵ. By hypothesis, we cannot have $S \subseteq B_\epsilon(s_1)$; i.e., there exists $s_2 \in S$ such that $\|s_1 - s_2\| \geq \epsilon$. By hypothesis again, we cannot have $S \subseteq B_\epsilon(s_1) \cup B_e(s_2)$ and so we can find $s_3 \in S$ such that $\|s_1 - s_3\|, \|s_2 - s_3\| \geq \epsilon$. Proceeding in this manner, we obtain a sequence $(s_n)_n$ in S such that $\|s_i - s_j\| \geq \epsilon$ for any distinct $i, j \in \mathbb{N}$. Since S is sequentially compact, there must exist a convergent subsequence, $(s_{n_k})_k$. But this is impossible since $s_{n_k} \to s$ implies that

$$\|x_{n_k} - x_{n_\ell}\| \leq \|x_{n_k} - x\| + \|x - x_{n_\ell}\| < \epsilon$$

for sufficiently large and distinct k and ℓ.

Question 4 (5 points) Suppose $f : X \to \mathbb{R}$ is strictly quasiconvex, where $X \subseteq \mathbb{R}^d$ is nonempty and convex. Show that any local minimum of f is a global minimum of f. Prove or disprove the same statement when f is only quasiconvex.

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Solution 4. Now suppose f is strictly quasiconvex and $\mathbf{x}^* \in X$ is a local minimum of f on X. Thus, there exists $\epsilon > 0$ such that $f(\mathbf{x}^*) \leq f(\mathbf{x})$ for all $\mathbf{x} \in B_\epsilon(\mathbf{x}^*)$. Toward a contradiction, suppose that $\mathbf{x}^*$ is not a global minimum; i.e., there exists $\mathbf{z} \in X$ such that $f(\mathbf{z}) < f(\mathbf{x}^*)$. But then since X is convex, for any $\alpha \in [0,1]$, $\alpha\mathbf{x}^* + (1-\alpha)\mathbf{z} \in X$ and by the quasiconvexity of f, we have

$$f\left(\alpha\mathbf{x}^* + (1-\alpha)\,\mathbf{z}\right) < \max\left\{f\left(\mathbf{x}^*\right), f\left(\mathbf{z}\right)\right\} = f\left(\mathbf{x}^*\right) \ \forall \alpha \in (0,1).$$

In particular, for sufficiently large $\alpha > 0$, $\alpha\mathbf{x}^* + (1-\alpha)\mathbf{z} \in B_r(\mathbf{x}^*)$ and hence we must have $f(\alpha\mathbf{x}^* + (1-\alpha)\mathbf{z}) < f(\mathbf{x}^*)$; a contradiction.

Example from class notes: consider $f : \mathbb{R}_+ \to \mathbb{R}$ defined as

$$f\left(x\right) := \begin{cases} x^3 & x \in [0,1] \\ 1 & x \in (1,2] \\ x^3 & x > 2 \end{cases}.$$

Observe that f is both quasi-concave and quasi-convex because f is a nondecreasing function. Observe that f is constant on the interval $(1,2)$ and so every point in this interval is a local maximum (as well as minimum) of f. However, no point in $(1,2)$ is either a global maximum or a global minimum.

Question 5 (5 points) Let $X \subseteq \mathbb{R}^d$ be a convex set with a nonempty interior. Let $\mathbf{x}_0$ be a boundary point of X. We wish to show that there is a supporting hyperplane at $\mathbf{x}_0$. The idea of the proof is to construct sequence of hyperplanes that converges to the hyperplane at $\mathbf{x}_0$. To that end, let $(\mathbf{x}_n)_n$ be a sequence in $\mathbb{R}^d$ that converges to $\mathbf{x}_0$.

(i) Write down the definition for X to have a supporting hyperplane at $\mathbf{x}_0$.

(ii) Suppose $\mathbf{x}_n \notin X$, given an example of a set X such that no hyperplane exists that *strongly* separates $\mathbf{x}_n$ and X? How can we modify X, say to $\tilde{X}$, to ensure the existence of a strongly separating hyperplane?

(iii) Suppose we found a sequence of hyperplanes that strongly separate $\mathbf{x}_n$ and $\tilde{X}$ for all $n \in \mathbb{N}$. Argue that the sequence of hyperplanes converges to a limit and that the limit is the supporting hyperplane at $\mathbf{x}_0$.

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Solution 5. (i) There exists $\mathbf{p} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ such that $\mathbf{p} \cdot \mathbf{x}_0 \geq \mathbf{p} \cdot \mathbf{x}$ for all $\mathbf{x} \in X$.

(ii) If X is not closed, then a hyperplane that strongly separates $\mathbf{x}_n$ and X need not exist. For example, suppose $d = 2$, $\mathbf{x} = (0, 1)$ and $X := \{(x, y) \in \mathbb{R}^2 : x > 0, xy > 1\}$. We therefore can let $\tilde{X} := \mathrm{cl}(X)$.

(iii) By the separating hyperplane theorem, there exists $\mathbf{p}_n \neq \mathbf{0}$ such that

$$\mathbf{p}_n \cdot \mathbf{x}_n > \mathbf{p}_n \cdot \mathbf{x} \; \forall \mathbf{x} \in \mathrm{cl}\,(X)$$

$\mathbf{x} \in X \subseteq \mathrm{cl}(X)$. Since we can normalise $\mathbf{p}^n$ without affecting the inequality, we can assume that $\|\mathbf{p}^n\| = 1$ for all $n \in \mathbb{N}$. Then, $(\mathbf{p}_n)_n$ is a bounded sequence and so has a convergent subsequence, $(\mathbf{p}_{n_k})_k$ (by what theorem?). That is, $\mathbf{p}_{n_k}$ converges to some $\mathbf{p}$ and since $\mathbf{x}_{n_k}$ converges to $\mathbf{x}_0$, we have

$$\mathbf{p} \cdot \mathbf{x}_0 \geq \mathbf{p} \cdot \mathbf{x} \; \forall \mathbf{x} \in X,$$

where the inequality is weak because $\mathbf{x}_0 \in X$. Note that the $H(\mathbf{p}, \mathbf{p} \cdot \mathbf{x}_0)$ is a hyperplane supported at $\mathbf{x}_0$.