Econ 6190 Problem Set 4

Fall 2024

1. Let $\{X_1, \ldots, X_n\}$ be a random sample from the uniform distribution on the interval $(\theta, \theta+1)$, $-\infty < \theta < \infty$. Find a minimal sufficient statistic for θ. This question shows that the dimension of a minimal sufficient statistic does not necessarily match the dimension of the unknown parameter.

2. [Mid term, 2022] Suppose $X \sim \mathrm{N}(\mu, \sigma^2)$ with an unknown mean μ and **known** variance $\sigma^2 > 0$. We draw a random sample $\mathbf{X} := \{X_1, X_2, \ldots X_n\}$ of size n from X. We are interested in estimating μ based on $\mathbf{X}$.

   (a) Find a minimal sufficient statistic for μ.

   (b) Suppose now $\sigma^2 = 1$ and $n = 1$. Consider the following estimator $\hat{\theta} = \frac{c^2}{c^2+1} X_1$ for some known $c > 0$.

      i. Find the MSE of $\hat{\theta}$. Is $\hat{\theta}$ unbiased?

      ii. Compare the MSE of $\hat{\theta}$ with the MSE of $\tilde{\theta} = X_1$. Which one is more efficient? (Hint: there is a range of values of μ for which $\hat{\theta}$ is more efficient).

      iii. Based on your answer to (ii), which of the two estimators, $\hat{\theta}$ or $\tilde{\theta}$, is more efficient when $\mu = c$?

3. Let $\{X_1, \ldots, X_n\}$ be a random sample from finite second moment, and let $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2$ be an estimator for $\sigma^2 = \mathrm{var}(X)$. Show $\mathbb{E}[\hat{\sigma}^2] = (1 - \frac{1}{n})\sigma^2$ and thus find the bias of $\hat{\sigma}^2$.

4. [Hong] Suppose $\{X_1, X_2 \ldots X_n\}$ is iid $\mathrm{N}(0, \sigma^2)$. Consider the following estimator for σ^2:

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n X_i^2$$

   Find:

   (a) the sampling distribution of $n\hat{\sigma}^2/\sigma^2$.

   (b) $\mathbb{E}\hat{\sigma}^2$.

   (c) $\mathrm{var}(\hat{\sigma}^2)$.

   (d) $\mathrm{MSE}(\hat{\sigma}^2)$.

5. Let $\{X_1, \ldots, X_n\}$ be a random sample from a Poisson distribution with parameter λ

$$P\{X_i = j\} = \frac{e^{-\lambda}\lambda^j}{j!}, j = 0, 1 \ldots$$

(a) Find a minimal sufficient statistic for λ, say T.

(b) Suppose we are interested in estimating probability of a count of zero $\theta = P\{X = 0\} = e^{-\lambda}$. Find an unbiased estimator for θ, say $\hat{\theta}_1$. (Note $P\{X = 0\} = \mathbb{E}[\mathbf{1}\{X = 0\}]$.)

(c) Is the estimator in (b) a function of the minimal sufficient statistics T?

(d) Use the definition of a sufficient statistic and an unbiased estimator, show that the estimator $\hat{\theta}_2 = \mathbb{E}[\hat{\theta}_1|T]$ is also unbiased and $\text{MSE}(\hat{\theta}_2) \leq \text{MSE}(\hat{\theta}_1)$.

(e) Based on (d), find an analytic form of $\hat{\theta}_2$.

1. The joint pdf of $\mathbf{X}$ is

$$f(\mathbf{x}|\theta) = \begin{cases} 1 & \theta < x_i < \theta + 1, i = 1 \ldots n, \\ 0 & \text{otherwise,} \end{cases}$$

equivalent to

$$f(\mathbf{x}|\theta) = \begin{cases} 1 & \max x_i - 1 < \theta < \min x_i, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, for two sample points $\mathbf{x}$ and $\mathbf{y}$, the numerator and denominator of ratio $\frac{f(\mathbf{x}|\theta)}{f(\mathbf{y}|\theta)}$ will be positive for the same values of θ if and only if $\max x_i = \max y_i$ and $\min x_i = \min y_i$. Furthermore, when $\max x_i = \max y_i$ and $\min x_i = \min y_i$, $\frac{f(\mathbf{x}|\theta)}{f(\mathbf{y}|\theta)} = 1$. Therfore, the minimal sufficient statistic is $(\max_i X_i, \min_i X_i)$.

2(a) Answer: For any two sample points $\mathbf{x}$ and $\mathbf{y}$

$$\begin{aligned} \frac{f(\mathbf{x}|\mu,\sigma^2)}{f(\mathbf{y}|\mu,\sigma^2)} &= \frac{\prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right)}{\prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right)} \\ &= \frac{\exp\left(-\sum_{i=1}^n \frac{(x_i-\mu)^2}{2\sigma^2}\right)}{\exp\left(-\sum_{i=1}^n \frac{(y_i-\mu)^2}{2\sigma^2}\right)} \\ &= \frac{\exp\left(-\sum_{i=1}^n \frac{(x_i-\bar{x})^2+(\bar{x}-\mu)^2}{2\sigma^2}\right)}{\exp\left(-\sum_{i=1}^n \frac{(y_i-\bar{y})^2+(\bar{y}-\mu)^2}{2\sigma^2}\right)} \\ &= \exp\left(\sum_{i=1}^n \frac{(y_i-\bar{y})^2-(x_i-\bar{x})^2}{2\sigma^2} + \frac{n(\bar{y}^2-\bar{x}^2) - 2n(\bar{x}-\bar{y})\mu}{2\sigma^2}\right), \end{aligned}$$

which does not depend on μ if and only if $\bar{x} = \bar{y}$. Thus, a minimal sufficient statistic is $T(\mathbf{X}) = \bar{X}$.

(b)

i. Find the MSE of $\hat{\theta}$. Is $\hat{\theta}$ unbiased?
Answer: $\text{MSE}(\hat{\theta}) = \mathbb{E}[(\hat{\theta}-\theta)^2] = \text{bias}^2(\hat{\theta}) + \text{var}(\hat{\theta})$.

$$\begin{aligned} \text{bias}(\hat{\theta}) &= \mathbb{E}[\hat{\theta}] - \mu \\ &= \frac{c^2}{c^2+1}\mu - \mu = -\frac{\mu}{c^2+1} \\ \text{var}(\hat{\theta}) &= \left(\frac{c^2}{c^2+1}\right)^2 \cdot 1. \end{aligned}$$

Thus, $\text{MSE}(\hat{\theta}) = \frac{\mu^2}{(c^2+1)^2} + \left(\frac{c^2}{c^2+1}\right)^2 = \frac{\mu^2+c^4}{(c^2+1)^2}$. Also, $\hat{\theta}$ is biased unless $\mu = 0$.

ii. Compare the MSE of $\hat{\theta}$ with the MSE of $\tilde{\theta} = X_1$. Which one is more efficient? (Hint: there is a range of values of μ for which $\hat{\theta}$ is more efficient).
Answer: $\text{MSE}(\tilde{\theta}) = \sigma^2 = 1$. Therefore,

$$\begin{aligned}\text{MSE}(\hat{\theta}) - \text{MSE}(\tilde{\theta}) &= \frac{\mu^2 + c^4}{(c^2+1)^2} - 1 \\ &= \frac{\mu^2 - 2c^2 - 1}{(c^2+1)^2}.\end{aligned}$$

Since $c^2 + 1 > 0$, $\text{MSE}(\hat{\theta}) - \text{MSE}(\tilde{\theta}) > 0$ if and only if

$$\mu^2 - 2c^2 - 1 > 0,$$

i.e., when $\mu > \sqrt{2c^2+1}$ or $\mu < -\sqrt{2c^2+1}$. Thus, $\tilde{\theta}$ is more efficient when $\mu \in (-\infty, -\sqrt{2c^2+1}) \cup (\sqrt{2c^2+1}, \infty)$. And $\hat{\theta}$ is more efficient when $\mu \in (-\sqrt{2c^2+1}, \sqrt{2c^2+1})$.

iii. Based on your answer to (ii), which of the two estimators, $\hat{\theta}$ or $\tilde{\theta}$, is more efficient when $\mu = c$?
Answer: since it always holds that $c \in (-\sqrt{2c^2+1}, \sqrt{2c^2+1})$, $\hat{\theta}$ is more efficient when $\mu = c$.

3.

$$E[\hat{\sigma}^2] = E\left[\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2\right]$$

$$= E\left[\frac{1}{n}\sum_{i=1}^{n}(X_i - EX + EX - \bar{X})^2\right]$$

$$= E\Big\{\underbrace{\frac{1}{n}\sum_{i=1}^{n}\big[(X_i - EX)^2}_{\text{part } ①} + \underbrace{2(X_i - EX)\cdot(EX - \bar{X})}_{②} + \underbrace{(EX - \bar{X})^2\big]}_{③}\Big\}$$

$$E\left[\frac{1}{n}\sum_{i=1}^{n}(X_i - EX)^2\right] = E(X_i - EX)^2 = \sigma^2$$

$$2E\left[\frac{1}{n}\sum_{i=1}^{n}(X_i - EX)\cdot(EX - \bar{X})\right] = -2E(EX - \bar{X})^2$$

$$E\left[\frac{1}{n}\sum_{i=1}^{n}(EX - \bar{X})^2\right] = E(EX - \bar{X})^2$$

$$-E[EX - \bar{X}]^2 = -E[\bar{X} - EX]^2$$

$$= -\frac{1}{n}\sigma^2$$

#

4.

(a) the sampling distribution of $n\hat{\sigma}^2/\sigma^2$.

(b) $\mathbb{E}\hat{\sigma}^2$.

(c) $\mathrm{var}(\hat{\sigma}^2)$.

(d) $\mathrm{MSE}(\hat{\sigma}^2)$.

$\Rightarrow$ (a). $n\hat{\sigma}^2/\sigma^2 = \sum_{i=1}^{n} \left(\frac{X_i}{\sigma}\right)^2$

$\frac{X_i}{\sigma} \sim N(0,1)$

indepence.

χ^2_n

(b). $E\hat{\sigma}^2 = E\frac{1}{n}\sum_{i=1}^{n} X_i^2$

$= E X_i^2$ $\quad$ $X_i \sim$ Normal.

1). p.d.f. integrate.

2). $\frac{\mathrm{Var}\, X_i}{\sigma^2} = \frac{E X_i^2}{\sigma^2} - (E X_i)^2$ ✓

$\qquad\qquad\qquad\qquad\qquad\qquad 0$

$\boxed{\text{if } Z \sim \chi^2_n \text{ then } EZ = n,\ \mathrm{Var}\, Z = 2n}$

3). $\frac{n\hat{\sigma}^2}{\sigma^2} \sim \chi^2_n$, $E\frac{\cancel{n}\hat{\sigma}^2}{\sigma^2} = \cancel{n}$, $E\hat{\sigma}^2 = \sigma^2$

(c). $\mathrm{Var}\,\hat{\sigma}^2 = E(\hat{\sigma}^2 - E\hat{\sigma}^2)^2$ $\qquad \hat{\sigma}^4$

$\mathrm{Var}\left(\frac{n\hat{\sigma}^2}{\sigma^2}\right) = 2n = \mathrm{Var}(\hat{\sigma}^2)\cdot\frac{n^2}{\sigma^4} \Rightarrow \mathrm{Var}(\hat{\sigma}^2) = \frac{2\sigma^4}{n}$.

$\hookrightarrow \chi^2_n$

(d). $\underline{\mathrm{MSE} = \mathrm{Var}} + \mathrm{Bias}^2$

$\mathrm{Bias} \to \underline{E\hat{\sigma}^2 - \sigma^2} = 0.$

5. (a). We know $P(X_i = j) = \frac{e^{-\lambda} \lambda^j}{j!}$

then $f(X) = \prod_{i=1}^{n} \frac{e^{-\lambda} \lambda^{x_i}}{x_i !}$

$= \frac{e^{-\lambda n} \lambda^{\Sigma x_i}}{\prod_{i=1}^{n} [x_i !]}$

then $g(T(x) | \lambda) = e^{-\lambda n} \lambda^{\Sigma x_i}$

$h(x) = \frac{1}{\prod_{i=1}^{n} [x_i !]}$

the $T(x) = \sum_{i=1}^{n} x_i$ by factorization theorem.

since $\frac{f(x, \lambda)}{f(y, \lambda)} = \lambda^{\Sigma x_i - \Sigma y_i} \cdot \frac{\prod_{i=1}^{n} [y_i !]}{\prod_{i=1}^{n} [x_i !]}$ does not depend on λ

$\Updownarrow$

$\Sigma x_i = \Sigma y_i$ then minimal s.s.

(b). $P(X=0) = e^{-\lambda}$ and

$P(X \neq 0) = 1 - e^{-\lambda}$

then The who process is like Binomial distribution $B(n, \theta)$

one natural estimator is $\frac{\sum_{i=1}^{n} I\{X_i = 0\}}{n}$

$E \frac{\sum_{i=1}^{n} I\{X_i = 0\}}{n} = E\, I\{X_i = 0\} = P(X_i = 0) = \theta$ #

(c). No.

(d). (i) $E\hat{\theta}_2 = E[E(\hat{\theta}_1 | T)] = E\hat{\theta} = \theta$.

(ii) $MSE(\hat{\theta}_1) = E[(\hat{\theta}_1 - \hat{\theta}_2 + \hat{\theta}_2 - \theta)^2]$

$$= E[\hat{\theta}_1 - \hat{\theta}_2]^2 + E[\hat{\theta}_2 - \theta]^2 + \underbrace{2E(\hat{\theta}_1 - \hat{\theta}_2)(\hat{\theta}_2 - \theta)}$$

we know $\hat{\theta}_2 = E[\hat{\theta}_1 | T]$

$$E(\hat{\theta}_1 - \hat{\theta}_2)(\hat{\theta}_2 - \theta) = E[E(\hat{\theta}_1 - \hat{\theta}_2)(\hat{\theta}_2 - \theta) | T]$$

$$= E[(\hat{\theta}_2 - \theta) \cdot \underbrace{E[\hat{\theta}_1 - \hat{\theta}_2] | T}_{=0}]$$

then $MSE(\hat{\theta}_1) \geq MSE(\hat{\theta}_2)$.

(e). $E\left[\frac{\sum_{i=1}^{n} 1\{X_i = 0\}}{n} \middle| \sum_{i=1}^{n} X_i = t\right]$

$$= E\left[\sum_{i=1}^{n} 1\{X_i = 0\} \middle| \sum_{i=1}^{n} X_i = t\right] / n$$

$$= \sum_{i=1}^{n} P(X_i = 0 | \sum_{i=1}^{n} X_i = t) / n$$

$$= P(X_i = 0 | \sum_{i=1}^{n} X_i = t).$$

w.l.o.g.

$$= P(X_1 = 0 | \sum_{i=1}^{n} X_i = t)$$

$$= \frac{P(X_1 = 0 \ \& \ \sum_{i=2}^{n} X_i = t)}{P(\sum_{i=1}^{n} X_i = t)}$$

$$= \frac{P(X_1 = 0) \cdot P(\sum_{i=2}^{n} X_i = t)}{P(\sum_{i=1}^{n} X_i = t)}$$

$$= \frac{e^{-\lambda} \cdot e^{-(n-1)\lambda} ((n-1)\lambda)^t / t!}{e^{-n\lambda} (n\lambda)^t / t!}$$

$$= \left(\frac{n-1}{n} \right)^t$$

$$\hat{\theta}_2 = \left(\frac{n-1}{n} \right)^{\Sigma x_i}$$