ECON6190 Section 13

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Hypothesis test

- Hypothesis test is a decision based on data
- The decision either accepts $\mathbb{H}_0$ or rejects $\mathbb{H}_0$ in favor of $\mathbb{H}_1$
- Procedures of hypothesis testing
  - Construct a real valued function of the data called **test statistic**
$$T = T(X_1, X_2 \ldots X_n) \in \mathbb{R}$$
  - Pick a **critical region** C → How to pick critical region?
    - One sided test: $C = \{x : x > c\}$ for **critical value** c
    - Two sided test: $C = \{x : |x| > c\}$ for **critical value** c
  - State hypothesis test as the decision rule
$$\begin{array}{ll} \text{accept } \mathbb{H}_0 & \text{if } T \notin C \\ \text{reject } \mathbb{H}_0 & \text{if } T \in C \end{array}$$

Evaluation of hypothesis Test

$\boxed{\text{DEF}}$ (power function) The probability of rejection

$$\pi(F) = P(\text{reject } H_0 \mid F) = P(T \in C \mid F)$$

$\boxed{\text{DEF}}$ Type I error: reject H_0 under truth H_0

Size of a test: probability of type I error $P(\text{reject } H_0 \mid F_0) = \pi(F_0)$

Type II error: Accept H_0 under truth H_1

Power of a test: probability of reject H_0 under truth H_1 (1 − Type II error)

$$P(\text{reject } H_0 \mid F_1) = \pi(F_1)$$

- Type I & II error cannot be reduce simultaneously.
- Pick C that control size and then maximize power s.t. size constraint.

1. Let $X \sim \text{binomial}(5, \theta)$ with θ unknown. Consider testing $\mathbb{H}_0 : \theta = \frac{1}{2}$ versus $\mathbb{H}_1 : \theta > \frac{1}{2}$.

   (a) Consider test alpha that rejects $\mathbb{H}_0$ if and only if all "successes" are observed. Derive the power function of this test. Calculate its type I error. Express its type II error as a function of θ where $\theta > \frac{1}{2}$.

   (b) Consider an alternative test beta that rejects $\mathbb{H}_0$ if we observe $X = 3, 4$, or 5. Write down the power function of this test. Calculate its type I error. Express its type II error as a function of θ where $\theta > \frac{1}{2}$.

   (c) Between tests alpha and beta, which test has a smaller type I error? Which test has a smaller type II error? Which test would you prefer?

$X \sim \text{binomial}(5, \theta) \quad \Rightarrow \text{pmf } P(X=k) = \binom{5}{k}\theta^k(1-\theta)^{5-k}$

(a) $\pi_\alpha(F) = P(\text{all success} \mid F)$

$= \binom{5}{5}\theta^5(1-\theta)^0$

$= \theta^5$

<u>Type I error:</u> $\pi_\alpha(F_0) = \theta^5\big|_{\theta=1/2} = (\frac{1}{2})^5 = \frac{1}{32}$

<u>Type II error:</u> $1 - \pi_\alpha(F_1) = 1 - \theta^5$, where $\theta > 1/2$.

↳ when $\theta > 1/2$, $\theta^5 < 0.0313 \Rightarrow$ Type II error is larger

(b) $\pi_\beta(F) = P(X = 3, 4, 5 \mid F)$

$= \binom{5}{3}\theta^3(1-\theta)^2 + \binom{5}{4}\theta^4(1-\theta) + \binom{5}{5}\theta^5\theta^0$

$= 10\theta^3(1-\theta)^2 + 5\theta^4(1-\theta) + \theta^5$

<u>Type I error:</u> $\pi_\beta(F_0) = 10(1/2)^3(1/2)^2 + 5(1/2)^4(1/2) + (1/2)^5 = 1/2$

<u>Type II error:</u> $1 - \pi_\beta(F_1) = 1 - 10\theta^3(1-\theta)^2 - 5\theta^4(1-\theta) - \theta^5$, for $\theta > 1/2$

(c) $\pi_\alpha(F_0) = \frac{1}{32} < \pi_\beta(F_0) = 1/2 \Rightarrow$ Test alpha has smaller Type I error

$1 - \pi_\alpha(F_1) = 1 - \theta^5 > 1 - 10\theta^3(1-\theta)^2 - 5\theta^4(1-\theta) - \theta^5 = 1 - \pi_\beta(F_1)$

$\Rightarrow$ Test β has smaller type II error.

3. Take the model $X \sim N(\mu, 4)$. We want to test the null hypothesis $\mathbb{H}_0 : \mu = 20$ against $\mathbb{H}_1 : \mu > 20$. A sample of $n = 16$ independent realizations of X was collected, and the sample mean $\bar{X} = 20.5$.

   (a) Propose a test with size α equal to 1%. What is the condition for rejecting $\mathbb{H}_0$ for this test?

   (b) What is the p value of this test?

   (c) What is the condition for rejecting $\mathbb{H}_0$ with $\alpha = 1\%$ if we increase the size of the sample to $n = 25$?

   (d) We want a test with power 90% if $\mu = 21$. What is the size of the sample n needed for that? Explain briefly how n affects the power of the test.

   (e) Now consider the two-sided test $\mathbb{H}_0 : \mu = 20$ against $\mathbb{H}_1 : \mu \neq 20$. Write down the power function of the test if $\mu = 21$. Compare with (d). Do you need a larger or smaller n in order to achieve 90% power?

(a) If σ^2 is known, under H_0, $T = \dfrac{\bar{X} - 20}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1) \Rightarrow z$ test

Since $\sigma^2 = 4$, $n = 16$, $T = \dfrac{20.5 - 20}{\frac{2}{4}} = 1$.

A test with size 1% $\Rightarrow$ reject H_0, under H_0 with 1%.

$\Leftrightarrow$ reject H_0 if $\underbrace{T}_{1} > \underbrace{Z_{0.99}}_{\text{0.99th quantile of } N(0,1)}$

2.33

$\Rightarrow$ Do not reject H_0 with $\alpha = 1\%$

(b) $\boxed{\text{DEF}}$ p-value is $p = 1 - G(t \mid F_0)$

↖ sampling distribution of T (CDF)

$\nearrow \mathcal{N}(0,1)$

$$p = 1 - G(1 \mid \mu = 20)$$

$$= 1 - \Phi(1) = 1 - 0.8419 = 0.1587$$

↖ CDF of $\mathcal{N}(0,1)$

$\phi(\cdot)$ pdf of $\mathcal{N}(0,1)$

(c) Under H_0, $T = \dfrac{20.5 - 20}{\frac{2}{\sqrt{25}}} = \dfrac{0.5}{0.4} = 1.25$

A test with size 1% reject H_0 if $\underbrace{T}_{1.25} > \underbrace{z_{0.99}}_{2.33}$

Do not reject H_0 under $\alpha = 1\%$.

(d) With $H_0: \mu = 20$ against $H_1: \mu > 20$, the rejection rule at 1% significance level is reject H_0 if $T = \dfrac{\bar{X} - 20}{\frac{\sigma}{\sqrt{n}}} > 2.33$.

Choose sample size n s.t. power of test $= 0.9$.

The power at $\mu = 21$ is 90% if

$$P(\text{reject } H_0 \mid \mu = 21) = 0.9$$

$$\Leftrightarrow \quad P\left(\frac{\bar{X} - 20}{\frac{2}{\sqrt{n}}} > 2.33 \mid \mu = 21 \right) = 0.9$$

Under $\mu = 21$, $\dfrac{\bar{X} - 21}{\frac{2}{\sqrt{n}}} \sim \mathcal{N}(0,1)$, so

$$P\left(\frac{\bar{X} - 21 + 1}{\frac{2}{\sqrt{n}}} > 2.33 \;\middle|\; \mu = 21 \right)$$

$$= \underbrace{P}_{\mathcal{N}(0,1)}\left(\frac{\bar{X} - 21}{\frac{2}{\sqrt{n}}} > 2.33 - \frac{\sqrt{n}}{2} \;\middle|\; \mu = 21 \right), \quad \frac{\bar{X} - 21}{\frac{2}{\sqrt{n}}} \sim \mathcal{N}(0,1)$$

$$= 1 - \Phi\left(2.33 - \frac{\sqrt{n}}{2}\right)$$

We choose n s.t. $1 - \Phi\left(2.33 - \frac{\sqrt{n}}{2}\right) = 0.9$

$$\Leftrightarrow \quad \Phi\left(2.33-\frac{\sqrt{n}}{2}\right)=0.1$$

$$\Leftrightarrow \quad 2.33-\frac{\sqrt{n}}{2}=-1.28$$

$$\Rightarrow \quad n=(7.22)^2 \approx 53.$$

As $n\uparrow$, $\Phi(2.33-\sqrt{n}/2)\downarrow$, power $\uparrow$.

(e) Consider two sided test with size 1%.

With $H_0: \mu=20$, $H_1: \mu\neq 20$, the rejection rule at 1% level is

reject H_0 if $T=\left|\frac{\bar{X}-20}{\frac{\sigma}{\sqrt{n}}}\right| > Z_{1-0.005}=2.57$

Power function at $\mu=21$ is

$$P\left(\left|\frac{\bar{X}-20}{\frac{2}{\sqrt{n}}}\right|>2.57 \,\middle|\, \mu=21\right)$$

$$=P\left(\frac{\bar{X}-20}{\frac{2}{\sqrt{n}}}>2.57 \text{ or } \frac{\bar{X}-20}{\frac{2}{\sqrt{n}}}<-2.57 \,\middle|\, \mu=21\right)$$

$$=P\left(\frac{\bar{X}-21+1}{\frac{2}{\sqrt{n}}}>2.57 \text{ or } \frac{\bar{X}-21+1}{\frac{2}{\sqrt{n}}}<-2.57 \,\middle|\, \mu=21\right)$$

$$=P\left(\frac{\bar{X}-21}{\frac{2}{\sqrt{n}}}>2.57-\frac{\sqrt{n}}{2} \text{ or } \frac{\bar{X}-21}{\frac{2}{\sqrt{n}}}<-2.57-\sqrt{n}/2 \,\middle|\, \mu=21\right)$$

$$=1-\Phi(2.57-\sqrt{n}/2)+\Phi(-2.57-\sqrt{n}/2) \quad \cdots\cdots \quad (\Delta)$$

Solve n s.t. $(\Delta)=0.9$. $\Rightarrow n\approx 60.$