Problem Set 3

Due: TA Discussion, 13 September 2023.

1 Exercises from class notes

All from "2. Euclidean Topology.pdf".

Exercise 2. Prove that (arbitrary) union of open sets is open and that intersection of finitely many open sets is open. What about arbitrary intersections of open sets?

Exercise 3. Prove that the closed interval $[a, b]$ is indeed closed. (Feel free to use Exercise 1 once you've convinced yourself of it!)

Exercise 4. Prove that: arbitrary intersection of closed sets is closed and that union of finitely many closed sets is closed.[1] What about arbitrary unions of closed sets?

2 Additional Exercises

Exercise 1. In class, we proved the Bolzano-Weierstrass theorem (i.e., that every bounded sequence has a convergent subsequence; see Module 1) for sequences in $\mathbb{R}$. Use this result to extend the statement to sequences in $\mathbb{R}^d$.

Related to the notion of *compactness* is another property called *sequential compactness*.

Definition 1. A set $S \subset \mathbb{R}^d$ is *sequentially compact* if every sequence of points in S has a convergent subsequence converging to a point in S.

Compare this to the definition of *compactness*... they don't look like they have anything in common. But it turns out that in $\mathbb{R}^d$ (and in metric spaces in general), compactness and sequential compactness are equivalent (i.e., a set is compact if and only if it is sequentially compact)!

There is a way of stating the Bolzano-Weierstrass theorem that parallels the Heine-Borel theorem:

Theorem (Bolzano-Weierstrass)**.** *A set $S \subseteq \mathbb{R}^d$ is sequentially compact if and only if it is closed and bounded.*

[1] **Hint:** Recall De Morgan's laws.

This looks a little different from the version we proved, which said that bounded sequences have a convergent subsequence. But you can get to this new way of stating the result straightforwardly from the result in class.

Exercise 2. Using the statement of Bolzano-Weierstrass from class, prove this new version of the Bolzano-Weierstrass theorem.

Hint 1*:* Remember, this is an "if and only if" statement, unlike the version of Bolzano-Weierstrass we proved in class (which said "if a sequence is bounded, then the sequence has a convergent subsequence"). So you have to prove that closed and bounded implies sequential compactness *and* vice versa.

Hint 2*:* The "closed and bounded $\Rightarrow$ sequentially compact" direction just amounts to unpacking the definition of closedness and applying the statement of Bolzano-Weierstrass from class we already proved. As for the other direction, "sequentially compact $\Rightarrow$ closed" is again a step of unpacking the definitions. The only part that takes work is "sequentially compact $\Rightarrow$ bounded." Argue by contradiction: Suppose a set is sequentially compact but *not* bounded. You should be able to construct a sequence in S that does not even have a convergent subsequence, contradicting the supposed sequential compactness of S.

Remark 1. In $\mathbb{R}^d$, Heine-Borel and Bolzano-Weierstrass are two sides of the same coin. The sequentially compact sets in $\mathbb{R}^d$ are precisely the closed and bounded sets in $\mathbb{R}^d$ which are precisely the compact sets in $\mathbb{R}^d$.