ECON6190 Section 9

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3. Consider a sample of data $\{X_1, \dots X_n\}$, where

$$X_i = \mu + \sigma_i e_i, i = 1 \dots n,$$

where $\{e_i\}_{i=1}^n$ are iid and $\mathbb{E}[e_i] = 0$, $\text{var}(e_i) = 1$, $\{\sigma_i\}_{i=1}^n$ are n finite and positive constants, and $\mu \in \mathbb{R}$ is the parameter of interest.

(a) Let

$$\hat{\mu}_1 = \frac{1}{n}\sum_{i=1}^n X_i$$

be the sample mean estimator. Under what condition is $\hat{\mu}_1$ a consistent estimator of μ? Under what condition is $\hat{\mu}_1 - \mu = O_p(\frac{1}{\sqrt{n}})$?

(a) consistency : $\hat{\mu}_1 \xrightarrow{p} \mu$.

In general, $P(|\hat{\mu} - \mu| > \delta) \leq \frac{E[(\hat{\mu}-\mu)^2]}{\delta^2}$ MSE($\hat{\mu}$), $\forall \delta > 0$

If $\hat{\mu}_1$ is unbiased, we know by chebyshev inequality, if $\text{var}(\hat{\mu}_1) \to 0$, then $\hat{\mu}_1$ is consistent. (equivalent as showing $MSE(\hat{\mu}) \to 0$)

1) check if $\hat{\mu}_1$ is unbiased.

$$E[\hat{\mu}_1] = E\left[\frac{1}{n}\sum_{i=1}^n X_i\right] \overset{E[\cdot]\text{ linear}}{=} \frac{1}{n}\sum_{i=1}^n E[X_i]$$

$$\overset{\text{plug in } X_i}{=} \underbrace{\frac{1}{n}\sum_{i=1}^n E[\mu + \sigma_i e_i]}$$

constant, but vary by i

constant, not indexed by i

$\frac{1}{n}\sum_{i=1}^n \mu = \frac{1}{n}(n\cdot\mu)$

$$= \mu = \mu + \frac{1}{n}\sum_{i=1}^n \sigma_i \underbrace{E[e_i]}_{iid\ =0}$$

$$\overset{iid}{=} \mu$$

$\Rightarrow \hat{\mu}_1$ is an unbiased estimator for μ.

2) Derive condition for $\text{var}(\hat{\mu}) \to 0$.

$$\text{var}(\hat{\mu}_1) = \text{var}\left(\frac{1}{n}\sum_{i=1}^n (\mu + \sigma_i e_i)\right)$$

$$= \text{var}\left(\underbrace{\mu}_{\text{constant}} + \frac{1}{n}\sum_{i=1}^n \sigma_i e_i\right)$$

$$= \text{var}\left(\frac{1}{n}\sum_{i=1}^n \sigma_i e_i\right)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} var(\underbrace{\sigma_i}_{\text{constant}} e_i) \quad \text{where } cov(e_i, e_j)=0, \text{ b/c iid}$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} \sigma_i^2 \underbrace{var(e_i)}_{=1,\ iid}$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} \sigma_i^2 .$$

By chebyshev inequality, $\hat{\mu}_1 \xrightarrow{P} \mu$ if $\frac{1}{n^2}\sum_{i=1}^{n}\sigma_i^2 \to 0$ or equivalently $\frac{1}{n^2}\sum_{i=1}^{n}\sigma_i^2 = o(1)$.

$$\hat{\mu}_1 - \mu = O_p\left(\sqrt{MSE(\hat{\mu}_1)}\right)$$

$$= O_p\left(\sqrt{\frac{1}{n^2}\sum_{i=1}^{n}\sigma_i^2}\right)$$

$$= O_p\left(\sqrt{\frac{1}{n}\cdot \frac{1}{n}\sum_{i=1}^{n}\sigma_i^2}\right)$$

If $O(1)$, asymptotically bounded, then $\hat{\mu}_1 - \mu = O_p(\frac{1}{\sqrt{n}})$

Alternatively, WTS: $\hat{\mu}_1 - \mu = O_p(\frac{1}{\sqrt{n}}) \iff \sqrt{n}(\hat{\mu}_1 - \mu) = O_p(1)$

$$\sqrt{n}(\hat{\mu}_1 - \mu) = O_p\left(\cancel{\sqrt{n}} \cdot \underbrace{\frac{1}{\cancel{\sqrt{n}}} \cdot \sqrt{\frac{1}{n}\sum_{i=1}^{n}\sigma_i^2}}_{RMSE(\hat{\mu}_1)}\right)$$

$$= O_p\left(\sqrt{\underbrace{\frac{1}{n}\sum_{i=1}^{n}\sigma_i^2}_{\text{as long as bounded}}}\right)$$

$$= O_p(1)$$

(b) Let

$$\hat{\mu}_2 = \frac{\frac{1}{n}\sum_{i=1}^{n}\frac{X_i}{\sigma_i^2}}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sigma_i^2}}$$

be an alternative estimator of μ. Under what condition is $\hat{\mu}_2$ a consistent estimator of μ? Under what condition is $\hat{\mu}_2 - \mu = O_p(\frac{1}{\sqrt{n}})$?

Similarly to (a), check if $\hat{\mu}_2$ is unbiased.

$$E[\hat{\mu}_2] = E\left[\frac{\frac{1}{n}\sum_{i=1}^{n}\frac{X_i}{\sigma_i^2}}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sigma_i^2}}\right] = \frac{1}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sigma_i^2}} E\left[\frac{1}{n}\sum_{i=1}^{n}\frac{X_i}{\sigma_i^2}\right]$$

$$= \frac{1}{\cancel{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sigma_i^2}}} \cancel{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sigma_i^2}} \underbrace{E[X_i]}$$

iid, $E[X_i] = E[\mu + \sigma_i e_i]$
$= \mu + \sigma_i \underbrace{E[e_i]}_{0}$
$= \mu$

$$= \mu$$

find variance of $\hat{\mu}_2$

$$var(\hat{\mu}_2) = var\left(\frac{\frac{1}{n}\sum_{i=1}^{n}\frac{X_i}{\sigma_i^2}}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sigma_i^2}}\right) \overset{iid}{=} \frac{1}{(\cancel{\frac{1}{n}}\sum_{i=1}^{n}\frac{1}{\sigma_i^2})^{\cancel{2}}}\cdot\cancel{\frac{1}{n^2}}\sum_{i=1}^{n}\cancel{\frac{1}{(\sigma_i^2)^2}}\underbrace{var(X_i)}$$

by iid

$$var(x_i) = var(\mu + \sigma_i e_i)$$
$$= \sigma_i^2 \underbrace{var(e_i)}_{=1}$$
$$= \cancel{\sigma_i^2}$$

$$= \frac{1}{\sum_{i=1}^{n}\frac{1}{\sigma_i^2}}$$

$\Rightarrow \hat{\mu}_2$ is consistent if $\frac{1}{\sum_{i=1}^{n}\frac{1}{\sigma_i^2}} \to 0$.

$$\hat{\mu}_2 - \mu = O_p\left(\sqrt{MSE(\hat{\mu}_2)}\right)$$
$$= O_p\left(\sqrt{\frac{1}{\sum_{i=1}^{n}\frac{1}{\sigma_i^2}}}\right)$$
$$= O_p\left(\sqrt{\frac{1}{n}\cdot\frac{n}{\sum_{i=1}^{n}\frac{1}{\sigma_i^2}}}\right)$$

If $O(1)$, then $\hat{\mu}_2 - \mu = O_p(\frac{1}{\sqrt{n}})$

5. Let $\{X_1 \dots X_n\}$ be a sequence of i.i.d random variables with mean μ and and variance σ^2. Let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

(a) If $\mu \neq 0$, how would you approximate the distribution of $(\bar{X})^2$ in large samples as $n \to \infty$?

(b) If $\mu = 0$, how would you approximate the distribution of $(\bar{X})^2$ in large samples as $n \to \infty$?

☆: When you see questions about "approximating distribution", "asymptotic distribution" of an estimator, assume looking for a nondegenerate dist. after appropriate rescaling.

(a) Define $h(x) = x^2$, we want to derive the asymptotic dist. of $h(\bar{x})$

[you can directly apply results of delta method].

Theorem [Delta Method]

If $\sqrt{n}(\hat{\mu} - \mu) \xrightarrow{d} \xi$, and $h(\cdot)$ cont. diff. in a nbhd of μ, then

$$\sqrt{n}\left(h(\hat{\mu}) - h(\mu)\right) \xrightarrow{d} \mathbf{H}^T \xi,$$

where $\mathbf{H} = \frac{\partial}{\partial u} h(u)\big|_{u=\mu}$.

In particular, if $\sqrt{n}(\hat{\mu} - \mu) \xrightarrow{d} \mathcal{N}(0, V)$, then

$$\sqrt{n}\left(h(\hat{\mu})-h(\mu)\right) \xrightarrow{d} \mathcal{N}(0, \mathbf{H}^T V \mathbf{H})$$

If μ and h are both scalar, then

$$\sqrt{n}\left(h(\hat{\mu})-h(\mu)\right) \xrightarrow{d} \mathcal{N}\left(0, \left(\frac{\partial}{\partial u} h(u)\Big|_{u=\mu}\right)^2 V\right).$$

In this problem, $\frac{\partial}{\partial x} h(x)\Big|_{x=\mu} = 2\mu$. and by CLT, $\sqrt{n}(\bar{x}-\mu) \xrightarrow{d} \mathcal{N}(0, \sigma^2)$.

$$\Rightarrow \sqrt{n}\left((\bar{x})^2-\mu^2\right) \xrightarrow{d} \mathcal{N}(0, 4\mu^2\sigma^2)$$

[or you can work out delta method step by step]

By first-order Taylor expansion :

$$h(\bar{x}) = h(\mu) + \underbrace{\frac{\partial h(x)}{\partial x}\Big|_{x=\tilde{x}}}_{h'(\tilde{x})} (\bar{x}-\mu), \text{ for some } \tilde{x} \text{ in between } \bar{x} \text{ and } \mu. \quad \cdots\cdots \text{①}$$

- By CLT, we know that $\sqrt{n}(\bar{x}-\mu) \xrightarrow{d} \mathcal{N}(0, \sigma^2)$ for iid data
- Since $\bar{x} \xrightarrow{P} \mu$, by WLLN, and $\tilde{x}$ is between $\bar{x}$ and $\mu \Rightarrow \tilde{x} \xrightarrow{P} \mu$.

  Since $h'(x) = 2x$ is continuous, By CMT, $h'(\tilde{x}) \xrightarrow{P} h'(\mu)$.

$$\begin{aligned}
\Rightarrow \sqrt{n}\left((\bar{x})^2-\mu^2\right) &= \sqrt{n}\left(h(\bar{x})-h(\mu)\right) \\
&= \underbrace{h'(\tilde{x})}_{\xrightarrow{P} h'(\mu)} \underbrace{\sqrt{n}(\bar{x}-\mu)}_{\xrightarrow{d} \mathcal{N}(0,\sigma^2)} \\
&\xrightarrow{d} \underbrace{\underbrace{h'(\mu)}_{\text{constant}} \mathcal{N}(0, \sigma^2)}
\end{aligned}$$

Rearrange ① and multiply by $\sqrt{n}$

Still normal with mean $E[h'(\mu)\cdot 0] = 0$

variance $\underbrace{(\underbrace{h'(\mu)}_{=2\mu})^2}_{4\mu^2}\sigma^2 = 4\mu^2\sigma^2$

$$\Rightarrow \sqrt{n}\left((\bar{x})^2-\mu^2\right) \xrightarrow{d} \mathcal{N}(0, 4\mu^2\sigma^2) \quad \cdots\cdots \circledast$$

(b) Apply $\circledast$ when $\mu=0$, $\sqrt{n}(\bar{x}^2-0) \xrightarrow{d} \mathcal{N}(0, 0)$

$\Rightarrow \sqrt{n}(\bar{x})^2 \xrightarrow{d} 0 \iff \sqrt{n}(\bar{x})^2 = o_p(1)$

$\Rightarrow$ need a different normalization factor

CLT for $\bar{x}$ still hold: $\sqrt{n}(\bar{x}-0) \xrightarrow{d} \mathcal{N}(0, \sigma^2)$

$$\Rightarrow \sqrt{n}\,\bar{X} \xrightarrow{d} \mathcal{N}(0,\sigma^2)$$

$$\Rightarrow \frac{\sqrt{n}}{\sigma}\bar{X} \xrightarrow{d} \mathcal{N}(0,1)$$

By CMT, $\left(\frac{\sqrt{n}}{\sigma}\bar{X}\right)^2 \xrightarrow{d} \big(\underbrace{\mathcal{N}(0,1)}_{\chi^2_1}\big)^2$

$$\Rightarrow \frac{n}{\sigma^2}(\bar{X})^2 \xrightarrow{d} \chi^2_1 .$$