ECON 6190

Problem Set 9

Gabe Sekeres

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1. Let $X \sim \text{Binom}(5, \theta)$ with θ unknown. Consider testing $\mathbb{H}_0 : \theta = \frac{1}{2}$ versus $\mathbb{H}_1 : \theta > \frac{1}{2}$.

   (a) The power function of α is the probability of rejection:

   $$\pi(\alpha) = \mathbb{P}\{X_1 = 1, \dots, X_5 = 1 \mid \theta\} = \theta^5$$

   This test's Type I error is the probability of rejecting the null hypothesis when it is actually true:

   $$\mathbb{P}\left\{X_1 = 1, \dots, X_5 = 1 \middle| \theta = \frac{1}{2}\right\} = \frac{1}{32}$$

   This test's Type II error is the probability of not rejecting the null hypothesis when the truth is $\mathbb{H}_1$:

   $$\mathbb{P}\left\{\min_i X_i = 0 \middle| \theta > \frac{1}{2}\right\} = 1 - \mathbb{P}\left\{X_i = 1 \;\forall\, i \middle| \theta > \frac{1}{2}\right\} = 1 - \theta^5$$

   (b) The power function of β is the probability of rejection:

   $$\pi(\beta) = \mathbb{P}\left\{X \geq 3 \mid \theta\right\} = 1 - \mathbb{P}\{X \leq 2 \mid \theta\} = 1 - (\mathbb{P}\{X = 2 \mid \theta\} + \mathbb{P}\{X = 1 \mid \theta\} + \mathbb{P}\{X = 0 \mid \theta\})$$

   Thus,

   $$\pi(\beta) = 1 - \binom{5}{2}\theta^2(1-\theta)^3 - \binom{5}{1}\theta(1-\theta)^4 - (1-\theta)^5 = 1 - 10 \cdot \theta^2(1-\theta)^3 - 5 \cdot \theta(1-\theta)^4 - (1-\theta)^5$$

   This test's Type I error is the probability of rejecting the null hypothesis when it is actually true:

   $$\mathbb{P}\left\{X \geq 3 \mid \theta = \frac{1}{2}\right\} = 1 - \frac{10}{32} - \frac{5}{32} - \frac{1}{32} = 1 - \frac{16}{32} = \frac{1}{2}$$

   This test's Type II error is the probability of not rejecting the null hypothesis when the truth is $\mathbb{H}_1$:

   $$\mathbb{P}\left\{X \leq 2 \mid \theta > \frac{1}{2}\right\} = 1 - \pi(\beta) = 10 \cdot \theta^2(1-\theta)^3 - 5 \cdot \theta(1-\theta)^4 - (1-\theta)^5$$

   (c) Test β has the higher Type I error, meaning that it (necessarily) has a lower Type II error. I would prefer test α, as when considering the effects of a false result, I prefer that a false null hypothesis not be rejected to a true null hypothesis erroneously being rejected. For more on this, see McCloskey & Michaillat (2024).

2. Take the model $X \sim \mathcal{N}(\mu, \sigma^2)$ with σ^2 unknown. A sample of size $n = 4$ yields $\sum_{i=1}^4 X_i = 40$, $\sum_{i=1}^4 (X_i - \bar{X})^2 = 48$, where $\bar{X}$ is the sample average.

   (a) I propose the t-statistic, which is constructed as

   $$t = \frac{|\bar{X}_n - \mu|}{\sqrt{\frac{s_n^2}{n}}} = \frac{|\bar{X}_4 - 9|}{\sqrt{\frac{s_4^2}{4}}} \sim t_3$$

In this model, we have that

$$t = \frac{|10-9|}{\sqrt{\frac{16}{4}}} = \frac{1}{2}$$

The critical value comes from the two-sided t distribution with 3 degrees of freedom, so it's 3.182. Since $0.5 < 3.182$, we cannot reject the null.

Not drawing to scale, the t_3 distribution is

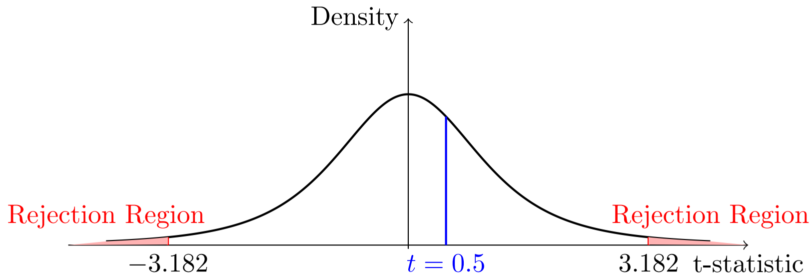

Figure 1: t_3 Distribution, when $t = 0.5$

(b) When $\mathbb{H}_0 : \mu = 7$ and $\mathbb{H}_1 : \mu > 7$, our test statistic becomes

$$t = \frac{\bar{X}_4 - \mu}{\sqrt{\frac{s_4^2}{4}}} = \frac{10-7}{2} = \frac{3}{2}$$

With a one-sided t test with 3 degrees of freedom, our critical value is now 2.353. Again, $\frac{3}{2} < 2.353$, so we fail to reject.

The one-sided t distribution is:

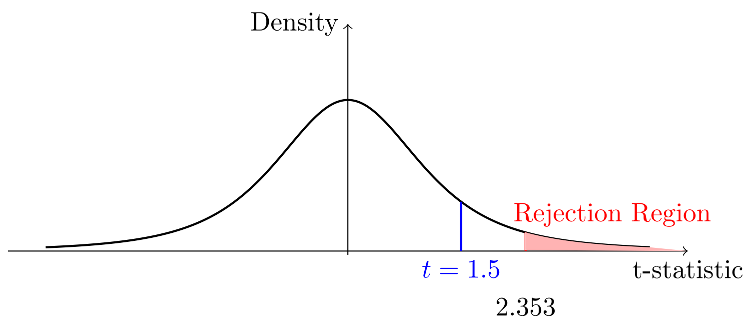

Figure 2: One-sided t_3 Distribution with $t = 1.5$

3. Take the model $X \sim \mathcal{N}(\mu, 4)$. We want to test the hypothesis $\mathbb{H}_0 : \mu = 20$ against $\mathbb{H}_1 : \mu > 20$. A sample of $n = 16$ independent realizations of X was collected, and the sample mean was $\bar{X} = 20.5$.

(a) Since $\sigma^2 = 4$ is known, we can use a z-test:

$$T = \frac{\bar{X}_n - \mu_0}{\sqrt{\frac{\sigma^2}{n}}} = \frac{0.5}{0.5} = 1$$

We can reject $\mathbb{H}_0$ under this test if $T > 2.33$, so this test does not reject the null.

(b) The p-value is:

$$p = 1 - G(T \mid F_0) = 1 - \Phi(1) = 1 - 0.84 = 0.16$$

(c) If we change n to 25, our critical value doesn't change, as it's based on the standard normal, so our test becomes

$$T = \frac{0.5}{\sqrt{\frac{4}{25}}} = \frac{5}{4} < 2.33$$

so we still cannot reject.

(d) We want to find n such that $\pi(T) = 0.9$, when $\mu = 21$. Note that the test statistic is

$$T = \frac{\bar{X} - 20}{\sqrt{4/n}} \sim \mathcal{N}\left(\frac{1}{\sqrt{4/n}}, 1\right)$$

We want the probability of rejecting $\mathbb{H}_0$ when $\mu = 21$ to be 0.9. That means that we want

$$\mathbb{P}\left\{\frac{\bar{X} - 20}{\sqrt{4/n}} > z_{0.01} \mid \mu = 21\right\} = 0.90$$

Substituting, we get

$$\mathbb{P}\left\{\frac{1}{\sqrt{4/n}} > 2.33\right\} = 0.90$$

From the standard tables, 0.90 corresponds to 1.28, so this becomes

$$\frac{1}{\sqrt{4/n}} - 2.33 = 1.28 \Longrightarrow n = 52.128$$

Thus, as long as $n \geq 53$, this test will be properly powered.

(e) The power function for a two-sided test is the same as in part (d), except that the critical value for a two-sided test is larger. It becomes

$$\pi(T) = \mathbb{P}\left\{\left|\frac{1}{\sqrt{4/n}}\right| > 2.575\right\}$$

Since all involved values are positive, we will need a strictly greater n to be powered at 90%.

4. To test whether the average rents in Madison and Ann Arbor are the same, denote r_m as the average rent in Madison and r_a as the average rent in Ann Arbor. Then we have that $\mathbb{H}_0 : r_m = r_a$, and $\mathbb{H}_1 : r_m \neq r_a$. Denote by $\bar{r}_m$ and $\bar{r}_a$ the sample means for Madison and Ann Arbor respectively. We define the pooled sample variance as follows:

$$s_p^2 = \frac{s_a^2 + s_m^2}{2}$$

where s_a^2 and s_m^2 are the sample variances for Ann Arbor and Madison respectively. Our estimator is

$$t = \frac{|\bar{r}_m - \bar{r}_a|}{\sqrt{s_p^2/n}} \sim t_{2n-2}$$

5. You design a statistical test of some hypothesis $\mathbb{H}_0$ which has asymptotic size 5% but you are unsure of the approximation in finite samples. You run a simulation experiment on your computer to check if the asymptotic distribution is a good approximation. You generate data which satisfies $\mathbb{H}_0$. On each simulated sample, you compute the test. Out of $B = 50$ independent trials you find 5 rejections and 45 acceptances.

(a) Our estimate of $\hat{p}$ is $\frac{5}{50} = 0.10$.

(b) This distribution follows a binomial distribution. The Central Limit Theorem tells us that

$$\hat{p} \xrightarrow{d} \mathcal{N}(p, p(1-p)/B)$$

Normalizing, we get that

$$\sqrt{B}(\hat{p} - p) \xrightarrow{d} \mathcal{N}(0, p(1-p))$$

(c) Our test statistic is

$$t = \frac{|\hat{p} - p|}{\sqrt{(\hat{p}(1-\hat{p}))/B}} = \sqrt{50}\frac{0.05}{\sqrt{0.09}} = 1.18$$

Since the critical value for a two-sided test is 1.96, we cannot reject the null.

6. One very striking abuse of hypothesis testing is to choose size α after seeing the data and to choose them in such a way as to force rejection (or acceptance) of a null hypothesis. To see what the true Type I and Type II error probabilities of such a procedure are, calculate size and power of the following two trivial tests.

(a) We have that the Type I error is:

$$\mathbb{P}\{\text{reject } \mathbb{H}_0 \mid \mathbb{H}_0\} = 1$$

and the Type II error is:

$$\mathbb{P}\{\text{accept } \mathbb{H}_0 \mid \mathbb{H}_1\} = 0$$

(b) We have that the Type I error is:

$$\mathbb{P}\{\text{reject } \mathbb{H}_0 \mid \mathbb{H}_0\} = 0$$

and the Type II error is:

$$\mathbb{P}\{\text{accept } \mathbb{H}_0 \mid \mathbb{H}_1\} = 1$$

7. Suppose $X \sim \mathcal{N}(\mu, \sigma^2)$ where both μ and σ^2 are unknown. We hope to use a random sample $\{X_i : i = 1, \ldots, n\}$ drawn from X to test hypothesis $\mathbb{H}_0 : \mu = \mu_0$ for some $\mu_0 \in \mathbb{R}$ against $\mathbb{H}_1 : \mu \neq \mu_0$.

(a) We have that $\beta = (\mu, \sigma^2)$. Under $\mathbb{H}_0$, $\beta = (\mu_0, \sigma^2)$. The log-likelihood is

$$\ell(\beta) = -\frac{n}{2}\log\left(2\pi\sigma^2\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i - \mu_0)^2$$