Part 3: Producer theory

ECON 6090
Cornell University
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Overview of next four lectures

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   - → Can you guess the key assumption we'll maintain here about firms' relation to the market?
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3. Intro to theory of non-price-taking firms
   - → In other words, intro to the theory of industrial organization.
   - → "Core IO" is the study of what happens when firms have some ability to affect prices.

Lecture 1: Producer theory

Technological feasibility

Assumptions 3.1:

(i) L commodities

(ii) Production plan $y \in \mathbb{R}^L$

- Net input: good i such that $y_i < 0$
- Net output: good j such that $y_j > 0$

(iii) Production possibility set, $Y \subseteq \mathbb{R}^L$ of feasible production plans

(iv) Prices, $p \geqslant 0$, are unaffected by the activity of the firm.

Assumptions 3.2:

(i) Y is nonempty, closed and (strictly) convex.

(ii) Free disposal: If $y \in Y$ and $y' \leqslant y$, then $y' \in Y$.

Efficiency

Definition: A production plan $y \in Y$ is *efficient* if there does not exist a $y' \in Y$ such that $y' \geqslant y$ and $y'_i > y_i$ for some i.

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Consider the case where there's only one output, i.e., $y = (q, -z)$ where $q \in \mathbb{R}_+$ and $z \in \mathbb{R}_+^{L-1}$.

Definition: The *production function* $f : \mathbb{R}^{L-1} \to \mathbb{R}_+$ is defined by

$$f(z) = \max_q q \quad \text{subject to} \quad (q, -z) \in Y$$

Related definitions

Definition: The *input requirement set*

$$V(q) \equiv \{z \in \mathbb{R}_+^{L-1} \mid (q, -z) \in Y\}$$

gives all the input vectors that can be used to produce output q.

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Definition: The *isoquant*

$$Q(q) \equiv \{z \in \mathbb{R}_{+}^{L-1} \mid z \in V(q) \text{ and } z \notin V(q') \text{ for any } q' > q\}$$

gives all the input vectors that can be used to produce at most q units of output.

Cost minimization

Assumptions 3.7:

(i) $L-1$ inputs in z

(ii) One output $q = f(z)$

(iii) $f \in C^2$

(iv) Input price $w \in \mathbb{R}_+^{L-1}$

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Definitions: The firm's *cost minimization problem* (CMP) is

$$C(w,q) = \min_{z \in \mathbb{R}^{L-1}} w \cdot z \text{ s.t. } f(z) = q$$

and the associated value function $C(w,q)$ is the **cost function**.

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Proposition 3.10 (Properties of the cost function)

(i) C is homogeneous degree 1 in w.

(ii) C is concave in w.

(iii) If we assume free disposal, then C is nondecreasing in q.

(iv) If f is homogeneous of degree k in z, the C is homogeneous of degree $1/k$ in q.

Properties of homogeneous functions

Proposition 3.12 If f is homogeneous degree k, then for $i = 1, \ldots, n$, $\frac{\partial f}{\partial x_i}$ is homogeneous of degree $k - 1$.

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$$\sum_i \frac{\partial f(x)}{\partial x_i} x_i = kf(x)$$

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$$\sum_i \frac{\partial f(x)}{\partial x_i} x_i = kf(x)$$

Proposition 3.14 If the production function f is homogeneous of degree k, then

$$\text{MRTS}_{ij}(z) \equiv \frac{\frac{\partial f(z)}{\partial z_i}}{\frac{\partial f(z)}{\partial z_j}} = \frac{\frac{\partial f(\alpha z)}{\partial z_i}}{\frac{\partial f(\alpha z)}{\partial z_j}} = \text{MRTS}_{ij}(\alpha z)$$

Profit maximization

The firm's *profit maximization problem* (PMP) is

$$\pi(p) \equiv \max_{y} p \cdot y \text{ subject to } y \in Y$$

and the associated value function $\pi(p)$ is the *profit function*.

Single-output case:

$$\pi(p, w) \equiv \max_{z \in \mathbb{R}^{L-1}} pf(z) - w \cdot z$$

Profit maximization

Proposition 3.16 (Properties of the profit function)

(i) Homogeneous of degree 1

(ii) Nondecreasing in output price p

(iii) Nonincreasing in input prices w

(iv) Convex in (p, w)

(v) Continuous

Profit maximization

Definitions: The *unconditional input demand function*

$$x(p, w) \equiv \arg \max_{z \in \mathbb{R}^{L-1}} pf(z) - w \cdot z$$

is the solution to the PMP. The *output supply function*

$$q(p, w) \equiv f(x(p, w))$$

is the output level when profit is maximized.

Profit maximization

Proposition 3.19 (Hotelling's lemma) If π is differentiable, then for $(p, w) \in \mathbb{R}^L_{++}$,

$$q(p, w) = \frac{\partial \pi(p, w)}{\partial p}$$
$$x_j(p, w) = -\frac{\partial \pi(p, w)}{\partial w_j}$$

Profit maximization

Definition The *conditional input demand function*

$$z(w, q) \equiv \arg \min_{z \in \mathbb{R}_+^{L-1}} w \cdot z \text{ s.t. } f(z) = q$$

is the solution to the CMP.

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Proposition 3.21 (Shephard's lemma). If C is differentiable, then for $w \in \mathbb{R}_{++}^{L-1}$

$$z_i(w, q) = \frac{\partial C(w, q)}{\partial w_i}$$

Profit maximization

Proposition 3.22 Suppose that the profit function is twice continuously differentiable. Then,

(i) $\frac{\partial q(p,w)}{\partial p} \geqslant 0$

(ii) $\frac{\partial x_j(p,w)}{\partial w_j} \leqslant 0$

(iii) $\frac{\partial x_j(p,w)}{\partial w_i} = \frac{\partial x_i(p,w)}{\partial w_j}$

Profit maximization

Proposition 3.23 Suppose that the cost function is twice continuously differentiable. Then,

(i) $\frac{\partial z_i(w,q)}{\partial w_i} \geqslant 0$

(ii) $\frac{\partial z_j(w,q)}{\partial w_i} = \frac{\partial z_i(w,q)}{\partial w_j}$

(iii) $\frac{\partial}{\partial w_i}\frac{\partial C(w,q)}{\partial q} = \frac{\partial z_i(w,q)}{\partial q} \Rightarrow \begin{cases} > 0 \text{ Normal input} \\ < 0 \text{ Inferior input} \end{cases}$

Comparative statics

Assumptions 3.24

(i) Two inputs (x_1, x_2)

(ii) One output $q = f(x)$

(iii) $f \in C^2$ and the Hessian H_f is negative definite.

(iv) $f(0, x_2) = f(x_1, 0) = 0$, i.e., both inputs are necessary.

(v) Inada conditions on x_1, x_2

(vi) Output price $p > 0$

(vii) Input prices $w >> 0$.

Comparative statics

Consider the profit maximization problem

$$\max_{x \in \mathbb{R}^2_{++}} \; pf(x) - w \cdot x$$

Exercise 1: Prove that $\partial x_1(p, w)/\partial w_1 < 0$.

Comparative statics

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First order conditions

$$pf_1(x) - w_1 = 0$$
$$pf_2(x) - w_2 = 0$$

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First order conditions

$$\begin{aligned} pf_1(x) - w_1 &= 0 \\ pf_2(x) - w_2 &= 0 \end{aligned}$$

Hessian of profit is

$$H(x) = pH_f(x)$$

which is invertible, so by Implicit Function Theorem, FOCs implicitly define $x(p, w) = (x_1(p, w), x_2(p, w))$, which is C^1 near (x, p, w).

Comparative statics

Then, we can rewrite FOCs as

$$pf_1(x(p, w)) - w_1 = 0$$
$$pf_2(x(p, w)) - w_2 = 0$$

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Taking derivatives with respect to w_1:

$$pf_{11}\frac{\partial x_1}{\partial w_1} + pf_{12}\frac{\partial x_2}{\partial w_1} = 1$$
$$pf_{21}\frac{\partial x_1}{\partial w_1} + pf_{22}\frac{\partial x_2}{\partial w_1} = 0$$

Comparative statics

In matrix form:

$$pH_f \begin{bmatrix} \frac{\partial x_1}{\partial w_1} \\ \frac{\partial x_2}{\partial w_1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Comparative statics

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Inverting gives

$$\begin{aligned} \begin{bmatrix} \frac{\partial x_1}{\partial w_1} \\ \frac{\partial x_2}{\partial w_1} \end{bmatrix} &= \frac{1}{p} \frac{1}{f_{11} f_{22} - f_{12} f_{21}} \begin{bmatrix} f_{22} & -f_{12} \\ -f_{21} & f_{11} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{p} \frac{1}{f_{11} f_{22} - f_{12} f_{21}} \begin{bmatrix} f_{22} \\ -f_{21} \end{bmatrix} \end{aligned}$$

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Note:

- $f_{11} f_{22} - f_{12} f_{21} > 0$. **Why?**

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Note:

- $f_{11} f_{22} - f_{12} f_{21} > 0$. **Why?**
- $f_{22} < 0$
- Therefore, $\frac{\partial x_1}{\partial w_1} < 0$

Comparative statics

Exercise 2: Prove that $\partial q/\partial w_1 > 0$.

Write output as

$$q(p, w) = f(x(p, w))$$

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Taking derivative with respect to w_1:

$$\begin{aligned} \frac{\partial q}{\partial w_1} &= f_1 \frac{\partial x_1}{\partial w_1} + f_2 \frac{\partial x_2}{\partial w_1} \\ &= \frac{1}{p} \frac{f_1 f_{22} - f_2 f_{21}}{f_{11} f_{22} - f_{12} f_{21}} \end{aligned}$$

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So

$$\text{sign}\left(\frac{\partial q}{\partial w_1} \right) = \text{sign}\left(f_1 f_{22} - f_2 f_{21} \right)$$

Comparative statics

To find $\text{sign}\left(f_1 f_{22} - f_2 f_{21}\right)$, we return to the cost minimization problem

$$\min_{x \in \mathbb{R}^2_{++}} \; w \cdot x \quad \text{s.t. } f(x) = q$$

Comparative statics

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Take FOCs of the Lagrangian

$$\begin{aligned} -w_1 + \lambda f_1(x) &= 0 \\ -w_2 + \lambda f_2(x) &= 0 \\ q - f(x) &= 0 \end{aligned}$$

where $\lambda = \lambda(w, q)$ is the Lagrange multiplier.

Comparative statics

Taking derivatives of these FOCs with respect to q

$$\begin{aligned}
\frac{\partial \lambda}{\partial q} f_1 + \lambda \frac{\partial f_1}{\partial x_1}\frac{\partial x_1}{\partial q} + \lambda \frac{\partial f_1}{\partial x_2}\frac{\partial x_2}{\partial q} &= 0 \\
\frac{\partial \lambda}{\partial q} f_2 + \lambda \frac{\partial f_2}{\partial x_1}\frac{\partial x_1}{\partial q} + \lambda \frac{\partial f_2}{\partial x_2}\frac{\partial x_2}{\partial q} &= 0 \\
1 - \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial q} - \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial q} &= 0
\end{aligned}$$

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$$1 - \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial q} - \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial q} = 0$$

In matrix form

$$\begin{bmatrix} \lambda f_{11} & \lambda f_{12} & f_1 \\ \lambda f_{21} & \lambda f_{22} & f_2 \\ f_1 & f_2 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial q} \\ \frac{\partial x_2}{\partial q} \\ \frac{\partial \lambda}{\partial q} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Comparative statics

And then by Cramer's rule,

$$\frac{\partial x_1}{\partial q} = \frac{\det\left(\begin{bmatrix} 0 & \lambda f_{12} & f_1 \\ 0 & \lambda f_{22} & f_2 \\ 1 & f_2 & 0 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} \lambda f_{11} & \lambda f_{12} & f_1 \\ \lambda f_{21} & \lambda f_{22} & f_2 \\ f_1 & f_2 & 0 \end{bmatrix}\right)} = \frac{\lambda\left(f_{12} f_2 - f_{22} f_1\right)}{\det\left(\begin{bmatrix} \lambda f_{11} & \lambda f_{12} & f_1 \\ \lambda f_{21} & \lambda f_{22} & f_2 \\ f_1 & f_2 & 0 \end{bmatrix}\right)}$$

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Note:

- The denominator is positive because the matrix is the Hessian of a convex function.
- We know that $\partial x_1 / \partial q$ is positive for "normal inputs". So in this case, $f_{12} f_2 - f_{22} f_1 > 0$.

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- Recall that $\operatorname{sign}\left(\frac{\partial q}{\partial w_1}\right) = \operatorname{sign}\left(f_1 f_{22} - f_2 f_{21}\right)$, so if input 1 is normal, $\frac{\partial q}{\partial w_1} > 0$.

Lecture 2: Producer theory review

Review of notation and definitions

Notation:

- Production plan $y \in \mathbb{R}^L$
  - Net input: good i such that $y_i < 0$
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$$f(z) = \max_q q \quad \text{subject to} \quad (q, -z) \in Y$$

- $y \in Y$ is *efficient* if $y = (f(z), -z)$ for some $z \in \mathbb{R}_+^{L-1}$. (This definition assumes that f is strictly increasing in every z_i.)

Two assumptions on firm behavior

The firm's **cost minimization problem** (CMP) is

$$\min_{z \in \mathbb{R}^{L-1}} w \cdot z \text{ s.t. } f(z) = q$$

The firm's **profit maximization problem** (PMP) is

$$\max_{z \in \mathbb{R}^{L-1}} pf(z) - w \cdot z$$

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- Value function: $C(w, q)$ is the *cost function*.

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- Value function: $\pi(p, w)$ is the *profit function*
- $q(p, w) = f(x(p, w))$ is the *output supply function*

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Two assumptions on firm behavior

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That is, assuming a firm minimizes cost is strictly weaker than assuming firm maximizes profit.

Profit maximization implies cost minimization

$$\pi(p, w) \equiv \max_{z \in \mathbb{R}^{L-1}} pf(z) - w \cdot z$$

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$$\begin{aligned}\pi(p, w) &\equiv \max_{z \in \mathbb{R}^{L-1}} pf(z) - w \cdot z \\ &= \max_q \left[\max_{z \in \mathbb{R}^{L-1}} pq - w \cdot z \text{ s.t. } f(z) = q \right]\end{aligned}$$

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Why is it worth studying CMP and PMP separately?

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In these cases, we may assume that choices of inputs are $z(w, q)$ for some q, but not necessarily $x(p, w)$.

Properties of CMP and PMP: Homogeneity

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Properties:

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- If f is homogeneous of degree k, then C is homogeneous degree $1/k$ in q.

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Properties of CMP and PMP: Convexity/concavity

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Proofs are symmetric. Only difference is that min versus max yields concavity versus convexity.

Suppose

$$x^*(\alpha) = \arg\max_x h(x, \alpha)$$

and the value function is

$$V(\alpha) = h(x^*(\alpha), \alpha).$$

Math review

Envelope Theorem

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Differentiating with respect to α gives

$$V'(\alpha) = h_x(x^*(\alpha), \alpha)x^{*\prime}(\alpha) + h_\alpha(x^*(\alpha), \alpha)$$

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Differentiating with respect to α gives

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Envelope Theorem with constraint

Suppose

$$x^*(\alpha) = \arg\max_x h(x, \alpha) \text{ s.t. } g(x) = 0$$

and the value function is

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Differentiating with respect to α gives

$$\begin{aligned} V'(\alpha) &= h_\alpha(x^*(\alpha), \alpha) + \lambda g_\alpha(x^*(\alpha)) \\ &= h_\alpha(x^*(\alpha), \alpha) \end{aligned}$$

Hotelling's Lemma

Statement

Proposition 3.19 (Hotelling's lemma) If π is differentiable, then for $(p, w) \in \mathbb{R}^L_{++}$,

$$q(p,w) = \frac{\partial \pi(p,w)}{\partial p}$$

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Both are simply applications of the Envelope Theorem!

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Theoretical implications:

- Symmetry of derivatives of unconditional input demand function:

$$\frac{\partial}{\partial w_i} x_j(p, w) = -\frac{\partial^2 \pi(p, w)}{\partial w_i \partial w_j} = -\frac{\partial^2 \pi(p, w)}{\partial w_j \partial w_i} = \frac{\partial}{\partial w_j} x_i(p, w)$$

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Theoretical implications:

- Signs of derivatives of unconditional input demand function:

$$\frac{\partial}{\partial w_j} x_j(p, w) = -\frac{\partial^2 \pi(p, w)}{\partial w_j^2} \leqslant 0$$

because π is convex, so its Hessian is positive definite, and positive definite matrices have non-negative diagonal entries.

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because π is convex, so its Hessian is positive definite, and positive definite matrices have non-negative diagonal entries. Likewise,

$$\frac{\partial}{\partial p} q(p, w) = \frac{\partial^2 \pi(p, w)}{\partial p^2} \geqslant 0$$

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Empirical implications

Proposition 3.19 (Hotelling's lemma) If π is differentiable, then for $(p, w) \in \mathbb{R}^L_{++}$,

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- Suppose you observe the response of profits to exogenous variation in input/output prices. Then, assuming profit maximization, you know the firm's input/quantity policy function.
- Or, vice versa.

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- Or, vice versa.
- Suppose there are two inputs and you observe how input choices respond to exogenous variation in w_1. You know how input choices respond to w_2.

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Statement

Proposition 3.21 (Shephard's lemma) If C is differentiable, then for $w \in \mathbb{R}^{L-1}_{++}$,

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- Sign of derivatives of conditional input demand function:

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Theoretical implications:

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If $\frac{\partial z_i(w,q)}{\partial q} > 0$, we call it a *normal input*; if $\frac{\partial z_i(w,q)}{\partial q} < 0$, we call it an *inferior input*.

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Empirical implications

Proposition 3.21 (Shephard's lemma) If C is differentiable, then for $w \in \mathbb{R}^{L-1}_{++}$,

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Empirical implications:

- If you observe how total costs respond to exogenous changes in input prices, then you know the input policy function (and under a weaker assumption than before!)

A broader view

Utility maximization

$$\max_{x \in \mathbb{R}^L_{++}} u(x) \text{ s.t. } p \cdot x \leqslant w$$

Cost minimization

$$\min_{z \in \mathbb{R}^{L-1}} w \cdot z \text{ s.t. } f(z) = q$$

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- They are both constrained optimization problems, so why don't the properties of utility maximization map directly onto cost minimization?

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- They are both constrained optimization problems, so why don't the properties of utility maximization map directly onto cost minimization?
- It matters whether it's the objective or the constraint that's linear and whether prices appear in the objective or in the constraint.
- But there are some direct analogs....

A broader view

Expenditure minimization

$$e(p, \bar{u}) = \min_{x \in \mathbb{R}^L_+} p \cdot x \text{ s.t. } u(x) \geqslant \bar{u}$$

Cost minimization

$$C(w, q) = \min_{z \in \mathbb{R}^{L-1}} w \cdot z \text{ s.t. } f(z) = q$$

A broader view

Expenditure minimization

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Proposition 2.55

(i) e is homogeneous degree 1 in p.

(ii) e is concave in p.

(iii) e is increasing in $\bar{u}$.

Cost minimization

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(iv) If f is homogeneous of degree k in z, the C is homogeneous of degree $1/k$ in q.

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(iii) C is nondecreasing in q.

(iv) If f is homogeneous of degree k in z, the C is homogeneous of degree $1/k$ in q.

— There's something called Shephard's Lemma for EMP and something called Shephard's Lemma for CMP. The two are exactly the same.

Non-price-taking firms

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- Input market power:

$$\max_{z \in \mathbb{R}^{L-1}} pf(z) - \mathbf{w}(\mathbf{z}) \cdot z \qquad \text{FOC: } pf_i(z) = \mathbf{w}_\mathbf{i}'(\mathbf{z}_\mathbf{i})\mathbf{z}_\mathbf{i} + \mathbf{w}_\mathbf{i}(\mathbf{z}_\mathbf{i})$$

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MRTS with and without market power

- No market power:

$$p \nabla f(z) = w \;\Rightarrow\; \underbrace{\frac{f_i(z)}{f_{i'}(z)}}_{\text{MRTS}} = \frac{w_i}{w_{i'}}$$

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$$pf_i(z) = w_i'(z_i)z_i + w_i(z_i) \;\Rightarrow\; \underbrace{\frac{f_i(z)}{f_{i'}(z)}}_{\text{MRTS}} = \frac{w_i'(z_i)z_i + w_i(z_i)}{w_{i'}'(z_{i'})z_{i'} + w_{i'}(z_{i'})}$$

Profit maximization implies cost minimization

$$
\begin{aligned}
\pi(p, w) &\equiv \max_{z \in \mathbb{R}^{L-1}} pf(z) - w \cdot z \\
&= \max_q \left[\max_{z \in \mathbb{R}^{L-1}} pq - w \cdot z \text{ s.t. } f(z) = q \right] \\
&= \max_q pq - \left[\underbrace{\min_{z \in \mathbb{R}^{L-1}} w \cdot z \text{ s.t. } f(z) = q}_{\text{CMP}} \right] \\
&= \max_q pq - C(w, q)
\end{aligned}
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Profit maximization implies cost minimization (with output market power)

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\pi(w) &\equiv \max_{z \in \mathbb{R}^{L-1}} \mathbf{p}(\mathbf{f}(\mathbf{z})) f(z) - w \cdot z \\
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Profit maximization implies cost minimization* (with input market power)

$$
\begin{aligned}
\pi(p, w) &\equiv \max_{z \in \mathbb{R}^{L-1}} pf(z) - \mathbf{w}(\mathbf{z}) \cdot z \\
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&= \max_q pq - \left[\underbrace{\min_{z \in \mathbb{R}^{L-1}} \mathbf{w}(\mathbf{z}) \cdot z \text{ s.t. } f(z) = q}_{\text{CMP}^*} \right] \\
&= \max_q pq - \mathbf{C}(\mathbf{q})
\end{aligned}
$$

Quantity choice under perfect competition

$$\pi(p, w) \equiv \max_q pq - C(w, q)$$

FOC:

$$p = \frac{\partial}{\partial q} C(w, q)$$

Price equals marginal cost. Zero profit on the marginal unit.

Quantity choice with output market power

Quantity choice:

$$\pi(w) \equiv \max_q p(q)q - C(w, q)$$

FOC:

$$\left[p(q^m) + p'(q^m)q^m\right] = \frac{\partial}{\partial q} C(w, q^m)$$

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Marginal revenue equals marginal cost.

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$$\Rightarrow \; p(q^m) = \frac{\partial}{\partial q}C(w,q^m) - \underbrace{p'(q^m)}_{<0}q^m$$

$$> \frac{\partial}{\partial q}C(w,q^m)$$

Positive profit on the marginal unit.

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Positive profit on the marginal unit. *How much profit?*

Price choice with output market power

Equivalently, price choice:

$$\max_{p} pD(p) - C\left(w, D(p)\right)$$

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$$\begin{aligned} p^m - \frac{\partial}{\partial q} C(w, q^m) &= -\frac{D(p^m)}{D'(p^m)} \\ \frac{p^m - \frac{\partial}{\partial q} C(w, D(p^m))}{p^m} &= -\frac{D(p^m)}{D'(p^m) p^m} \\ \text{"Lerner Index":} \qquad L &= -\frac{1}{\epsilon} \end{aligned}$$

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Question: What happens in the limiting cases (perfectly elastic and perfectly inelastic demand)?

Input choice with input market power

Let's make our lives easier by simplifying the problem: Suppose there's only one input (or at least, there's only one input market in which the firm has market power).

$$\max_{z} pf(z) - w(z)z$$

Since $w(z)$ is increasing, we can define its inverse $z(w)$ and rewrite the problem as

$$\max_{w} pf\left(z(w)\right) - w \cdot z(w)$$

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FOC:

$$\begin{aligned}
pf'(z(w))z'(w) &= z'(w)w + z(w) \\
p\frac{f'(z(w))}{w} &= \frac{z(w)}{z'(w)w} + 1 \\
p\frac{f'(z(w))}{w} &= \frac{1}{\epsilon_{z,w}} + 1 = \frac{1+\epsilon_{z,w}}{\epsilon_{z,w}} \\
w &= \left(\frac{\epsilon_{z,w}}{1+\epsilon_{z,w}}\right) pf'(z(w)) < pf'(z(w))
\end{aligned}$$

where $\epsilon_{z,w} > 0$ is the elasticity of input supply.

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  - If $D(p)$ is the demand for pickup trucks, then no.
  - But it is useful for thinking about oligopoly if $D(p)$ is the *residual demand* for pickup trucks taking other products' prices as fixed.
  - In this case, we can use monopoly pricing to derive Ford's *best-response* pricing of F-150 taking all other trucks' prices as given.

Oligopolist as monopolist facing residual demand curve

Suppose the only two pickup trucks available are Ford F-150 and Chevy Silverado. Demand curve is

$$D_k(p_k, p_{-k}) = 1 - p_k + 0.5p_{-k}$$

for each $k \in \{F, C\}$.

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Assuming constant marginal costs, Ford's choice taking p_C as given is

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FOC is

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This is Ford's *best response* to Chevy's choice of price.

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— What point on the demand curve does monopolist choose?

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$$p^m = \left(\frac{1+\epsilon}{\epsilon}\right) c'$$

$$p^m > 0 \Leftrightarrow \frac{1+\epsilon}{\epsilon} > 0$$

$$\Leftrightarrow \epsilon < -1$$

Elastic part of the demand curve.

$\rightarrow$ As long as demand is inelastic, $\frac{\partial \pi}{\partial p} > 0$, so increase price (i.e., decrease quantity) until you get to an elastic part of the demand curve.

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— p^m is weakly increasing in marginal cost.

Proof: p^m is weakly increasing in marginal cost.

- Suppose $c_2'(q) > c_1'(q)$ for all $q > 0$.
- Let (p_1, q_1) and (p_2, q_2) denote the corresponding monopoly prices and quantities.
  - **Key idea**: Both (p_1, q_1) and (p_2, q_2) are points on the demand curve, so both feasible for both monopolists.

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- Combining the two:

$$[c_2(q_1) - c_1(q_1)] - [c_2(q_2) - c_1(q_2)] \geqslant 0$$

which implies

$$\int_{q_2}^{q_1} \underbrace{[c_2'(x) - c_1'(x)]}_{>0\ \forall x} dx \geqslant 0$$

so $q_1 \geqslant q_2$, which means $p_1 \leqslant p_2$.