ECON 6170 Section 13

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Theorem 1 (Tarski). *Suppose*

(i) $(X, \geq)$ *is a complete lattice,*

(ii) and $f : X \to X$ *is nondecreasing.*

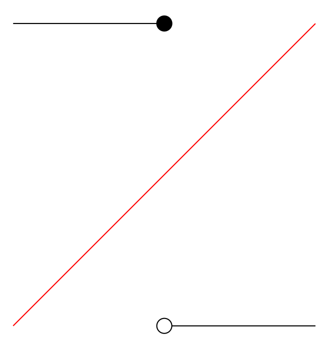
Then

(i) f *has a fixed point,*

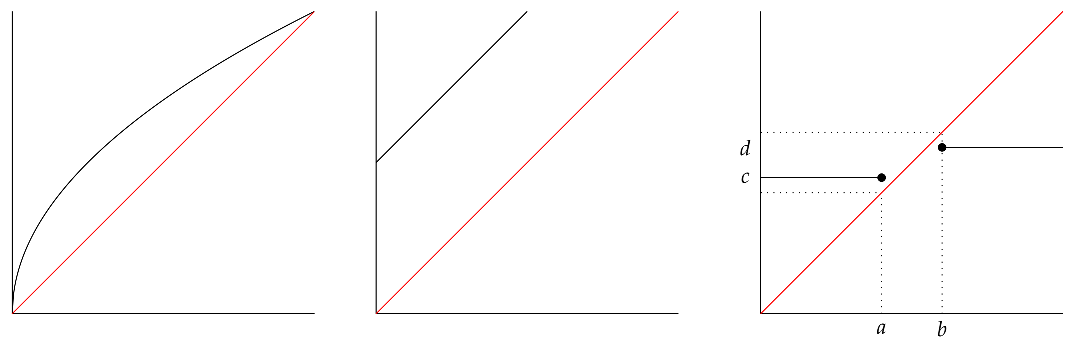
(ii) and the set of fixed points is a complete lattice.

Section Exercise 1. Show, by way of counterexample, that conclusion (i) does not necessarily follow if assumption (ii) is replaced by "$f : X \to X$ is nonincreasing".

An example is $f : [0, 1] \to [0, 1]$ given by $f(x) = \mathbf{1}\{x \leq 1/2\}$. More generally, any nonincreasing function defined on a subset of $\mathbb{R}$ is a counterexample if it's graph does not intersect that of $g(x) = x$.



Section Exercise 2. Explain why none of the following mappings into $[0, 1]$ are counterexamples to Tarski (in each case the red line is the graph of $g(x) = x$):



The first mapping has fixed points at 0 and 1. The second is not a self-mapping: it maps $[1/2, 1]$ onto $[0, 1/2]$. The third is also not as self-mapping: it maps $[0, 1] \setminus (a, b)$ onto $\{c, d\}$ where $b > d > c > a$.

Remark 1. The following material may be best reviewed after the lectures later this week.

Theorem 6 (Kakutani)**.** *Suppose*

(i) $X \subseteq \mathbb{R}^d$ is nonempty, compact and convex,

(ii) and $F : X \rightrightarrows X$ is nonempty-, closed-, convex-valued, and upper hemicontinuous.

Then F has a fixed point.

Section Exercise 3. For each of the following domains, identify a nonempty-, closed-, convex-valued, and upper hemicontinuous correspondence that does not have a fixed point:

(i) $X = \mathbb{R}$.
$F(x) := \{1 + x\}$

(ii) $X = (0, 1)$.
$F(x) := \{\sqrt{x}\}$

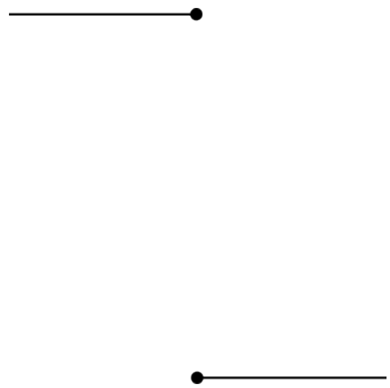
(iii) $X = [0, 1] \cup [2, 3]$.
$F(x) := \begin{cases} 5/2 & \text{if } x \in [0,1] \\ 1/2 & \text{if } x \in [2,3] \end{cases}$

Section Exercise 4. None of the following correspondences $F : [0, 1] \rightrightarrows [0, 1]$ have fixed points. For each, identify which hypothesis of Kakutani's theorem is violated.

(i)

$$F(x) := \begin{cases} \{1\} & \text{if } x < 1/2 \\ \{0, 1\} & \text{if } x = 1/2 \\ \{0\} & \text{if } x > 1/2 \end{cases}$$

F is not convex-valued at $1/2$.



(ii)

$$F(x) := \begin{cases} \emptyset & \text{if } 1/4 < x < 3/4 \\ \{1/2\} & \text{otherwise} \end{cases}$$

F is empty-valued on $(1/4, 3/4)$.

(iii)

$$F(x) := \begin{cases} [3/4, 1] & \text{if } x \leq 1/2 \\ [0, 1/4] & \text{if } x > 1/2 \end{cases}$$

F is not upper hemicontinuous at $x = 1/2$.