Econ 6190 Problem Set 5

Fall 2024

1. Consider a random variable Z_n with the probability distribution

$$Z_n = \begin{cases} -n & \text{with probability } \frac{1}{n} \\ 0 & \text{with probability } 1 - \frac{2}{n} \\ 2n & \text{with probability } \frac{1}{n} \end{cases}.$$

   (a) Does $Z_n \xrightarrow{p} 0$ as $n \to \infty$? Give your reasoning clearly.

   (b) Calculate $\mathbb{E}Z_n$. Does $\mathbb{E}Z_n \to 0$ as $n \to \infty$?

   (c) Calculate $\text{var}[Z_n]$.

2. Let X_n and Y_n be sequences of random variables, and let X be a random variable.

   (a) If $X_n \xrightarrow{p} c$ and $X_n - Y_n \xrightarrow{p} 0$, show $Y_n \xrightarrow{p} c$.

   (b) If $X_n \xrightarrow{p} X$ and a_n is a deterministic sequence such that $a_n \to a$, show that $a_n X_n \xrightarrow{p} aX$.

   (c) If $X_n \xrightarrow{p} 0$, show that $\frac{\sin X_n}{X_n} \xrightarrow{p} 1$.

3. Let X be a random variable and let A be a set in $\mathbb{R}$. Show that $\mathbb{E}[\mathbf{1}\{X \in A\}] = P\{X \in A\}$, where

$$\mathbf{1}\{X \in A\} = \begin{cases} 1 & \text{if } X \in A \\ 0 & \text{if } X \notin A \end{cases}.$$

4. Let $\{X_1 \dots X_n\}$ be random sample.

   (a) Suppose X_i has pdf $f(x) = e^{-x+\theta}\mathbf{1}\{x \geq \theta\}$ for some constant θ. Show that

$$\min(X_1, X_2, \dots X_n) \xrightarrow{p} \theta.$$

   (b) Suppose X_i is $U[0, \theta]$ for some constant $\theta > 0$. Show that

$$\max(X_1, X_2, \dots X_n) \xrightarrow{p} \theta.$$

5. [Hansen 7.6] Take a random sample $\{X_1, ..., X_n\}$. Which of the following statistics converge in probability by the weak law of large numbers and continuous mapping theorem? For each, which moments are needed to exist?

(a) $\frac{1}{n}\sum_{i=1}^n X_i^2$,

(b) $\frac{1}{n}\sum_{i=1}^n X_i^3$,

(c) $\max_{i\le n} X_i$,

(d) $\frac{1}{n}\sum_{i=1}^n X_i^2 - \left(\frac{1}{n}\sum_{i=1}^n X_i\right)^2$,

(e) $\frac{\sum_{i=1}^n X_i^2}{\sum_{i=1}^n X_i}$ (assuming $\mathbb{E}X > 0$),

(f) $\mathbf{1}\{\frac{1}{n}\sum_{i=1}^n X_i > 0\}$,

(g) $\frac{1}{n}\sum_{i=1}^n X_i Y_i$.

6. [Hansen 7.7] A weighted sample mean takes the form $\bar{X}_n^* = \frac{1}{n}\sum_{i=1}^n w_i X_i$ for some non negative constants w_i satisfying $\frac{1}{n}\sum_{i=1}^n w_i = 1$. Assume X_i is i.i.d.

(a) Show that $\bar{X}_n^*$ is unbiased for $\mu = \mathbb{E}[X]$,

(b) Calculate $\operatorname{var}\left(\bar{X}_n^*\right)$,

(c) Show that a sufficient condition for $\bar{X}_n^* \xrightarrow{p} \mu$ is that $n^{-2}\sum_{i=1}^n w_i^2 \to 0$,

(d) Show that a sufficient condition for the condition in part (c) is $\frac{\max_{i\le n} w_i}{n} \to 0$ as $n \to \infty$.

1. (a). $\forall \varepsilon > 0$, $\forall \delta > 0$

$$P(|Z_n - 0| > \varepsilon)$$

$$\leq \frac{2}{n} < \delta \quad \text{when} \quad n > \frac{2}{\delta}$$

then we know $Z_n \xrightarrow{p} 0$

(b). $$E Z_n = -n \cdot \frac{1}{n} + 0 \cdot \left(1 - \frac{2}{n}\right) + 2n \cdot \frac{1}{n}$$

$$= -1 + 2 = 1 \neq 0 \quad \text{as} \quad n \to \infty$$

(c). $$\text{Var } Z_n = E Z_n^2 - (E Z_n)^2$$

$$E Z_n^2 = n^2 \cdot \frac{1}{n} + 0 \cdot \left(1 - \frac{2}{n}\right) + 4n^2 \cdot \frac{1}{n}$$

$$= 5n$$

$$\text{Var } Z_n = 5n - 1$$

2. (a) let $Z_n = Y_n - X_n$, then $Y_n = X_n + Z_n$

$$Y_n = X_n + Z_n \xrightarrow{p} \text{plim}(X_n) + \text{plim}(Z_n)$$

$$= \text{plim}(X_n) + \text{plim}(Y_n - X_n)$$

$$= c + 0 = c$$

(b). $a_n \to a \Rightarrow a_n \xrightarrow{p} a$

since $X_n \xrightarrow{p} X$ and $a_n X_n$ is continuous in both a_n and X_n, then

$$a_n X_n \xrightarrow{p} a X$$

(c). firstly $g(x) = \dfrac{\sin x}{x}$ is continuous at $x = 0$

secondly since $X_n \xrightarrow{p} 0$

when $X_n \xrightarrow{p} 0$, $\dfrac{\sin X_n}{X_n} \xrightarrow{p} \dfrac{\cos(0)}{1} = 1$

3. $E[1\{X \in A\}] = 1 \cdot P(X \in A) + 0 \cdot P(X \notin A)$

$= P(X \in A)$

4. (a). let $Y_n = \min(X_1, \dots, X_n)$

$P(|Y_n - \theta| > \delta) = 1 - P(\theta - \delta \leq Y_n \leq \theta + \delta)$

$= 1 - F_{Y_n}(\theta + \delta) + F_{Y_n}(\theta - \delta)$

$P(\min(X_1, \dots, X_n) \leq c)$

$= 1 - P(\min(X_1, \dots, X_n) > c)$

$= 1 - P(X_1 > c) \cdot P(X_2 > c) \cdots P(X_n > c)$

$= 1 - \prod_{i=1}^{n} P(X_i > c)$

$= 1 - \prod_{i=1}^{n} (1 - P(X_i \leq c))$

$P(X_i \leq c) = \int_{-\infty}^{c} f(x)\, dx$

$= \int_{\theta}^{c} e^{-x+\theta}\, dx$

$= e^{\theta}(-e^{-c} + e^{-\theta})$ $[\text{if } c \geq \theta]$

$= 1 - e^{\theta - c}$

then $P(\min(X_1, \dots, X_n) \leq c)$ $[F_{Y_n}(c)]$

$= 1 - e^{n(\theta - c)}$

Then $F_{Y_n}(\theta - \delta) = 0$ (since $\theta - \delta < \theta$)

$F_{Y_n}(\theta + \delta) = 1 - e^{-n\delta} \to 1$

then $P(|Y_n - \theta| > \delta) \to 0$ #

(b). let $Z_n = \max(X_1, X_2, \dots, X_n)$

$P(|Z_n - \theta| > \delta) = 1 - F_{Z_n}(\theta + \delta) + F_{Z_n}(\theta - \delta)$

$F_{Z_n}(c) = P(\max(X_1, \dots, X_n) \leq c)$

$= \prod_{i=1}^{n} P(X_i \leq c)$

$= \prod_{i=1}^{n} \left(\frac{c}{\theta}\right)$ if $c \in [\theta, \theta]$

since $\theta + \delta > \theta$, then $F_{Z_n}(\theta + \delta) = 1$

$F_{Z_n}(\theta - \delta) = \left(\frac{\theta - \delta}{\theta}\right)^n \to 0$ (as $n \to \infty$)

$P(|Z_n - \theta| > \delta) \to 0$.

then $Z_n \xrightarrow{P} \theta$. #.

5.

The WLLN (or specifically, Khinchine's Weak Law of Large Numbers) says $\frac{1}{n}\sum_{i=1}^n X_i \xrightarrow{p} \mathbb{E}[X]$ if $\{X_i, i = 1\ldots n\}$ are i.i.d and $\mathbb{E}|X_i| = \mathbb{E}|X| < \infty$. Hence

(a) $\frac{1}{n}\sum_{i=1}^n X_i^2 \xrightarrow{p} \mathbb{E}[X^2]$ if $\mathbb{E}X_i^2 < \infty$. That is, we require the second moment to be finite

(b) $\frac{1}{n}\sum_{i=1}^n X_i^3 \xrightarrow{p} \mathbb{E}[X^3]$ if $\mathbb{E}|X_i|^3 < \infty$. We need third moment to be finite.

(c) $\max_{i\le n} X_i$ can not be written as an average and does not converge. If the support of X_i is bounded, say $|X_i| < \infty$, then for sure $\max_{i\le n} X_i$ is bounded too. In this case, we can say $\max_{i\le n} X_i = O_p(1)$.

(d) If $\mathbb{E}X_i^2 < \infty$,

$$\frac{1}{n}\sum_{i=1}^n X_i^2 \xrightarrow{p} \mathbb{E}[X^2], \frac{1}{n}\sum_{i=1}^n X_i \xrightarrow{p} \mathbb{E}[X]$$

and by continuous mapping theorem: $\frac{1}{n}\sum_{i=1}^n X_i^2 - \left(\frac{1}{n}\sum_{i=1}^n X_i\right)^2 \xrightarrow{p} \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \text{var}(X)$

(e) Similarly, if $\mathbb{E}X_i^2 < \infty$ and by WLLN and CMT:

$$\frac{\sum_{i=1}^n X_i^2}{\sum_{i=1}^n X_i} = \frac{\frac{1}{n}\sum_{i=1}^n X_i^2}{\frac{1}{n}\sum_{i=1}^n X_i} \xrightarrow{p} \frac{\mathbb{E}[X^2]}{(\mathbb{E}[X])^2},$$

provided $\mathbb{E}X > 0$

(f) If $\mathbb{E}|X_i| < \infty$, $\frac{1}{n}\sum_{i=1}^n X_i \to \mathbb{E}X$. Note the function $1\{u > 0\}$ is continuous for all points except 0. By CMT (specifically in this case, Slutsky's Theorem), as long as $\mathbb{E}X \neq 0$,

$$\mathbf{1}\{\frac{1}{n}\sum_{i=1}^n X_i > 0\} \xrightarrow{p} \mathbf{1}\{\mathbb{E}X > 0\}$$

(g) $\frac{1}{n}\sum_{i=1}^n X_iY_i \xrightarrow{p} \mathbb{E}XY$ if $\mathbb{E}|XY| < \infty$. Since by Cauchy-Schwarz inequality

$$\mathbb{E}|XY| \le \sqrt{\mathbb{E}X^2}\sqrt{\mathbb{E}Y^2},$$

a sufficient condition for $\mathbb{E}|XY| < \infty$ is $\mathbb{E}X^2 < \infty$ and $\mathbb{E}Y^2 < \infty$. That is, we require both X and Y to have finite second moment.

6.

(a) Note $\mathbb{E}\bar{X}_n^* = \mathbb{E}\frac{1}{n}\sum_{i=1}^n w_iX_i = \frac{1}{n}\sum_{i=1}^n w_i\mathbb{E}X_i = \frac{1}{n}\sum_{i=1}^n w_i\mu = \mu\frac{1}{n}\sum_{i=1}^n w_i = \mu\cdot 1 = \mu$, where the first equality is by definition of $\bar{X}_n^*$, the second equality holds by linearity of expectations and because $w_i, i = 1\ldots n$ are constants, the third equality holds by random sampling assumption $\mathbb{E}X_i = \mathbb{E}X = \mu$, the fourth equality holds since μ is a constant so we can take it out of the summation, and fifth equality holds by assumption $\frac{1}{n}\sum_{i=1}^n w_i = 1$. Thus $\bar{X}_n^*$ is unbiased.

(b)

$$\begin{aligned}
\operatorname{var}\left(\bar{X}_n^*\right) &= \operatorname{var}(\frac{1}{n}\sum_{i=1}^n w_i X_i)) \\
&= \frac{1}{n^2}\operatorname{var}(\sum_{i=1}^n w_i X_i) \\
&= \frac{1}{n^2}\sum_{i=1}^n \operatorname{var}(w_i X_i) \\
&= \frac{1}{n^2}\sum_{i=1}^n w_i^2 \operatorname{var}(X_i) \\
&= \frac{\sigma^2}{n^2}\sum_{i=1}^n w_i^2,
\end{aligned}$$

where the first equality holds by definition of $\bar{X}_n^*$, the second equality uses algebra of variance, the third equality holds because by random sampling, $w_i X_i$ and $w_j X_j$ are independent for $i \neq j$ so all covariance terms are zero. The fourth equality uses variance algebra again, and the final equality holds by assuming $\operatorname{var}(X_i) = \sigma^2$ for some constant σ^2.

(c) By Chebyshev's inequality, $\bar{X}_n^* \xrightarrow{p} \mu$ if $\mathbb{E}[(\bar{X}_n^* - \mu)^2] \to 0$ as $n \to 0$. Since

$$\begin{aligned}
\mathbb{E}[(\bar{X}_n^* - \mu)^2] &= \operatorname{mse}(\bar{X}_n^*) \\
&= \left(\operatorname{bias}(\bar{X}_n^*)\right)^2 + \operatorname{var}(\bar{X}_n^*) \\
&= 0 + \frac{\sigma^2}{n^2}\sum_{i=1}^n w_i^2
\end{aligned}$$

where the last equality holds by answers to (a) and (b). Hence $\mathbb{E}[(\bar{X}_n^* - \mu)^2] \to 0$ if $\frac{1}{n^2}\sum_{i=1}^n w_i^2 \to 0$ as $n \to \infty$.

(d) Note $w_i, i = 1 \dots n$ are non-negative constants and $\frac{1}{n}\sum_{i=1}^n w_i = 1$. It follows

$$\begin{aligned}
\frac{1}{n^2}\sum_{i=1}^n w_i^2 &= \frac{1}{n^2}\sum_{i=1}^n w_i \cdot w_i \\
&\leq \frac{1}{n^2}\sum_{i=1}^n w_i \left(\max_{i\leq n} w_i\right) \\
&= \left(\max_{i\leq n} w_i\right)\frac{1}{n^2}\sum_{i=1}^n w_i \\
&= \left(\max_{i\leq n} w_i\right)\frac{1}{n}\frac{1}{n}\sum_{i=1}^n w_i \\
&= \left(\max_{i\leq n} w_i\right)\frac{1}{n}
\end{aligned}$$

Hence a sufficient condition for $n^{-2}\sum_{i=1}^n w_i^2 \to 0$ is $(\max_{i\leq n} w_i)\frac{1}{n} \to 0$, or $(\max_{i\leq n} w_i) = o(n)$.