ECON6190 Section 3

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3. [Hansen, 5.2, 5.3] For the standard normal density $\phi(x)$, show that $\phi'(x) = -x\phi(x)$. Then, use integration by parts to show that $\mathbb{E}[Z^2] = 1$ for $Z \sim \mathrm{N}(0,1)$.

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

(a) $$\phi'(x) = \frac{d}{dx}\left(\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\right)$$

$$\overset{\text{chain rule}}{=} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\left(-\tfrac{1}{2}\cdot 2x\right)$$

$$= -x \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$= -x\,\phi(x)$$

(b) WTS: $Z \sim \mathcal{N}(0,1)$, $E[Z^2] = 1$

$$\int u\,dv = uv - \int v\,du$$

Notice: $\frac{d(-\phi(z))}{dz} = z\phi(z)$

$\Rightarrow d(-\phi(z)) = z\phi(z)dz$

$$E[z^2] = \int \underbrace{z^2\phi(z)}_{z\cdot z\phi(z)}\,dz$$

$$= \int \underbrace{z}_{u}\, d\underbrace{(-\phi(z))}_{v}$$

$$= \underbrace{-z\phi(z)\Big|_{-\infty}^{\infty}}_{?} - \underbrace{\int -\phi(z)dz}_{\int \phi(z)dz = 1 \text{ b/c } \phi(z) \text{ density}} \quad \cdots\ (\star)$$

$$-\lim_{z\to\infty} z\phi(z) = -\lim_{z\to\infty} z\left(\frac{1}{\sqrt{2\pi}}e^{-z^2/2}\right) = -\lim_{z\to\infty} \frac{\overset{\to\infty}{z}}{\underset{\to\infty}{\sqrt{2\pi}e^{z^2/2}}}$$

By L'Hopital's rule, $\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$

$$-\lim_{z\to\infty}\frac{z}{\sqrt{2\pi}e^{z^2/2}} = -\lim_{z\to\infty}\frac{1}{\sqrt{2\pi}\underbrace{e^{z^2/2}\cdot z}_{\to\infty}} = 0$$

Similarly, can show $\lim_{z\to-\infty} -z\phi(z) = 0$.

So by $(\star)$, $E[z^2] = 1$.

4. [Mid term, 2023] If X is normal with mean μ and variance σ^2, it has the following pdf

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right), \text{ for } x \in \mathbb{R}.$$

Let X and Y be jointly normal with the joint pdf

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - 2\frac{\rho xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2}\right)\right), \text{ for } x, y \in \mathbb{R} \qquad (1)$$

where $\sigma_X > 0, \sigma_Y > 0$ and $-1 \leq \rho \leq 1$ are some constants.

(a) Without using the properties of jointly normal distributions, show that the marginal distribution of Y is normal with mean 0 and variance σ_Y^2.

$h(x) \to (a+b)^2$

$$f_Y(y) = \int f(x,y)dx$$

$$= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \int \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - 2\frac{\rho xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2}\right)\right) dx$$

$$= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \int \exp\left(-\frac{1}{2(1-\rho^2)}\left(\underbrace{\frac{x^2}{\sigma_X^2} - 2\frac{\rho xy}{\sigma_X\sigma_Y} + \frac{\rho^2y^2}{\sigma_Y^2}}_{\left(\frac{x}{\sigma_x}\right)^2 - 2\left(\frac{x}{\sigma_x}\right)\left(\frac{\rho y}{\sigma_Y}\right) + \left(\frac{\rho y}{\sigma_Y}\right)^2} - \frac{\rho^2y^2}{\sigma_Y^2} + \frac{y^2}{\sigma_Y^2}\right)\right) dx$$

$$= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \int \exp\left(-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x}{\sigma_X} - \frac{\rho y}{\sigma_Y}\right)^2 + \frac{(1-\rho^2)\,y^2}{\sigma_Y^2}\right)\right) dx$$

$$\exp\left(-\frac{1}{\cancel{2(1-\rho^2)}}\,\frac{\cancel{(1-\rho^2)}y^2}{\sigma_Y^2}\right)$$

$$= \frac{1}{2\pi\cancel{\sigma_X}\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2}\frac{y^2}{\sigma_Y^2}\right) \underbrace{\int \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x}{\sigma_X} - \frac{\rho y}{\sigma_Y}\right)^2\right) dx}_{\text{almost a density}}$$

Let $\frac{x}{\sigma_x} = t$. $\frac{dt}{dx} = \frac{1}{\sigma_x} \Rightarrow dt = \frac{1}{\sigma_x}dx$

$$\cancel{\sigma_x} \int \exp\left(-\frac{1}{2(1-\rho^2)}\left(t - \frac{\rho y}{\sigma_Y}\right)^2\right) dt$$

$$= \frac{1}{2\pi\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2}\frac{y^2}{\sigma_Y^2}\right) \int \exp\left(-\frac{1}{2\underbrace{(1-\rho^2)}_{\sigma^2}}\left(t - \overset{\mu}{\frac{\rho y}{\sigma_Y}}\right)^2\right) dt \text{(by change of variable)}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_Y} \exp\left(-\frac{1}{2}\frac{y^2}{\sigma_Y^2}\right) \underbrace{\frac{1}{\sqrt{2\pi}\underbrace{\sqrt{1-\rho^2}}_{=\sigma}} \int \exp\left(-\frac{1}{2(1-\rho^2)}\left(t - \frac{\rho y}{\sigma_Y}\right)^2\right) dt}_{}$$

$= 1$ b/c it's integrating a density

$T \sim N\left(\frac{\rho y}{\sigma_y}, 1-\rho^2\right)$ $\quad f(t) = \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2}\frac{\left(t - \frac{\rho y}{\sigma_y}\right)^2}{(1-\rho^2)}\right).$

$$= \frac{1}{\sqrt{2\pi}\sigma_Y} \exp\left(-\frac{1}{2}\left(\frac{y^2}{\sigma_Y^2}\right)\right),$$

(b) If you cannot work (a) out, assume it is true and move on. Derive the conditional distribution of X given $Y = y$. (Hint: it should be normal with mean $\frac{\sigma_X}{\sigma_Y}\rho y$ and variance $(1-\rho^2)\sigma_X^2$).

$$
\begin{aligned}
f_{X|Y}(x|y) &= \frac{f(x,y)}{f_Y(y)} \\
&= \frac{\frac{1}{\cancel{2}\pi\sigma_X\cancel{\sigma_Y}\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2}-2\frac{\rho xy}{\sigma_X\sigma_Y}+\frac{y^2}{\sigma_Y^2}\right)\right)}{\frac{1}{\cancel{\sqrt{2\pi}\sigma_Y}}\exp\left(-\frac{1}{2}\left(\frac{y^2}{\sigma_Y^2}\right)\right)} \\
&\quad \sqrt{2\pi} \\
&= \frac{1}{\sqrt{2\pi}\sigma_X\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2}-2\frac{\rho xy}{\sigma_X\sigma_Y}+\frac{y^2}{\sigma_Y^2}\right)+\frac{1}{2}\left(\frac{y^2}{\sigma_Y^2}\right)\right)
\end{aligned}
$$

$$\frac{1}{2}\left(\frac{y^2}{\sigma_Y^2}\right) = -\frac{1}{2(1-\rho^2)}\left(\frac{-(1-\rho^2)y^2}{\sigma_Y^2}\right)$$

$$-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2}-2\frac{\rho xy}{\sigma_X\sigma_Y}+\frac{y^2-(1-\rho^2)y^2}{\sigma_Y^2}\right)$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}\sigma_X\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2}-2\frac{\rho xy}{\sigma_X\sigma_Y}+\frac{y^2\rho^2}{\sigma_Y^2}\right)\right) \qquad (a+b)^2 \\
&= \frac{1}{\sqrt{2\pi}\sigma_X\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}\underbrace{\left(\frac{x}{\sigma_X}-\frac{y\rho}{\sigma_Y}\right)^2}\right)
\end{aligned}
$$

$$\left(\frac{1}{\sigma_X}\left(x-\frac{y\sigma_X\rho}{\sigma_Y}\right)\right)^2 = \frac{1}{\sigma_X^2}\left(x-\frac{y\sigma_X\rho}{\sigma_Y}\right)^2$$

$$= \frac{1}{\sqrt{2\pi}\sigma_X\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)\sigma_X^2}\left(x-\frac{y\sigma_X\rho}{\sigma_Y}\right)^2\right)$$

normal density with mean $= \dfrac{y\sigma_X\rho}{\sigma_Y}$

variance $= (1-\rho^2)\sigma_X^2$

(c) Let $Z = \frac{X}{\sigma_X} - \frac{\rho}{\sigma_Y}Y$. Show Y and Z are independent. Clearly state your reasoning. (Hint: For this question, you can use the properties of jointly normal distributions.)

Since Z is a linear transformation of X, Y,

and (X, Y) are jointly normal $\Rightarrow$ Z, Y are jointly normal.

$$
\begin{aligned}
\operatorname{cov}(Z, Y) &= \operatorname{cov}\left(\frac{X}{\sigma_X}-\frac{\rho}{\sigma_Y}Y,\ Y\right) \\
&= \frac{1}{\sigma_X}\underbrace{\operatorname{cov}(X,Y)}_{\rho\sigma_X\sigma_Y} - \frac{\rho}{\sigma_Y}\underbrace{\operatorname{cov}(Y,Y)}_{\operatorname{var}(Y)=\sigma_Y^2}
\end{aligned}
$$

$= 0.$

$\Rightarrow$ Since Z, Y jointly normal and $\text{cov}(Z,Y)=0$, $Z \perp Y$.

Question:

① Are two random variables, both marginally normally distributed, always jointly normal? NO.

② Are the linear combination of two random variables, both marginally normally distributed, always jointly normal? NO.

counterexample: $X \sim N(0,1)$

Let $W = \begin{cases} 1 & \text{w/ prob } 1/2 \\ -1 & \text{w/ prob } 1/2 \end{cases}$ and $W \perp X$.

$Y = WX$ is normal

can show $\text{cov}(X,Y)=0$: $\text{cov}(X,Y) = E[XWX] - \underbrace{E[X]}_{0}E[W]$

$$\overset{\text{LIE}}{=} E[E[X^2W|W]]$$

$$= E[W\underbrace{E[X^2|W]}_{=E[X^2] \text{ b/c } X \perp W}]$$

$$= E[X^2]\underbrace{E[W]}_{0} = 0$$

but note X, Y are not independent!

Consider $X+Y$. $P(X+Y=0) \overset{\text{by Law of total Probability}}{=} \underbrace{P(X+Y=0|W=1)}_{\substack{P(X+X=0) \\ =P(2X=0)=P(X=0)=0}}\underbrace{P(W=1)}_{1/2} + \underbrace{P(X+Y=0|W=-1)}_{\substack{P(X-X=0) \\ =1}}\underbrace{P(W=-1)}_{1/2} = 1/2.$

cont. variable

A normal distribution never has point mass 1/2 at 0.

$\Rightarrow$ $X+Y$ is def not normally distributed

$\Rightarrow$ X, Y are not jointly normal.

③ Are two independent random variables, both marginally normally distributed, always jointly normal? YES.

6. [Hansen 6.13] Let $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2$. Find the covariance of $\hat{\sigma}^2$ and $\bar{X}$. Under what condition is this zero? [Hint: This exercise shows that the zero correlation between the numerator and the denominator of the t ratio does not always hold when the random sample is not from a normal distribution].

quick answer: $s^2 = \frac{1}{n-1}\sum_{i=1}^n (x_i - \bar{x})^2$

standard result: $\operatorname{cov}(\bar{x}, s^2) = \frac{1}{n} E[(x-\mu)^3]$

$$\operatorname{cov}(\bar{x}, \underbrace{\tfrac{n-1}{n} s^2}_{\hat{\sigma}^2}) = \frac{n-1}{n} \cdot \frac{1}{n} E[(x-\mu)^3]$$

$$= \frac{n-1}{n^2} E[(x-\mu)^3]$$

$\operatorname{cov}(\bar{x}, \hat{\sigma}^2)$

$= E[(\bar{x} - \mu + \mu)\hat{\sigma}^2] - \overbrace{E[\bar{x}]}^{\mu} E[\hat{\sigma}^2]$

$= E[(\bar{x}-\mu)\hat{\sigma}^2] + \mu \cancel{E[\hat{\sigma}^2]} - \mu \cancel{E[\hat{\sigma}^2]}$

$= E\left[\left(\frac{1}{n}\sum_{i=1}^n (x_i-\mu)\right)\left(\underline{\frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2}\right)\right]$

$$= \frac{1}{n}\sum_{i=1}^n (x_i - \mu + \mu - \bar{x})^2$$

$$= \frac{1}{n}\sum_{i=1}^n (x_i-\mu)^2 + \underbrace{2(\mu-\bar{x})(x_i-\mu)}_{2(\mu-\bar{x})(\frac{1}{n}\sum x_i - \mu)} + \underline{(\mu-\bar{x})^2}$$

$$2(\mu-\bar{x})(\underbrace{\tfrac{1}{n}\sum x_i}_{\bar{x}} - \mu)$$

$$= \underline{-2(\mu-\bar{x})^2}$$

$$= \frac{1}{n}\sum_{i=1}^n (x_i-\mu)^2 - (\mu-\bar{x})^2$$

$= E\left[\left(\frac{1}{n}\sum_{i=1}^n (x_i-\mu)\right)\left(\frac{1}{n}\sum_{i=1}^n (x_i-\mu)^2 - (\mu-\bar{x})^2\right)\right]$ .... (1)

Notice the product of ∿∿ have terms with same index i, and cross terms with index i, j

<u>claim</u> $E[(x_i-\mu)(x_j-\mu)^2] = 0$ for $i \neq j$

<u>proof</u> $E[(x_i-\mu)(x_j-\mu)^2]$

$$= E[(x_i-\mu)x_j^2 - 2\mu x_j(x_i-\mu) + \mu^2(x_i-\mu)]$$

$$= \underbrace{E[(x_i-\mu)x_j^2]} - 2\mu \underbrace{E[x_j(x_i-\mu)]} + \mu^2 \underbrace{E[x_i-\mu]}_{=0 \text{ def.}}$$

$$\overset{LIE}{=} E[E[(x_i-\mu)x_j^2 \mid x_j]]$$

$$= E[x_j^2 \underbrace{E[(x_i-\mu)\mid x_j]}]$$

$$\text{iid} \;= E[(x_i-\mu)] = 0$$

$$= 0$$

b/c iid $cov(x_i, x_j) = 0$

$$= E[x_j]\underbrace{E[x_i-\mu]}_{=0}$$

$$= 0$$

$$= 0. \qquad \blacksquare$$

Alternatively, by theorem 4.4 in Hansen's book, since $x_i \perp x_j$,

$(x_i-\mu) \perp (x_j-\mu)^2$. $\Rightarrow E[(x_i-\mu)(x_j-\mu)^2] = \underbrace{E[(x_i-\mu)]}_{=0}E[(x_j-\mu)^2]$

$$= 0.$$

Back to equation ①:

$$= \frac{1}{n^2} E\left[\sum_{i=1}^{n}(x_i-\mu)(x_i-\mu)^2 - n\sum_{i=1}^{n}(x_i-\mu)(\mu-\bar{x})^2\right].$$

$$= \frac{1}{n^2} E\left[\sum_{i=1}^{n}(x_i-\mu)^3 - \underbrace{n^2 \cdot \underbrace{\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)}_{=(\bar{x}-\mu)}(\mu-\bar{x})^2}_{(\bar{x}-\mu)^3}\right]$$

$$= \frac{1}{n^2}\left(\underbrace{\sum_{i=1}^{n}E[(x_i-\mu)^3]}_{\text{iid} \Rightarrow nE[(x-\mu)^3]} - n^2 \underbrace{E[(\bar{x}-\mu)^3]}_{?}\right) \quad \cdots\cdots (\triangle)$$

$E[(\bar{x}-\mu)^3]$

$$= E\left[\left(\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)\right)^3\right]$$

$$= \frac{1}{n^3} E\left[\left(\sum_{i=1}^{n}(x_i-\mu)\right)^3\right]$$

$$= \frac{1}{n^3} E\left[\sum_{i=1}^{n}(x_i-\mu)^3 + 3\sum_{i<j}(x_i-\mu)^2(x_j-\mu) + 3\sum_{i<j}(x_i-\mu)(x_j-\mu)^2 + 6\sum_{i<j<k}(x_i-\mu)(x_j-\mu)(x_k-\mu)\right]$$

$$= \frac{1}{n^3}\left(E\left[\sum_{i=1}^{n}(X_i-\mu)^3\right] + 3\sum_{i<j} E\Big[\underbrace{(X_i-\mu)^2(X_j-\mu)}_{0}\Big] + 3\sum_{i<j} E\Big[\underbrace{(X_i-\mu)(X_j-\mu)^2}_{0}\Big] \right.$$

$$\left. + 6\sum_{i<j<k} \underbrace{E[(X_i-\mu)]}_{0}\underbrace{E[(X_j-\mu)]}_{0}\underbrace{E[(X_k-\mu)]}_{0} \right)$$

$$= \frac{1}{n^3}\left(nE[(X-\mu)^3]\right) \quad iid$$

$$= \frac{1}{n^2}E[(X-\mu)^3].$$

Back to (Δ)

$$= \frac{1}{n^2}\left(nE[(X-\mu)^3] - n^2\left(\frac{1}{n^2}E[(X-\mu)^3]\right)\right)$$

$$= \frac{n-1}{n^2}E[(X-\mu)^3]$$

$\mathrm{cov}(\bar{X}, \hat{\sigma}^2) = 0$ if the third central moment $= 0$.