ECON 6190

Problem Set 10

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1. We want to construct a 98% confidence interval, meaning a number s such that

$$\mathbb{P}\{\bar{X} - s \leq \mu \leq \bar{X} + s\} = 0.98$$

Since $n = 16$, this becomes

$$\mathbb{P}\left\{-2s \leq \frac{\bar{X} - \mu}{\sqrt{4/16}} \leq 2s\right\} = 0.98 \qquad \text{where } \frac{\bar{X} - \mu}{\sqrt{4/16}} \sim \mathcal{N}(0,1)$$

Using the quantiles of a standard normal distribution, we get that the critical value is $2.33 = 2s$, meaning that the confidence interval for μ is

$$[20.5 - 2.33/2, 20.5 + 2.33/2] = [19.335, 21.665]$$

2. You have the point estimate $\hat{\theta} = 0.45$ and standard errors $s(\hat{\theta}) = 0.28$. You are interested in $\beta = \exp(\theta)$.

   (a) Since exponentiation is continuous, by the Continuous Mapping Theorem we have that $\hat{\beta} = \exp(\hat{\theta}) = \exp(0.45) \approx 1.568$.

   (b) Using the Delta Method, we have that

   $$s(\hat{\beta}) = \frac{\partial}{\partial \hat{\theta}}\left[\exp(\hat{\theta})\right] s(\hat{\theta}) \approx 1.568 \cdot 0.28 = 0.43904$$

   (c) The asymptotic confidence interval is

   $$\left[\hat{\beta} - z_{0.975}s(\hat{\beta}), \hat{\beta} + z_{0.975}s(\hat{\beta})\right] = [1.568 - 1.96 \cdot 0.43904, 1.568 + 1.96 \cdot 0.43904] = [0.7056, 2.4304]$$

   (d) The confidence interval for θ is given by

   $$\left[\hat{\theta} - z_{0.975}s(\hat{\theta}), \hat{\theta} + z_{0.975}s(\hat{\theta})\right] = \left[0.45 - 1.96 \cdot 0.28, 0.45 + 1.96 \cdot 0.28\right] = \left[-0.0988, 0.9988\right]$$

   Based on this, the alternative confidence interval is

   $$\left[\exp(-0.0988), \exp(0.9988)\right] = \left[0.905, 2.715\right]$$

   This confidence interval is asymptotically valid since

   $$\mathbb{P}\{\exp(L) \leq \beta \leq \exp(U)\} = \mathbb{P}\{\exp(L) \leq \exp(\theta) \leq \exp(U)\} = \mathbb{P}\{L \leq \theta \leq U\} = 1 - \alpha$$

3. Answer the following

   (a) Note that

   $$\mathbb{P}\{L \leq X \leq U\} = \mathbb{P}\left\{\frac{L}{1000} \leq Y \leq \frac{U}{1000}\right\}$$

   So the interval is

   $$\left[\frac{L}{1000}, \frac{U}{1000}\right]$$

(b) Note that

$$\mathbb{P}\{\theta \in [L, U]\} = \mathbb{P}\{h(\theta) \in [h(L), h(U)]\} = \mathbb{P}\{\beta \in [h(L), h(U)]\} = 1 - \alpha$$

so the coverage probability is $1 - \alpha$, where the equality holds since h is monotonic. Since the coverage probability is $1 - \alpha$, C_β is a $1 - \alpha$ confidence interval.

4. Let the random variable X be normally distributed with mean μ and variance 1. You are given a random sample of 16 observations.

(a) We know that

$$\frac{\bar{X} - \mu}{1/\sqrt{16}} \sim \mathcal{N}(0, 1)$$

Since $z_{0.95} = 1.65$, we have that

$$\mathbb{P}\left\{\frac{\bar{X} - \mu}{1/\sqrt{16}} \leq 1.65\right\} = 0.95 \Longrightarrow \mathbb{P}\left\{\mu \geq \bar{X} - \frac{1.65}{4}\right\} = 0.95$$

Thus, our 95% confidence interval is

$$\left[\bar{X} - \frac{1.65}{4}, \infty\right)$$

(b) Similarly, since $z_{0.975} = 1.96$, we have that

$$\mathbb{P}\left\{\bar{X} - \frac{1.96}{4} \leq \mu \leq \bar{X} + \frac{1.96}{4}\right\} = 0.95$$

So our confidence interval is

$$\left[\bar{X} - \frac{1.96}{4}, \bar{X} + \frac{1.96}{4}\right]$$

(c) When $\mathbb{H}_0 : \mu = 0$ is rejected with a t-test with size 0.05, we have that since the t statistic is normally distributed under the null, we choose to reject if

$$\left|\frac{\bar{X} - \mu}{\sqrt{\sigma^2/n}}\right| > 1.96 \Longrightarrow \frac{|\bar{X}|}{1/4} > 1.96 \Longrightarrow |\bar{X}| > \frac{1.96}{4}$$

This is equivalent to the statement that $\bar{X} \notin [-1.96/4, 1.96/4]$, which is the confidence interval from above.

(d) Since we do not know the variance, instead of relying on the quantiles of a standard normal distribution, we would use an estimator for the sample variance s^2 instead of 1, and we would use quantiles from t_{99}.

5. Same as 3

6. We have that $U \sim U[0, 1]$, and

$$C = \begin{cases} \mathbb{R} & \text{if } U \leq 0.95 \\ \varnothing & \text{if } U > 0.95 \end{cases}$$

(a) We have that

$$\mathbb{P}\{\theta \in C\} = 0.95 \cdot \mathbb{P}\{\theta \in \mathbb{R}\} + 0.05 \cdot \mathbb{P}\{\theta \in \varnothing\}$$

So the coverage probability is 0.95.

(b) C is not a good choice for a confidence interval. Though it has a reasonable coverage probability, its length is often infinite. It is much larger than it needs to be.