Econ 6190 Mid Term Exam

10:10 am - 11:30 am, 5 October 2023

Instructions

This exam consists of 3 questions, not of equal length or difficulty. Answer all questions. Remember to always explain your answer. Good luck!

1. **[20 pts]** By using the axioms of probability and properties of set operations, show that Bonferroni's inequality holds, i.e., for any events A and B, $P\{A \cap B\} \geq P\{A\} + P\{B\} - 1$. If you need to use $P\{A \cup B\} = P\{A\} + P\{B\} - P\{A \cap B\}$, prove it first.

2. **[25 pts]** Let X and Y be any two random variables.

   (a) **[10 pts]** Show that as long as both X and Y have finite variances, the following relation holds: $Var(X - \mathbb{E}[X|Y]) = \mathbb{E}[Var(X|Y)]$.

   (b) **[15 pts]** Now, let X have pdf $f_X(x) = \frac{2}{9}(x+1), -1 \leq x \leq 2$. Find the pdf of $Y = X^2$.

3. **[55 pts]** If X is normal with mean μ and variance σ^2, it has the following pdf

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right), \text{ for } x \in \mathbb{R}.$$

   Let X and Y be jointly normal with the joint pdf

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - 2\frac{\rho xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2}\right)\right), \text{ for } x, y \in \mathbb{R} \quad (1)$$

   where $\sigma_X > 0, \sigma_Y > 0$ and $-1 \leq \rho \leq 1$ are some constants.

   (a) **[10 pts]** Without using the properties of jointly normal distributions, show that the marginal distribution of Y is normal with mean 0 and variance σ_Y^2.

   (b) **[10 pts]** If you cannot work (a) out, assume it is true and move on. Derive the conditional distribution of X given $Y = y$. (Hint: it should be normal with mean $\frac{\sigma_X}{\sigma_Y}\rho y$ and variance $(1-\rho^2)\sigma_X^2$).

(c) **[10 pts]** Let $Z = \frac{X}{\sigma_X} - \frac{\rho}{\sigma_Y} Y$. Show Y and Z are independent. Clearly state your reasoning. (Hint: For this question, you can use the properties of jointly normal distributions.)

(d) Now, suppose I observe a random sample $\{(X_i, Y_i)_{i=1}^n\}$ from the population distribution (1).

i. **[10 pts]** Find a sufficient statistic for the parameters of interest $(\sigma_X^2, \sigma_Y^2, \rho)$. Clearly state your reasoning.

ii. **[8 pts]** Let $\hat{\sigma}_Y^2 = \frac{1}{n-1} \sum_{i=1}^n Y_i^2$. Find the mean of $\hat{\sigma}_Y^2$ and the finite-sample distribution of $\hat{\sigma}_Y^2$.

iii. **[7 pts]** Let $s_Y^2 = \frac{1}{n-1} \sum_{i=1}^n \left(Y_i - \bar{Y}\right)^2$, where $\bar{Y} = \frac{1}{n} \sum_{i=1}^n Y_i$. Find the mean of s_Y^2 and the finite-sample distribution of s_Y^2.