ECON 6190

Problem Set 5

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1. Consider a random variable Z_n with probability distribution

$$Z_n = \begin{cases} -n & \text{with probability } \frac{1}{n} \\ 0 & \text{with probability } 1 - \frac{2}{n} \\ 2n & \text{with probability } \frac{1}{n} \end{cases}$$

   (a) We have that, as $n \to \infty$, for some $\delta > 0$,

$$\lim_{n\to\infty} \mathbb{P}\{|Z_n - 0| > \delta\} = \lim_{n\to\infty} \frac{1}{n} = 0$$

   and

$$\lim_{n\to\infty} \mathbb{P}\{|Z_n - 0| \le \delta\} = \lim_{n\to\infty} 1 - \frac{2}{n} + \frac{1}{n} = 1$$

   so $Z_n \xrightarrow{p} 0$.

   (b) We have that, for some n,

$$\mathbb{E}[Z_n] = \frac{1}{n} \cdot (-n) + \left(1 - \frac{2}{n}\right) \cdot 0 + \frac{1}{n} \cdot 2n = -1 + 2 = 1$$

   Since this is not dependent on n, as $n \to \infty$, $\mathbb{E}[Z_n] = 1 \neq 0$.

   (c) We have that

$$\text{Var}(Z_n) = \mathbb{E}[Z_n^2] - (\mathbb{E}[Z_n])^2 = \frac{1}{n} \cdot n^2 + \left(1 - \frac{2}{n}\right) \cdot 0 + \frac{1}{n} \cdot 4n^2 - 1 = 5n - 1$$

2. Let X_n and Y_n be sequences of random variables, and let X be a random variable.

   (a) We have that, from the Triangle Inequality

$$\lim_{n\to\infty} \mathbb{P}\{|Y_n - c| > \delta\} \le \lim_{n\to\infty} \mathbb{P}\{|Y_n - X_n| + |X_n - c| > \delta\}$$

   and since $X_n \xrightarrow{p} Y_n$ and $X_n \xrightarrow{p} c$, we have that

$$\lim_{n\to\infty} \mathbb{P}\{|Y_n - c| > \delta\} \le \lim_{n\to\infty} \mathbb{P}\{|X_n - Y_n| > \delta\} + \lim_{n\to\infty} \mathbb{P}\{|X_n - c| > \delta\} = 0$$

   So $Y_n \xrightarrow{p} c$.

   (b) Since $a_n \to a$, the function $a_n x$ approaches the continuous function $f(x) = ax$ as $n \to \infty$, so by Slutsky's Theorem, $a_n X_n \xrightarrow{p} aX$.

   (c) We have that $X_n \xrightarrow{p} 0$, so from Slutsky's Theorem, $\sin X_n \xrightarrow{p} \sin 0 = 0$. From the Continuous Mapping Theorem, we have that $\frac{\sin X_n}{X_n} \xrightarrow{p} \frac{\sin 0}{0} = \cos 0 = 1$.

3. We have that

$$\mathbb{E}[\mathbb{1}_{x\in A}] = \mathbb{P}\{x \in A\} \cdot 1 + \mathbb{P}\{x \notin A\} \cdot 0 = \mathbb{P}\{x \in A\}$$

4. Let $\{X_1, \ldots, X_n\}$ be a random sample.

   (a) We have that $f(x) = e^{-x+\theta}\mathbb{1}_{x\geq\theta}$. Note that

   $$F(x) = \int_{\infty}^{x} e^{-x+\theta}\mathbb{1}_{x\geq\theta}dx = e^{\theta}\int_{\theta}^{x} e^{-x}dx = e^{\theta}\left[-e^{-x}\right|_{\theta}^{x} = \left(1 - e^{\theta-x}\right)\mathbb{1}_{x\geq\theta}$$

   Thus,

   $$\mathbb{P}\left\{\min_i X_i \leq x\right\} = 1 - \mathbb{P}\left\{\min_i X_i > x\right\} = 1 - \prod_{i=1}^{n}\mathbb{P}\{X_i > x\} = 1 - e^{n(\theta-x)}\mathbb{1}_{x\geq\theta}$$

   So fixing $\delta > 0$,

   $$\mathbb{P}\left\{\left|\min_i X_i - \theta\right| \leq \delta\right\} = \mathbb{P}\left\{\min_i X_i \geq \theta + \delta\right\} + \mathbb{P}\left\{\min_i X_i \leq \theta - \delta\right\} = \mathbb{P}\left\{\min_i X_i \geq \theta + \delta\right\} = 1 - e^{-n\delta}$$

   and as $n \to \infty$, $1 - e^{-n\delta} \to 1$, so $\min_i X_i \xrightarrow{p} \theta$.

   (b) We have that $X_i \sim U[0, \theta]$. This means that

   $$\mathbb{P}\{X_i \leq x\} = \int_{-\infty}^{\infty}\frac{1}{\theta}\mathbb{1}_{0\leq x\leq\theta}dx = \int_{0}^{x}\frac{1}{\theta}dx = \frac{x}{\theta}\mathbb{1}_{x\in[0,\theta]}$$

   Thus,

   $$\mathbb{P}\left\{\max_i X_i \leq x\right\} = \mathbb{1}_{x\in[0,\theta]}\prod_{i=1}^{n}\frac{x}{\theta} = \left(\frac{x}{\theta}\right)^n\mathbb{1}_{x\in[0,\theta]}$$

   And fixing $\delta > 0$,

   $$\mathbb{P}\left\{\left|\max_i X_i - \theta\right| > \delta\right\} = \mathbb{P}\left\{\max_i X_i > \theta + \delta\right\} + \mathbb{P}\left\{\max_i X_i < \theta - \delta\right\} = \left(\frac{\theta - \delta}{\theta}\right)^n$$

   So as $n \to \infty$, $\mathbb{P}\{|\max_i X_i - \theta| > \delta\} \to 0$, meaning that $\max_i X_i \xrightarrow{p} \theta$.

5. Which of the following statistics converge in probability by the weak law of large numbers and the continuous mapping theorem? For each, which moments are required to exist?

   (a) $\frac{1}{n}\sum_{i=1}^{n} X_i^2$. Since $f(y) = y^2$ is a continuous function, by the continuous mapping theorem this converges in probability as long as the first moment exists.

   (b) $\frac{1}{n}\sum_{i=1}^{n} X_i^3$. Since $f(y) = y^3$ is a continuous function, by the continuous mapping theorem this converges in probability as long as the first moment exists.

   (c) $\max_{i\leq n} X_i$. Since $f(Y) = \max_i Y_i$ is a continuous function, by the continuous mapping theorem this converges in probability as long as the first and second moments exist.

   (d) $\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2$. If the first and second moments exist and are finite, this converges in probability, since variance is continuous as long as it is finite.

   (e) $\frac{\sum_{i=1}^{n} X_i^2}{\sum_{i=1}^{n} X_i}$ (Assuming that $\mathbb{E}\,X > 0$). This does not converge in probability, as it is not a continuous function.

   (f) $\mathbb{1}\left\{\frac{1}{n}\sum_{i=1}^{n} X_i > 0\right\}$. This does not converge in probability, as it is not a continuous function.

   (g) $\frac{1}{n}\sum_{i=1}^{n} X_iY_i$. As long as $\bar{Y}_n \xrightarrow{p} Y$ for some Y, this converges in probability as long as the first moment exists.

6. A weighted sample mean takes the form $\bar{X}_n^\star = \frac{1}{n}\sum_{i=1}^n w_i X_i$ for some non-negative constants w_i satisfying $\frac{1}{n}\sum_{i=1}^n w_i = 1$. Assume X_i is iid.

   (a) We have that

$$\text{bias}(\bar{X}_n^\star) = \mathbb{E}[\bar{X}_n^\star] - \mu = \frac{1}{n}\sum_{i=1}^n w_i\, \mathbb{E}[X_i] - \mu = \mu\frac{1}{n}\sum_{i=1}^n w_i - \mu = \mu - \mu = 0$$

   (b) We have that, defining $\sigma^2 = \text{Var}(X_i)\ \forall\ i$, since X_i are iid,

$$\text{Var}(\bar{X}_n^\star) = \frac{1}{n^2}\sum_{i=1}^n \text{Var}(w_i X_i) = \frac{1}{n^2}\sum_{i=1}^n w_i^2\, \text{Var}(X_i)$$

   (c) We want to show that $\bar{X}_n^\star \xrightarrow{p} \mu = \mathbb{E}[\bar{X}_n^\star]$. From the slides, a sufficient condition is that $\text{Var}(\bar{X}_n^\star) \to 0$. If $\frac{1}{n^2}\sum_{i=1}^n w_i^2 \to 0$, then since our sample is iid, we have that $\text{Var}(\bar{X}_n^\star) \to 0$, and thus by Chebyshev's Inequality, $\bar{X}_n^\star \xrightarrow{p} \mu$.

   (d) We have that $\frac{1}{n}\max_i w_i \to 0$ as $n \to \infty$. This means that as $n \to \infty$,

$$\frac{1}{n^2}\sum_{i=1}^n w_i^2 \le \frac{1}{n^2}\left(n \cdot \max_i w_i^2\right) = \left(\frac{1}{n}\max_i w_i\right)\max_i w_i \to 0 \cdot \max_i w_i = 0$$

   so thus, $\frac{1}{n^2}\sum_{i=1}^n w_i^2 \to 0$.