ECON6190 Section 14

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2. [Hansen 14.4] You have the point estimate $\hat{\theta} = 0.45$ and standard errors $s(\hat{\theta}) = 0.28$. You are interested in $\beta = \exp(\theta)$.

   (a) Find $\hat{\beta}$.

   (b) Use the delta method to find a standard error $s(\hat{\beta})$.

   (c) Use the above to calculate a 95% asymptotic confidence interval for $\hat{\beta}$.

   (d) Calculate a 95% asymptotic confidence interval $[L, U]$ for the original parameter θ. Calculate a 95% asymptotic confidence interval for β as $[\exp(L), \exp(U)]$. Can you explain why this is a valid choice? Compare this interval with your answer in (c).

(a) use plug-in estimator for β

$$\hat{\beta} = \exp(\hat{\theta}) = \exp(0.45) \approx 1.57$$

(b) Assume $\hat{\theta}$ is asymptotically normal, ie. $\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} \mathcal{N}(0, \underbrace{V_\theta}_{\text{asymptotic variance of } \theta})$

By delta method,

$$\sqrt{n}(\hat{\beta} - \beta) = \sqrt{n}\left(\exp(\hat{\theta}) - \exp(\theta)\right)$$

by Taylor expansion

$$= \sqrt{n} \exp(\tilde{\theta})(\hat{\theta} - \theta) \quad , \text{where } \tilde{\theta} \text{ in between } \hat{\theta} \text{ and } \theta$$

$$\xrightarrow{d} \mathcal{N}(0, \underbrace{(\exp(\theta))^2}_{= \exp(2\theta)V_\theta} V_\theta)$$

The standard error of $\hat{\beta}$ is

$$s(\hat{\beta}) = \sqrt{\frac{\exp(2\hat{\theta})\hat{V}_\theta}{n}} = \exp(\hat{\theta})\underbrace{\sqrt{\frac{\hat{V}_\theta}{n}}}_{s(\hat{\theta})} = \exp(\hat{\theta})\, s(\hat{\theta})$$

comes from $\sqrt{n}$ rescaling

$$= \exp(0.45)(0.28) \approx 0.44$$

(c) from (b) $\Rightarrow \sqrt{n}(\hat{\beta} - \beta) \xrightarrow{d} \mathcal{N}(0, \exp(2\theta)V_\theta)$

$\Rightarrow$ construct asymptotic CI by asymptotic pivotal quantities

$$\frac{\hat{\beta} - \beta}{\sqrt{\frac{\exp(2\theta)V_\theta}{n}}} \xrightarrow{d} \mathcal{N}(0,1)$$

asymptotic pivot

Denote $Z_{1-0.025}$ as the $(1-0.025)$th quantile of $\mathcal{N}(0,1)$

Then as $n \to \infty$,

$$P\left(-Z_{1-0.025} \leq \frac{\hat{\beta} - \beta}{\sqrt{\frac{\exp(2\theta)V_\theta}{n}}} \leq Z_{1-0.025}\right) \longrightarrow 1 - 0.05$$

$$P\left(\hat{\beta} - Z_{1-0.025}(0.44) \le \beta \le \hat{\beta} + Z_{1-0.025}(0.44)\right) \to 0.95$$

$\Rightarrow$ A 95% asymptotic CI for β is

$$\left[\hat{\beta} - Z_{1-0.025}(0.44),\ \hat{\beta} + Z_{1-0.025}(0.44)\right].$$

or approximately $[0.71, 2.43]$.

(d) ① Construct asymptotic 95% CI for θ.

Similarly, $[\hat{\theta} - Z_{1-0.025} S(\hat{\theta}),\ \hat{\theta} + Z_{1-0.025} S(\hat{\theta})] \approx [\overbrace{-0.099}^{L}, \overbrace{0.999}^{U}]$

② $\Rightarrow$ An alternative CI for β is

$$[\exp(L), \exp(U)] \approx [0.91, 2.72]$$

This is a valid 95% asymptotic CI because

$$P(\exp(L) \le \beta \le \exp(U))$$

$$= P(\exp(L) \le \exp(\theta) \le \exp(U)) \qquad \exp(\cdot) \text{ monotonically } \uparrow$$

$$= P(L \le \theta \le U) \to 1 - \alpha = 95\%.$$

$[\exp(L), \exp(U)] = [0.91, 2.72]$ is longer than $[0.71, 2.43]$.

4. Let the random variable X be normally distributed with mean μ and variance 1. You are given a random sample of 16 observations.

(a) Construct a one sided 95% confidence interval for μ that has form $[\hat{L}, \infty)$ for some statistic $\hat{L}$.

(b) Construct a two sided 95% confidence interval for μ.

(c) Show that the rejection of the null $\mathbb{H}_0 : \mu = 0$ against $\mathbb{H}_1 : \mu \neq 0$ with size 5% based on t test corresponds to the rejection of $\mathbb{H}_0 : \mu = 0$ when zero does not lie in the 95% confidence interval for μ constructed in part (b).

(d) How would your answers be affected when you would not have known the variance of the random variable?

$X \sim \mathcal{N}(\mu, 1) \qquad n = 16$

(a) Denote the sample average $\bar{X}_n$ as an estimator for μ.

$$\frac{\bar{X}_n - \mu}{\sqrt{\frac{\sigma^2}{n}}} \sim \mathcal{N}(0,1) \quad \Rightarrow \text{Construct CI using Pivotal quantity.}$$

Note: $Z_{1-0.05} = 1.65$

$$P\left(\frac{\bar{X}_n - \mu}{\sqrt{\frac{\sigma^2}{n}}} \le 1.65\right) = 0.95$$

$$P\left(\frac{\bar{X}_n-\mu}{\sqrt{\frac{1}{16}}} \leq 1.65\right) = 0.95$$

$$\Rightarrow P\left(\mu \geq \bar{X}_n - \frac{1.65}{4}\right) = 0.95$$

A one sided 95% CI for μ is $\left[\underbrace{\bar{X}_n - \frac{1.65}{4}}_{\hat{L}}, \infty\right)$

(b) Similarly, $\frac{\bar{X}_n-\mu}{1/4} \sim N(0,1)$, and $Z_{1-0.025} = 1.96$.

$$P\left(-Z_{1-0.025} \leq \frac{\bar{X}_n-\mu}{1/4} \leq Z_{1-0.025}\right) = 0.95$$

$$P\left(\bar{X}_n - \frac{1.96}{4} \leq \mu \leq \bar{X}_n + \frac{1.96}{4}\right) = 0.95$$

$\Rightarrow$ A two sided 95% CI for μ is $\left[\bar{X}_n - \frac{1.96}{4}, \bar{X}_n + \frac{1.96}{4}\right]$.

(c) WTS: the rejection rule is the same.

For the second part, 0 doesn't lie in 95% CI for μ in (b)

$$\Rightarrow 0 \notin \left[\bar{X}_n - \frac{1.96}{4}, \bar{X}_n + \frac{1.96}{4}\right]$$

$$\Leftrightarrow 0 < \bar{X}_n - \frac{1.96}{4} \quad \text{or} \quad 0 > \bar{X}_n + \frac{1.96}{4}$$

$$\Leftrightarrow \frac{\bar{X}_n}{1/4} > 1.96 \quad \text{or} \quad \frac{\bar{X}_n}{1/4} < -1.96 \quad \cdots\cdots ①$$

For the first part: conduct test $H_0: \mu=0$, $H_1: \mu \neq 0$, with size 5%.

reject H_0 under H_0.

Under H_0: $T = \frac{\bar{X}_n - 0}{\sqrt{\frac{1}{16}}} \sim N(0,1)$, $Z_{1-0.025} = 1.96$

reject if $\frac{\bar{X}_n}{1/4} < -1.96$ or $\frac{\bar{X}_n}{1/4} > 1.96 \quad \cdots\cdots ②$

①② are the same.

(d) variance unknown.

If variance unknown, we can estimate the variance using

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2.$$

Hence, by results from sampling chapter, we get the following finite sample distribution

$$\frac{\bar{X}_n-\mu}{\sqrt{S^2/n}} \sim t_{n-1}$$

$\Rightarrow$ By similar construction, 95% CI is

$$\left[\bar{X}_n - t_{15,1-0.025}\left(\frac{S}{4}\right),\ \bar{X}_n + t_{15,1-0.025}\left(\frac{S}{4}\right)\right].$$

(e) How would your answers be affected when you would not have known the variance of the random variable but the sample size is 100?

If $n = 100 \Rightarrow$ we have "large" sample,

$$\underbrace{\frac{\bar{X}_n - \mu}{\sqrt{S^2/n}} \xrightarrow{d} N(0,1)}_{\text{asymptotic approximation.}} \text{ when } n \text{ is large.}$$

A valid 95% CI is $\left[\bar{X}_n - \underbrace{1.96}_{Z_{1-0.025}}\left(\frac{S}{10}\right),\ \bar{X}_n + 1.96\left(\frac{S}{10}\right)\right]$