Econ 6190 Final Exam

7:00 pm - 9:00 pm, Sunday, Dec 11, 2022

Instructions

This exam consists of two questions. Answer all questions. Remember to always explain your answer. Good luck!

1. **[50 pts]** Consider a random vector $(X, Y)'$ that has a bivariate normal distribution

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix} \right).$$

That is, the joint density of X and Y is

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)} \left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 \right] \right),$$

where $\sigma_X > 0$, $\sigma_Y > 0$. Suppose μ_Y is **known** to the statistician, as well as all the parameters in the covariance matrix. The goal is to learn about μ_X given a random sample drawn from $(X, Y)'$.

(a) **[10 pts]** Find the MLE estimator of μ_X, say, $\hat{\mu}_{MLE}$. Is $\hat{\mu}_{MLE}$ the same as $\hat{\mu} = \frac{1}{n}\sum_{i=1}^n X_i$?

(b) **[10 pts]** Compare the MSE of $\hat{\mu}_{MLE}$ and $\hat{\mu}$. Which one is more efficient?

(c) **[10 pts]** Is $\hat{\mu}_{MLE}$ Cramer-Rao efficient? If yes, provide a proof. If no, provide a counterexample.

(d) **[10 pts]** The covariance matrix in fact is usually unknown. Suppose we want to test the following hypothesis:

$$\mathbb{H}_0 : \rho = 0 \text{ v.s. } \mathbb{H}_1 : \rho \neq 0.$$

Using data $\{(X_i, Y_i)\}_{i=1}^n$, design a two-sided T test that has **asymptotic** size control of 5%. Carefully state your reasoning.

(e) **[10 pts]** Describe how to find a minimum sample size to guarantee that the asymptotic power of the test you just designed is at least 99% when $\rho \geq 0.5$ and $\sigma_X = \sigma_Y = 1$.

2. **[50 pts]** Let $\{X_1, \dots X_n\}$ be i.i.d with pdf $f(x \mid \theta) = e^{-(x-\theta)}\mathbf{1}\{x \geq \theta\}$. We hope to construct a confidence interval for θ.

   (a) **[10 pts]** Show $Y = \min\{X_1, \dots X_n\}$ is a sufficient statistic for θ **without** using the Factorization Theorem.

   (b) **[10 pts]** Show $Y - \theta = O_p(\frac{1}{n})$.

   (c) Now, consider testing $\mathbb{H}_0 : \theta = \theta_0$ v.s. $\mathbb{H}_1 : \theta \neq \theta_0$ for some θ_0. Note since Y is a sufficient statistic for θ, you only need to consider test procedures that use Y only.

      i. **[10 pts]** As a first step, derive the likelihood ratio statistic $LR(\theta_0)$ for testing $\mathbb{H}_0 : \theta = \theta_0$ v.s. $\mathbb{H}_1 : \theta \neq \theta_0$. Draw a picture of $LR(\theta_0)$ as a function of θ_0.

      ii. **[5 pts]** Then, derive a likelihood ratio test that is **finite-sample valid** with size α.

      iii. **[5 pts]** Derive a valid $1 - \alpha$ confidence interval by inverting the LRT you just constructed.

   (d) We now aim to construct a possibly different confidence interval by trying to find a pivotal quantity.

      i. **[5 pts]** Derive a pivotal quantity of Y and θ, say, $F(Y, \theta)$, by using probability integral transformation.

      ii. **[5 pts]** Then, construct a two-sided $1-\alpha$ confidence interval for θ based on the pivotal quantity $F(Y, \theta)$ you just found.