Econ 6170: Mid-Term 2

14 November 2024

You have the full class time to complete the following problems. You are to work alone. You are allowed one sheet of notes (double sided on any dimension of paper). Please write out your answer neatly below each question, and use a new sheet of paper if you need more space than provided. When using extra sheets, make sure to write out your name and the relevant question number. In your answers, you are free to cite results that you can recall from class or previous problem sets unless explicitly stated otherwise. The exam is out of 26 points.

Question 1 (6 points) Let X, Y and Z be some Euclidean space (possibly of different dimensions). Suppose that $F : X \rightrightarrows Y$ and $G : Y \rightrightarrows Z$ are compact-valued and continuous. Define $G \circ F : X \rightrightarrows Z$ as

$$x \mapsto \{z \in Z : \exists y \in F(x),\ z \in G(y)\}.$$

(i) Show that $G \circ F$ is upper hemi-continuous.

(ii) Show that $G \circ F$ is lower hemi-continuous.

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Solution 1. (i) Fix x and sequences $(x_n)_n$, $(y_n)_n$ and $(z_n)_n$ in X, Y and Z, respectively, such that $x_n \to x$, and $y_n \in F(x_n)$ and $z_n \in G(y_n)$ for all $n \in \mathbb{N}$. By upper hemicontinuity of F, there is a convergent subsequence of $(y_{n_k})_k$ with a limit point y is in $F(x)$. By upper hemicontinuity of G, there is a sub-subsequence of $(z_{n_{k_\ell}})_\ell$ with a limit point z in $G(y)$. Hence, $(z_{n_{k_\ell}})_\ell$ is a convergent subsequence of (z_n) that converges to $z \in (G \circ F)(x)$.

(ii) Pick any $z \in (G \circ F)(x)$ and a sequence $(x_n)_n$ such that $x_n \to x$. We wish to find $N \in \mathbb{N}$ and a sequence $(z_n)_n$ such that $z_n \to z$ and $z_n \in (G \circ F)(x_n)$ for all $n > N$. Since $z \in (G \circ F)(x)$, there exist $y \in F(x)$ such that $z \in G(y)$. By lower hemicontinuity of F, there exists $N_1 \in \mathbb{N}$ and $(y_n)_n$ such that $y_n \to y$ and $y_n \in F(x_n)$ for all $n > N_1$. By lower hemicontinuity of G, there exist $N_2 \in \mathbb{N}$ and $(z_n)_n$ such that $z_n \to z$ and $z_n \in G(y_n)$ for all $n > N_2$. Letting $N = \max\{N_1, N_2\}$, the sequence sequence $(z_n)_n$ satisfies the desired property.

Question 1 continued

Question 2 (5 points) Assume the supplies of *s*oy and *c*orn are affected only by the amount of rainfall (denoted r) and the prices of both soy and corn (denoted p_s and p_c respectively). For simplicity, assume that the supply of both goods increases with more rainfall.

(i) Letting S_i and D_i denote the supply and demand of $i \in \{s, c\}$ respectively, write down the two conditions that characterise the competitive equilibrium prices of both goods.
Hint: Don't think too hard!

(ii) By appealing to the implicit function theorem, give an expression for how the equilibrium prices of each good is affected by r. Be sure to state the conditions on S_i and D_i required to apply the implicit function theorem.

(iii) Suppose the supply and demand for good $i \in \{s, c\}$ is unaffected by the price of good $j \in \{s, c\} \backslash \{i\}$. Use the expression from (ii) to show that equilibrium price of both goods are decreasing with more rainfall.
Hint: You may assume that demand is decreasing in own price and supply is increasing in own price.

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Solution 2. (i)

$$S_i\left(p_s^*, p_c^*, r\right) = D_i\left(p_s^*, p_c^*\right) \ \forall i \in \{s, c\}.$$

(ii) Let $\mathbf{p} = (p_s, p_c)$ and, for each $i \in \{s, c\}$, define $f_i : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ as

$$f_i\left(\mathbf{p}, r\right) := S_i\left(\mathbf{p}, r\right) - D_i\left(\mathbf{p}\right).$$

Define $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ as $f := (f_s, f_c)$. Given any r, equilibrium price vector $\mathbf{p}^*$ solves

$$f\left(\mathbf{p}^*, r\right) = \mathbf{0}.$$

To apply IFT, we need that f is $\mathbf{C}^1$ and

$$D_{\mathbf{p}} f\left(\mathbf{p}, r\right) = \begin{bmatrix} \frac{\partial f_s}{\partial p_s} & \frac{\partial f_s}{\partial p_c} \\ \frac{\partial f_c}{\partial p_s} & \frac{\partial f_c}{\partial p_c} \end{bmatrix}$$

to be invertible. In this case, there exist a function $\mathbf{p}^*(\cdot)$ in the neighbourhood of r that is a competitive equilibrium and

$$\begin{aligned} D_r \mathbf{p}^*\left(r\right) = \begin{bmatrix} \frac{\mathrm{d}p_s^*(r)}{\mathrm{d}r} \\ \frac{\mathrm{d}p_c^*(r)}{\mathrm{d}r} \end{bmatrix} &= -\left(D_{\mathbf{p}} f\left(\mathbf{p}^*\left(r\right), r\right)\right)^{-1} \begin{bmatrix} \frac{\mathrm{d}f_s(\mathbf{p}^*(r), r)}{\mathrm{d}r} \\ \frac{\mathrm{d}f_c(\mathbf{p}^*(r), r)}{\mathrm{d}r} \end{bmatrix} \\ &= -\begin{bmatrix} x_{11} & x_{21} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} \frac{\mathrm{d}S_s(\mathbf{p}^*(r), r)}{\mathrm{d}r} \\ \frac{\mathrm{d}S_c(\mathbf{p}^*(r), r)}{\mathrm{d}r} \end{bmatrix}. \end{aligned}$$

Thus,

$$\frac{\mathrm{d}p_i^*\left(r\right)}{\mathrm{d}r} = -x_{i1}\frac{\mathrm{d}S_s\left(\mathbf{p}^*\left(r\right), r\right)}{\mathrm{d}r} - x_{i2}\frac{\mathrm{d}S_c\left(\mathbf{p}^*\left(r\right), r\right)}{\mathrm{d}r} \ \forall i \in \{s, c\}.$$

(iii) If supply and demand for one good is unaffected by the price of the other, then

$$D_{\mathbf{p}} f(\mathbf{p}, r) = \begin{bmatrix} \frac{\partial f_s}{\partial p_s} & \frac{\partial f_s}{\partial p_c} \\ \frac{\partial f_c}{\partial p_s} & \frac{\partial f_c}{\partial p_c} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_s}{\partial p_s} & 0 \\ 0 & \frac{\partial f_c}{\partial p_c} \end{bmatrix},$$

where

$$\frac{\partial f_s}{\partial p_s} = \frac{\partial S_i(\mathbf{p}, r)}{\partial p_s} - \frac{\partial D_i(\mathbf{p})}{\partial p_s} > 0.$$

Then,

$$\frac{\mathrm{d} p_i^*(r)}{\mathrm{d} r} = -\frac{1}{\frac{\partial f_s}{\partial p_s}} \frac{\mathrm{d} S_s(\mathbf{p}^*(r), r)}{\mathrm{d} r} < 0.$$

Question 2 continued

Question 3 (7 points) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is convex. Define a correspondence $\partial f : \mathbb{R} \rightrightarrows \mathbb{R}$ via

$$x_0 \mapsto \{s \in \mathbb{R} : f(x) \geq f(x_0) + s(x - x_0)\ \ \forall x \in \mathbb{R}\}.$$

(i) Suppose $f(x) := x^2$. Show that $\partial f(0) = \{0\}$.
Hint: Try drawing a graph of $f(x)$. A graphical argument will suffice (for this and the next part of the question only).

(ii) Suppose $f(x) := |x|$. Show that $\partial f(0) = [-1, 1]$.

(iii) Show that ∂f is closed- and convex-valued.

(iv) Show that f attains a global minimum at $x^* \in \mathbb{R}$ if and only if $0 \in \partial f(x^*)$.
Hint: Start by writing down the inequality that ensures that $x^* \in \mathbb{R}$ is a global minimum.

(v) Suppose f is differentiable. Show that ∂f is singleton-valued. What is the (unique) element of $\partial f(x_0)$ for any $x_0 \in \mathbb{R}$?

Remark: $\partial f(x_0)$ is called the *subdifferential* of f at $x_0 \in \mathbb{R}$.

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Solution 3. (i) Since $x_0 = 0$ and $f(x_0) = 0 \leq x^2 = f(x)$ for all $x \in \mathbb{R}$, $0 \in \partial f(0)$. By way of contradiction, suppose $s \in \partial f(0)$ with $s \neq 0$ so that

$$f(x) = x^2 \geq sx\ \forall x \in \mathbb{R}.$$

If $s > 0$, we must have that, for any $x > 0$,

$$f(x) = (x)^2 \geq sx \Leftrightarrow x \geq s$$

and we can always find $x \in (0, s)$ that violates the inequality to get a contradiction. If $s < 0$, then we must have that, for any $x < 0$,

$$(x)^2 \geq sx \Leftrightarrow x \leq s$$

and we can always find $x \in (s, 0)$ that violates this inequality to get a contradiction.

(ii) Set $x_0 = 0$ and note that $f(x_0) = |0| = 0$. Suppose $x > 0$, then

$$f(x) = x \geq sx \Leftrightarrow s \leq 1$$

and if $x < 0$,

$$f(x) = -x \geq sx \Leftrightarrow s \geq -1.$$

If $x = 0$, then any s satisfies the constraint. Hence, $\partial f(0) = [-1, 1]$.

(iii) Fix some $x_0 \in \mathbb{R}$. Let $(s_n)_n$ be a sequence in $\partial f(x_0)$ converging to some $s \in \mathbb{R}$. Then,

$$f(x) \geq f(x_0) + s_n(x - x_0)\ \ \forall x \in \mathbb{R}\ \forall n \in \mathbb{N}.$$

Since the right-hand side is affine in s_n, clearly,

$$f(x) \geq f(x_0) + s(x - x_0) \ \ \forall x \in \mathbb{R}.$$

Hence, $\partial f(x_0)$ is closed. Since $x_0$was chosen arbitrarily, we conclude that ∂f is closed-valued.

To see that ∂f is convex-valued. Fix some $x_0 \in \mathbb{R}$ and $s, s' \in \partial f(x_0)$. Then, for any $x \in \mathbb{R}$,

$$f(x) \geq f(x_0) + s(x - x_0), f(x_0) + s'(x - x_0)$$

and so for any $\alpha \in [0, 1]$,

$$f(x) \geq f(x_0) + \left(\alpha s + (1 - \alpha) s'\right)(x - x_0).$$

Hence, ∂f is convex-valued.

(iv) $x^* \in \mathbb{R}$ is a global minimum if and only if $f(x) \geq f(x^*)$ for all $x \in \mathbb{R}$. The inequality holds if and only if

$$f(x) \geq f(x^*) = f(x^*) + 0(x - x^*) \ \ \forall x \in \mathbb{R}.$$

That is, $0 \in \partial f(x^*)$.

(v) Suppose s satisfies

$$f(x) \geq f(x_0) + s(x - x_0) \ \ \forall x \in \mathbb{R}.$$

Then,

$$\frac{f(x) - f(x_0)}{x - x_0} \geq s \ \forall x > x_0$$

so that taking limit as $x \searrow x_0$ gives that

$$f'(x_0) \geq s.$$

Since

$$\frac{f(x) - f(x_0)}{x - x_0} \leq s \ \forall x < x_0,$$

so that taking limit as $x \nearrow x_0$ gives that

$$f'(x_0) \leq s.$$

hence $s = f'(x_0)$; i.e., $\partial f(x_0) = \{f'(x_0)\}$.

Question 3 continued

Question 4 (8 points) Consider the following linear programming problem:

$$p^* = \max_{\mathbf{x}\in\mathbb{R}^d} \mathbf{f}^\top \mathbf{x} \text{ s.t. } A\mathbf{x} \geq \mathbf{b},$$

where $\mathbf{f} \in \mathbb{R}^d$, $\mathbf{b} \in \mathbb{R}^J$ and $A \in \mathbb{R}^{J\times d}$.

(i) Write the "KKT necessary conditions" for the problem. Use $\boldsymbol{\lambda}$ to denote the Lagrangian multipliers. What is its dimension?

(ii) What is the constraint qualification for this problem? What condition(s) are the necessary conditions for the existence of Lagrangian multipliers that satisfy the KKT necessary conditions?

(iii) Suppose you found the Lagrangian multipliers $\boldsymbol{\lambda}^*$ that satisfy the KKT necessary conditions. Suppose a unique solution $\mathbf{x}^*$ exists. Given an expression for p^* in terms of components of A and $\mathbf{b}$.
Hint: Assume the first $J^* \leq J$ constraints are binding.

(iv) Write the Lagrangian, $\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda})$, for this problem. For any $\boldsymbol{\lambda}$, what is the solution to

$$d^*(\boldsymbol{\lambda}) = \sup_{\mathbf{x}\in\mathbb{R}^d} \mathcal{L}(\mathbf{x}, \boldsymbol{\lambda})?$$

Hint: Collect the terms on $\mathbf{x}$ and remember that sup can be $\pm\infty$.

(v) Show that $d^*(\boldsymbol{\lambda}^*) = p^*$.

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Solution 4. (i) Nonnegativity: $\boldsymbol{\lambda} \geq \mathbf{0}$. Constraints: $A\mathbf{x} \geq \mathbf{b}$. Complementary slackness:

$$\lambda_j \left(A_{j\cdot}\mathbf{x} - b_j\right) = 0 \; \forall j \in \{1, 2, \ldots, J\},$$

where $A_{j\cdot}$ denote the jth row of A; FOC:

$$\mathbf{f}^\top + \boldsymbol{\lambda}^\top A = \mathbf{0}.$$

(ii) The usual constraint qualification that rank$(A^\top)$ equals the number of binding constraints at an optimal. Note that the condition does not depend on the value of $\mathbf{x}$ and so it either holds everywhere or it doesn't hold anywhere.

(iii) An optimal $\mathbf{x}^* \in \mathbb{R}^d$ must satisfy constraints, $A\mathbf{x}^* \geq \mathbf{b}$, and complementary slackness conditions. Thus, for any $\lambda_j^* > 0$, we must have

$$A_{j\cdot}\mathbf{x}^* = b_j.$$

Let us denote the first J^* constraints as binding ones (i.e., $\lambda_j^* > 0$ if and only if $j \in \{1, \ldots, J^*\}$).

Then,

$$\underbrace{\begin{bmatrix} A_{1\cdot} \\ \vdots \\ A_{J^*\cdot} \end{bmatrix}}_{A^*} \mathbf{x}^* = \underbrace{\begin{bmatrix} b_{1\cdot} \\ \vdots \\ b_{J^*\cdot} \end{bmatrix}}_{\mathbf{b}^*}.$$

Since we assume $\mathbf{x}^*$ is unique, then $\mathbf{x}^* = (A^*)^{-1}\mathbf{b}^*$ and so

$$p^* = \mathbf{f}^\top \mathbf{x}^* = \mathbf{f}^\top (A^*)^{-1}\mathbf{b}^*$$

(iv) The Lagrangian is given by

$$\begin{aligned} \mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) &= \mathbf{f}^\top \mathbf{x} + \boldsymbol{\lambda} \cdot (A\mathbf{x} - \mathbf{b}) \\ &= \left(\mathbf{f}^\top + \boldsymbol{\lambda}^\top A\right) \mathbf{x} - \boldsymbol{\lambda} \cdot \mathbf{b}. \end{aligned}$$

Thus,

$$\begin{aligned} \sup_{\mathbf{x} \in \mathbb{R}^d} \mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) &= \sup_{\mathbf{x} \in \mathbb{R}^d} \left(\mathbf{f}^\top + \boldsymbol{\lambda}^\top A\right) \mathbf{x} - \boldsymbol{\lambda}^\top \mathbf{b} \\ &= \begin{cases} -\boldsymbol{\lambda}^\top \mathbf{b} & \text{if } \mathbf{f}^\top + \boldsymbol{\lambda}^\top A = \mathbf{0} \\ \infty & \text{if } \mathbf{f}^\top + \boldsymbol{\lambda}^\top A \neq \mathbf{0} \end{cases} \end{aligned}$$

(v) Since $\boldsymbol{\lambda}^*$ solves the KKT necessary conditions:

$$\mathbf{f}^\top + (\boldsymbol{\lambda}^*)^\top A = \mathbf{0}.$$

Recall that the last $J - J^*$ components of $\boldsymbol{\lambda}^*$ are zeros and so

$$\mathbf{f}^\top + (\boldsymbol{\lambda}^*)^\top A = \mathbf{f}^\top + \left[\lambda_1^*, \cdots, \lambda_{J^*}^*\right] A^* = \mathbf{0}$$

so that

$$\left[\lambda_1^*, \cdots, \lambda_{J^*}^*\right] = -\mathbf{f}^\top (A^*)^{-1}.$$

Hence,

$$d^*(\boldsymbol{\lambda}^*) = -(\boldsymbol{\lambda}^*)^\top \mathbf{b} = \mathbf{f}^\top (A^*)^{-1} \mathbf{b}^* = p^*.$$

Question 4 continued