Problem Set 10

Due: TA Discussion, 8 November 2023.

1 Additional Exercises

Exercise 1. Consider the problem of maximising $u : \mathbb{R}^2 \to \mathbb{R}$ given by $u(x_1, x_2) := x_1^{0.5} + x_2^{0.5}$ subject to the budget constraint; i.e.,

$$\Gamma := \left\{ (x_1, x_2) \in \mathbb{R}^2 : px_1 + x_2 \leq m,\ x_1, x_2 \geq 0 \right\},$$

where $p, m > 0$.

(i) Prove that a solution to the utility maximisation problem exists.

(ii) Prove that a solution must lie on the boundary of the set Γ.

(iii) Solve the Lagrangian as an equality-constrained one while ignoring the nonnegativity constraints. Does the solution to the Lagrangian identify a solution to the original problem? Why or why not?

Exercise 2. Consider the equality-constrained optimisation problem from class notes:

$$\max_{\mathbf{x} \in \mathbb{R}^d} f(\mathbf{x}) \text{ s.t. } h_k(\mathbf{x}) = 0 \ \forall k \in \{1, \dots, K\}, \qquad (1)$$
$$g_j(\mathbf{x}) \geq 0 \ \forall j \in \{1, \dots, J\}$$

were $f : \mathbb{R}^d \to \mathbb{R}$, $h_k : \mathbb{R}^d \to \mathbb{R}$ for each $k \in \{1, \dots, K\}$, and $g_j : \mathbb{R}^d \to \mathbb{R}$ for each $j \in \{1, \dots, J\}$ are all $\mathbf{C}^1$ functions. Define the *Lagrangian*, $\mathcal{L} : \mathbb{R}^d \times \mathbb{R}^K \times \mathbb{R}^J \to \mathbb{R}$, as

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \sum_{k=1}^{K} \mu_k h_k(\mathbf{x}) + \sum_{j=1}^{J} \lambda_j g_j(\mathbf{x}). \qquad (2)$$

Say that a vector $(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) \in \mathbb{R}^d \times \mathbb{R}^K \times \mathbb{R}^J$ is a *critical point* of $\mathcal{L}$ if it satisfies the following set of equations:

(i) for all $i \in \{1, \dots, d\}$,

$$\frac{\partial \mathcal{L}}{\partial x_i}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) = 0;$$

(ii) for all $k \in \{1, \dots K\}$,

$$\frac{\partial \mathcal{L}}{\partial \mu_k}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) = 0;$$

(iii) for all $j \in \{1, \dots, J\}$,

$$\lambda_j \geq 0 \, \frac{\partial \mathcal{L}}{\partial \lambda_j}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) \geq 0, \; \lambda_j \frac{\partial \mathcal{L}}{\partial \lambda_j}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) = 0.$$

Let S denote the set of all critical points of $\mathcal{L}$ and S_X denote the projection of S onto the first d components of S; i.e.,

$$S_X := \left\{ \mathbf{x} \in \mathbb{R}^d : \exists \, (\boldsymbol{\mu}, \boldsymbol{\lambda}) \in \mathbb{R}^K \times \mathbb{R}^J, \, (\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) \in S \right\}.$$

Now consider the following problem:

$$\max_{\mathbf{x} \in S_X} f(\mathbf{x}). \tag{3}$$

Show that if a the problem (1) attains a global maximum at some $\mathbf{x}^* \in \mathbb{R}^d$ such that $h_k(\mathbf{x}^*) = 0$ for all $k \in \{1, \dots, K\}$, $g_j(\mathbf{x}^*) \geq 0$ for all $j \in \{1, \dots, J\}$, *and* the constraint qualification holds at $\mathbf{x}^*$,[1] then a $\mathbf{x}^\circ \in S_X$ that solves (3) is also a global maximum.

Remark 1. This should convince you that the "usual" approach you use to solve for constrained optimisation using Lagrangian works if (i) a global maximum exists and (ii) the constraint qualification is met at global maxima.

Exercise 3. Consider the consumer's problem of maximising utility $u : \mathbb{R}^2 \to \mathbb{R}$ given by $u(x_1, x_2) := x_1 + x_2$ subject to the budget set

$$B(p_1, p_2, m) := \left\{ (x_1, x_2) \in \mathbb{R}^2_+ : m - p_1 x_1 - p_2 x_2 \geq 0 \right\},$$

where $m, p_1, p_2 > 0$.

(i) Write the constrained optimisation problem and the associated Lagrangian.

(ii) Show that the constraint qualification is satisfied. Hint: Argue that any optimum must exhaust income and then that the constraint qualification constraint holds no matter what other constraints bind.

(iii) What can you say about critical points that solve the Lagrangian in this case?

Exercise 4. Suppose a firm's production function is given by $g : \mathbb{R}^3 \to \mathbb{R}$, where

$$f(x_1, x_2, x_3) := x_1 (x_2 + x_3).$$

The unit price of firm's output is $p > 0$ and the price of each inputs are $w_i > 0$ for $i \in \{1, 2, 3\}$.

(i) Describe the firm's profit-maximisation problem, and derive the equations that define the critical points of the Lagrangian of this problem.

(ii) Show that the Lagrangian as has multiple critical points for any choice of $(p, w_1, w_2, w_3) \in \mathbb{R}^4_{++}$.

(iii) Show that none of these critical points identifies a solution of the profit-maximisation problem. Explain why.

[1] The relevant constraint qualification is the one that allows for both equality and inequality constraints.