About TA sections:

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Our plan for today:[1]

[1] Materials adapted from notes provided by a previous Teaching Assistant, Zhuoheng Xu.

1 Value Function Iterations

1.1 Key Steps of the Method

Key Result: Contraction Mapping Theorem (CMT)

Under certain assumptions (which will be discussed in the coming weeks), the contraction mapping theorem ensures the existence of a unique v^* to which a sequence of value functions $\{v_n\}_{n=0}^{\infty}$ will converge.

Implications of the Contraction Mapping Theorem:

- **Existence of v^***: The solution v^* exists.
- **Uniqueness of v^***: We can begin with any arbitrary initial guess, v_0, to obtain the solution v^*.

Step 1: Discretize the State Space
Identify all **endogenous state variables** and select a suitable grid for these variables. In most economic models, the state space (i.e., the possible values of state variables) is continuous, and we need to discretize the state space into a grid with finite points. The grid discretization allows us to approximate the continuous state space and perform numerical optimization using a finite set of points.

Remark 1: Fow now we only consider the deterministic case, meaning there is no uncertainty. That's why we only need to pick a grid for endogenous state variables. If there is an exogenous state variable with uncertainty, for example, the productivity level with the law of motion $\ln(z_{t+1}) = \rho \ln(z_t) + \epsilon_{t+1}$, then we also need to pick a grid for the exogenous state variable by applying some discretization methods, e.g., Tauchen's method. The challenge here is that the distribution of the productivity shock ϵ_t is assumed to be continuous.

Remark 2: We also select the control variable(s) from the discretized states. That is, the space of control variable(s) coincides with the discretized state space.

Example 1: In the simple neoclassical model, the endogenous variable is k_t. In this context, we can select a suitable grid ranging from $k_{\min}$ to $k_{\max}$, with, for example, 50 equally spaced points. We will also select the optimal k_{t+1} from one of these 50 points.

Example 2: In the Guess and Verify example from the last time, the endogenous variables are k_t and c_{t-1}. Then we have to create grids for both k_t and c_{t-1}. Suppose there are N_k points for k and N_c points for c_{t-1}, then there will be $N_k \times N_c$ combinations of (k, c_{t-1}).

Step 2: Initialize the Value Function
Set the initial guess of the value function v_0. Normally, we set $v_0 = \vec{0}$.

Step 3: Set the Stopping Criterion/Convergence Criterion
As the Continuous Mapping Theorem indicates, the sequence of value functions will converge to the limit v^* but will never reach the limit. To ensure the algorithm doesn't run indefinitely, we need to set the stopping criterion. For example, the stopping criterion can be "stop running if $|v_{s+1} - v_s| \leq 1 \times 10^{-6}$," where "$1 \times 10^{-6}$" is called the "tolerance level."

Step 4: Iteration (the Main Step)
Recall the Bellman Equation takes the form of (here we consider the simple neoclassical model)

$$v(k) = \max_{0 \leq k' \leq k^\alpha} \{\log(k^\alpha - k') + \beta v(k')\}$$

Suppose now we have completed m cycles of iteration, which means we have obtained the value of $v_m(k)$ and the policy function $g_m(k)$ for each state variable k.

For each possible value of state variable k in the discretized state space (in general, we will have "for each combination of state variables in the discretized state space" if we have multiple state variables.)

- **Evaluate the Objective Function:** For each possible control variable(s) (in this example, k') on the grid, substitute it into the objective function and calculate the resulting value. Suppose a particular value of the control variable is denoted by $\tilde{k}$, we can calculate the value of

  $$\log(k^\alpha - \tilde{k}) + \beta v_m(\tilde{k})$$

  where $v_m(\tilde{k})$ is known as we have completed m cycles of iteration.

- **Track the Optimal Control Variable:** We have calculated the value of the objective function for each possible control variable, and we can now find the control variable that maximizes the objective function, denoting it as k^*.

- **Update the Value Function and the Policy Function**

$$v_{m+1}(k) = \log(k^\alpha - k^*) + \beta v_m(k^*)$$

$$g_{m+1}(k) = k^*$$

Repeat the process above for **all possible values of the state variable**.

Remark 3: It is possible that conditional on a specific value of the state variable k, some control variables are not feasible. For example, in the simple neoclassical model, we have the constraint $0 \leq k' \leq k^\alpha$. Then, for a given k_1, it is possible that there exists some k_2 such that $k_2 > k_1^\alpha$. In such cases, we need to eliminate these k_2 values.

Remark 4: Tradeoff: More grid points yield a more precise solution but can be computationally costly.

Step 5: Check the Stopping Criterion/Convergence Criterion
Repeat the steps described above in Step 4 until the stopping criterion is met.

1.2 Value Function Iterations in MATLAB

1. Clear workspace and set paremter values.

```
clc; %clears the command window
clear all; %clears variables
close all; %close all figures
format compact %Set the output format to the short engineering format
    with compact line spacing

alpha = 0.25;  % share of capital from total output
bbeta  = 0.8;   % discount factor
its    = 1;     % Initialize the number of iterations for value
   function iteration
diff   = 1;     % Initial difference between the old and new value
   function
tol    = 1e-6;  % Tolerance level to stop the iteration (controls the
   accuracy of the solution)
```

```
s      = 2;      % Utility function curvature (reflects risk aversion
    in utility calculation)

u = @(c) (c>0).*(1/(1-s)).*(c.^(1-s)-1) +(c<=0).*(-1e18); % utility
    function with Inada!
%(c>0) indicates which values are greater than zero.
```

2. Set grids on the state variable, namely k.

Remark: Choosing bounds of k can be tricky. Trial/error or solve for steady state and binds k around steady state value.

```
kss = (alpha*bbeta)^(1/(1-alpha));% steady state capital stock
nk = 100; % number of data points in the the capital grid
kmin = 0.25*kss; % minimum value in the capital grid ... 75% lower
    than the steady state
kmax = 1.75*kss; % maximum value in the capital grid ... 75% more
    than the steady state
kgrid = linspace(kmin,kmax,nk); % capital grid
%y = linspace(x1,x2,n) generates n points. The spacing between the
    points is (x2-x1)/(n-1).
```

3. Initialize value function and policy function on each grid of k.

```
val_fun = zeros(1,nk);                 % initial value functions
pol_fun_idx = zeros(1,nk);             % indexes for the policy function
```

4. Key Step: Iteration ("While loop")

While the difference between two (sets of) value functions is still above the tolerance, we are gonna keep updating at iteration n:

- For each value of k in the grid, define policy function $k' = g(k)$
- Save the value function and the policy function

- Update the value function
- Compute the difference between value function vectors v_{n+1} and v_n. Compare with the tolerance level and decide whether to continue
- If the difference between value function vectors v_{n+1} and v_n is smaller than the tolerance level, the while loop stopped and the problem is solved

```
while diff>tol
    for i=1:length(kgrid)
        c = (kgrid(i)^alpha)-kgrid; %We use scalar (kgrid(i)^alpha)
           minus vector kgrid to get the result of c corresponding to
            every element of k' on the grid.
        % Note that in MATLAB, if we subtract scalar from array, the
           scalar is subtracted from each entry of A.
        [val_new(i), pol_fun_idx(i)] = max(u(c)+bbeta*val_fun); %
           Bellman equation
        %[M,I] = max(A)  returns the index into the operating
           dimension that corresponds to the maximum value of A
    end
    diff= max(abs((val_new-val_fun)));
    %Y = abs(X) returns the absolute value of each element in array X
       .
    %M = max(A) returns the maximum elements of an array.
    val_fun=val_new;
    its = its+1;
end
```

6. Now, save the (final) optimal policy function and compute the related consumption decisions.

```
pol_fun = kgrid(pol_fun_idx);    % This collects the points on the
   grid that resulted in the maximal value function
cons = (kgrid.^alpha)-pol_fun;
```

7. Make plots.

```
figure(1)
```

```
plot(kgrid,pol_fun,'linewidth',1.8); title('Policy Function (k_{t+1})'); ...
    xlabel('k_t'); ylabel('k_{t+1}'); grid on ; hold on; plot([0 kmax],[0 kmax]); ...
    xlim([0 kmax]); saveas(gcf,'pol_fun_k.png')
%plot(X,Y) plots a 2-D line plot of the data in Y versus the corresponding values in X.
```

2 Neoclassical Growth Model

2.1 Balanced Growth Path

In class, we considered the neoclassical growth model with exogenous growth, including both population growth and technological progress.

The **steady state** refers to a condition where key economic variables expressed **per unit of effective labor** remain constant over time. However, *aggregate quantities* can still grow due to population growth (n) and technological progress (g).

The **balanced growth path** (BGP) is a state where all economic variables grow at constant (possibly different) rates. On the BGP, per capita variables grow at the rate of technological progress g, while aggregate variables grow at a combined rate of population growth n and technological progress g.

Remember the notations we used in class:

$$c_t \quad \text{(consumption per capita)}$$
$$C_t = (1+n)^t c_t \quad \text{(aggregate consumption)}$$
$$\tilde{c}_t = \frac{c_t}{(1+g)^t} \quad \text{(consumption per unit of effective labor)}$$

On the BGP, at steady state:

- $\tilde{c}_t$ (consumption per unit of effective labor) is constant.
- c_t (consumption per capita) is growing at the rate of technological progress g.
- C_t (aggregate consumption) is growing at the combined rate $n+g$ (exogenous growth in population and technology).

2.2 Decentralizing Growth Model: Firm's Problem

Profit maximization:

$$\max_{k_t,n_t\geq 0} \pi_t \equiv p_t \left(F(k_t,n_t) - r_t k_t - w_t n_t\right)$$

First order conditions:

$$r_t = F_k(k_t,n_t)$$
$$w_t = F_n(k_t,n_t)$$

Remember that the production function exhibits constant returns to scale (CRS), which implies that the production function is homogeneous of degree 1 (HOD1). For a HOD1 function of the form $F(x_1, x_2, \ldots, x_N)$ for N factors of production, Euler's Theorem states that:

$$x_1 \frac{\partial F}{\partial x_1}(x) + x_2 \frac{\partial F}{\partial x_2}(x) + \ldots + x_N \frac{\partial F}{\partial x_N}(x) = F(x_1, x_2, \ldots, x_N)$$

Intuition: This means that in a competitive equilibrium with CRS, the firm's revenue is fully distributed among capital and labor inputs. No economic profit is left because all output is used to pay the factors of production.

Using Euler's theorem:

$$\pi_t = p_t \left(F(k_t,n_t) - F_k(k_t,n_t)k_t - F_n(k_t,n_t)n_t\right) = 0$$

This means that the firm's profit π_t is zero under CRS, as all the output is used to compensate capital and labor according to their marginal products. Therefore, in competitive equilibrium, no economic profit remains for the firm.

Remark: For HOD1 functions, the first partial derivatives are homogeneous of degree 0 (HOD0), meaning they do not scale with output. One implication is the indeterminacy of number of firms in the model.

2.3 Decentralizing Growth Model: Household's Problem (simplified for optimality)

$$\max_{\{c_t,k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

subject to

$$\sum_{t=0}^{\infty} p_t(c_t + k_{t+1} - (1-\delta)k_t) = \sum_{t=0}^{\infty} p_t(r_t k_t + w_t)$$

$$c_t, k_{t+1} \geq 0, \quad k_0 \text{ given}$$

Note that we used the following:

- $n_t = 1$: Labor supply is normalized to 1 and there is no disutility from labor in this model. Households inelastically supply one unit of labor without having to make labor-leisure trade-offs. This assumption simplifies the household's problem by removing labor from the set of decision variables.

- $k_t = x_t$: The household rents out all available capital because it maximizes utility by producing as much as possible with the available resources. In a perfectly competitive capital market, there is no incentive to leave capital idle, since unused machines or capital would result in forgone production or rental income.

- $i_t = k_{t+1} - (1-\delta)k_t$: With $k_t = x_t$, investment simply equals the difference between next-period capital and depreciated current capital (the capital accumulation equation).

2.4 Competitive Equilibriums in Growth Model: Arrow-Debreu Competitive Equilibrium

An **Arrow-Debreu Competitive Equilibrium** is a set of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$ and allocations for the firm $\{n_t^d, k_t^d\}_{t=0}^{\infty}$ and the household $\{c_t, k_t^s, n_t^s\}_{t=0}^{\infty}$ such that:

1. Given prices, the allocation of the representative firm solves the profit maximization problem.

$$\max_{k_t, n_t \geq 0} \pi_t \equiv p_t \left(F(k_t, n_t) - r_t k_t - w_t n_t \right)$$

2. Given prices, the allocation of the representative household solves the utility maximization problem.

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

subject to

$$\sum_{t=0}^{\infty} p_t (c_t + k_{t+1} - (1-\delta) k_t) = \sum_{t=0}^{\infty} p_t (r_t k_t + w_t n_t)$$

3. Markets clear:

$$F(k_t, n_t) = c_t + k_{t+1} - (1-\delta) k_t \quad \text{(Goods market)}$$

$$n_t^d = n_t^s \quad \text{(Labor market)}$$

$$k_t^d = k_t^s \quad \text{(Capital services market)}$$

2.5 Competitive Equilibriums in Growth Model: Sequence Market Competitive Equilibrium

A **Sequential Markets Equilibrium** is prices $\{w_t, r_t\}_{t=0}^{\infty}$, allocations for representative household $\{c_t, k_t^s, n_t^s\}_{t=0}^{\infty}$ and for representative firm $\{n_t^d, k_t^d\}_{t=0}^{\infty}$ such that:

1. Given prices, households solve the utility maximization problem.

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

subject to

$$c_t + k_{t+1} - (1-\delta) k_t = w_t n_t + r_t k_t$$

2. Given prices, firm solves the profit maximization problem.

$$\max_{k_t, n_t \geq 0} F(k_t, n_t) - w_t n_t - r_t k_t$$

3. Markets clear:

$$n_t^d = n_t^s$$

$$k_t^d = k_t^s$$

$$F(k_t, n_t) = c_t + k_{t+1} - (1 - \delta) k_t$$

3 Competitive Equilibriums in Growth Model: Recursive Competitive Equilibrium

Take the simple neoclassical model as the example. Let's consider the household's problem first. More specifically, consider a tiny household within the population. This assumption implies that the decisions made by individual households have no influence on aggregate-level outcomes. It is analogous to the situation in a perfectly competitive market where each producer is a price taker. In general, we normalize the mass of households to 1. For each household, the time endowment in each period is also normalized to 1.

The household's utility maximization problem (with incomplete constraints) is

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

subject to

$$c_t + k_{t+1} = w(K_t) + (1 + r(K_t) - \delta)\, k_t$$

where k_t represents the capital of the household, and K_t represents the aggregate capital.

Remark 1: Since the household is tiny, the wage rate w and the rental rate r are determined by the aggregate capital K_t rather than the household capital k_t. **The household takes the aggregate capital K_t as given. Moreover, the household takes all aggregate variables as given.**

Remark 2: The budget constraint is not sufficient to solve the problem. To make the consumption-investment decision, the household also needs to know the rental rate in the next period r_{t+1}. Since the rental rate is a function of aggregate capital in the next period K_{t+1}, the household has to know the path of aggregate capital. Suppose the household believes the aggregate capital evolves according to the mapping H, that is, $K_{t+1} = H(K_t)$.

Remark 3: The interpretation of the mapping H: knowing the aggregate capital today K_t enables the household to project the aggregate capital path into the future aggregate capital and the path for prices.

Remark 4: State variables are (k, K), rather than just k. Control variables are (c, k').

Since we have identified all state variables, we can write the household's utility maximization problem into the recursive form

$$v(k, K) = \max_{c,k'} U(c) + \beta v(k', K')$$

subject to

$$c + k' = w(K) + (1 + r(K) - \delta)\, k$$

$$K' = \mathcal{H}(K)$$

Solution: value function v and policy functions $c = C(k, K)$ and $k' = G(k, K)$.

Next, consider the firm's profit maximization problem. FOCs are

$$w(K) = F_n(K, 1)$$

$$r(K) = F_k(K, 1)$$

The third condition, which is the distinctive feature of the recursive formulation of competitive equilibrium, is the **Consistency Condition**

$$K' = G(K, K)$$

Interpretation of the Consistency Condition: Suppose the household possesses a quantity of capital equal to the aggregate capital, then the household's individual behavior in equilibrium will be exactly the same as the aggregate behavior. That is, the aggregate law of motion perceived by the agent (i.e., $\mathcal{H}(K)$) must be consistent with the actual behavior of individuals (i.e., $G(K, K)$). This links individual decisions to the aggregate outcome, ensuring equilibrium.

Finally, we need to have the market clearing condition

$$C(K, K) + G(K, K) = F(K, 1) + (1 - \delta)K$$

where $C(K,K)$ and $G(K,K)$ are aggregate consumption today and aggregate capital tomorrow since the first argument in these two policy functions is K rather than k.

Finally, combining all these equations together, we have the following definition of RCE: A **RCE** is a value function $v:\mathbb{R}_+^2\to\mathbb{R}$ and policy functions $C,G:\mathbb{R}_+^2\to\mathbb{R}_+$ for the representative household, pricing functions $w,r:\mathbb{R}_+\to\mathbb{R}_+$ and an aggregate law of motion $H:\mathbb{R}_+\to\mathbb{R}_+$ such that:

1. Given w,r and H, v solves the Bellman equation and C,G are the associated policy functions.
2. The pricing functions satisfy the firms FOC.
3. Consistency: $\mathcal{H}(K)=G(K,K)$.
4. For all $K\in\mathbb{R}_+$:

$$C(K,K)+G(K,K)=F(K,1)+(1-\delta)K$$

Remark 5: If we assume households are identical, then in equilibrium, each household will have the same level of consumption, that is, $c_{it}=c_t$ for all t, then we can conclude

$$\begin{aligned}
C_t &\equiv \int_0^1 c_{it}\,di \,(\text{by definition})\\
&= \int_0^1 c_t\,di \,(\text{by } c_{it}=c_t)\\
&= c_t\int_0^1 1\,di \,(\text{by integrating w.r.t } i)\\
&= c_t
\end{aligned}$$

Moreover, since the mass of the households is equal to 1, we can calculate the average consumption as

$$\bar{C}_t=\frac{C_t}{1}=C_t$$

Hence we have the following relationship:

Household Consumption(c_t) = Aggregate Consumption(C_t) = Average Consumption$(\bar{C}_t)$

The same relationship also holds for other variables, e.g., Household Capital = Aggregate Capital = Average Capital.

4 Practice question: Pollution in the neoclassical growth model (Q 2021)

We will investigate how pollution distorts a market economy and how intervention by a policymaker can bring the economy closer to the efficient allocation. Consider a neoclassical growth model in which the production of output creates pollution as a byproduct. The aggregate amount of pollution s_t created in period t is proportional to aggregate output in period t, so that

$$s_t = \theta f(k_t, n_t),$$

where k_t is the aggregate stock of capital, n_t is the aggregate labor supply, and the aggregate production function f exhibits constant returns to scale. There is a representative consumer whose preferences are given by

$$\sum_{t=0}^{\infty} \beta^t u(c_t, s_t),$$

where c_t is consumption in period t and where $0 < \beta < 1$. The function u is strictly increasing and strictly concave in its first argument and strictly decreasing in its second argument. The consumer is endowed with one unit of time in each period. Initial capital $k_0 > 0$ is given, and capital depreciates at a constant rate $0 < \delta < 1$.

Note that pollution is an externality, i.e. the representative consumer does not understand her own impact on pollution when making decisions.

1. (10 pt) Define a sequential markets equilibrium for this economy. In any equilibrium, what is the amount of labor n_t traded in period t? Why?

2. (10 pt) In general, does the path followed by the capital stock depend on θ in equilibrium? Does the steady-state aggregate capital stock depend on θ? Prove your answers and explain them.

3. (10 pt) Write down the problem of a social planner for this economy (the planner

understands its impact on pollution). Do the social planner and equilibrium allocations coincide at the steady state? If not, is the capital stock higher or lower in the efficient allocation? Prove all your answers. (From now on, you can assume that $n_t = 1$).