ECON 6090

Problem Set 2

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1. Video submitted on Canvas.
2. We have a consumer whose utility function is $u(x,t) = \left(x^{\frac{1}{2}} + (w-t)\right)^2$
   (a) We have that the consumer is solving
   $$\max_{x,l \in \mathbb{R}_+} u(x,l) = \left(x^{\frac{1}{2}} + l\right)^2$$
   subject to
   $$px \leq w - l \equiv px + l \leq w$$
   i.e., they are maximizing consumption and leisure subject to consumption not exceeding their wage for the total hours worked.

   (b) Our Lagrangian is
   $$\mathcal{L} = \left(x^{\frac{1}{2}} + l\right)^2 + \lambda(w - l - px)$$
   For the first order conditions, we get
   $$\frac{\partial \mathcal{L}}{\partial x} = \frac{x^{1/2}+l}{x^{1/2}} - p\lambda = 0 \implies \lambda = \frac{x^{1/2}+l}{p \cdot x^{1/2}}$$
   $$\frac{\partial \mathcal{L}}{\partial l} = 2x^{1/2} + 2l - \lambda = 0 \implies \lambda = 2x^{1/2} + 2l$$
   $$\frac{\partial \mathcal{L}}{\partial \lambda} = w - l - px = 0 \implies px + l = w$$
   Setting them equal and solving, we get that
   $$\frac{x^{1/2}+l}{p \cdot x^{1/2}} = 2x^{1/2} + 2l \implies 2px^{1/2}(x^{1/2}+l) = x^{1/2} + l$$
   so we get that the Walrasian demand for x is
   $$x^\star = \frac{1}{4p^2}$$
   and inputting into the budget constraint, we get
   $$\frac{p}{4p^2} + l = w \implies l^\star = w - \frac{1}{4p}$$
   Note that this might create a corner solution – if $w < \frac{1}{4p}$, then the consumer always work. Formally, our Walrasian demand functions are
   $$x^\star(p,w) = \begin{cases} \frac{1}{4p^2} & w \geq \frac{1}{4p} \\ w & \text{otherwise} \end{cases}$$

and

$$l^{\star}(p,w) = \begin{cases} w - \frac{1}{4p} & w \geq \frac{1}{4p} \\ 0 & \text{otherwise} \end{cases}$$

(c) Going back to the originally stated utility function, the indirect utility function is defined by

$$V(p,w) := \max_{x,t\in\mathbb{R}_+} \left(x^{\frac{1}{2}} + (w-t)\right)^2$$

subject to

$$px \leq t$$

We first solve for the Walrasian demand functions. Our Lagrangian is

$$\mathcal{L} = \left(x^{\frac{1}{2}} + (w-t)\right)^2 + \lambda(t - px)$$

and our first order conditions are

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{x^{1/2} + w - t}{x^{1/2}} - p\lambda = 0 \Longrightarrow \lambda = \frac{x^{1/2} + w - t}{px^{1/2}}$$

$$\frac{\partial \mathcal{L}}{\partial t} = -2\left(x^{1/2} + w - t\right) + \lambda = 0 \Longrightarrow \lambda = 2\left(x^{1/2} + w - t\right)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = t - px = 0 \Longrightarrow t = px$$

Which implies that

$$2\left(x^{1/2} + w - t\right) = \frac{x^{1/2} + w - t}{px^{1/2}} \Longrightarrow x^{\star} = \frac{1}{4p^2}$$

and

$$t^{\star} = px^{\star} = \frac{1}{4p}$$

which is the same as above, a confirmation that this formulation also works. As above, we have admitted a corner, where the worker will not take any time off if $\frac{1}{4p} > w$. Formally, our Walrasian Demand is

$$x^{\star}(p,w) = \begin{cases} \frac{1}{4p^2} & w \geq \frac{1}{4p} \\ w & \text{otherwise} \end{cases}$$

and

$$t^{\star}(p,w) = \begin{cases} \frac{1}{4p} & w \geq \frac{1}{4p} \\ w & \text{otherwise} \end{cases}$$

From the definition of the indirect value function, we have that

$$V(p,w) = u(x^{\star}, t^{\star}) = \left(\left(\frac{1}{4p^2}\right)^{\frac{1}{2}} + \left(w - \frac{1}{4p}\right)\right)^2 = \left(\frac{1}{2p} + w - \frac{1}{4p}\right)^2 = \left(\frac{1}{4p} + w\right)^2$$

(d) For leisure to be strictly positive, we will assume that $w \geq \frac{1}{4p}$. We can find the expenditure function by inverting the value function, since $e(p, V(p,w)) = w$. We get that

$$\bar{u} = \left(\frac{1}{4p} + e(p,\bar{u})\right)^2 \Longrightarrow e(p,\bar{u}) = \sqrt{\bar{u}} - \frac{1}{4p}$$

From Shephard's Lemma, since $u'' = -\frac{1}{x^{3/2}} < 0$, the implied preferences $\succsim$ are strictly convex, we have that

$$h_x(p,\bar{u}) = \frac{\partial e(p,\bar{u})}{\partial p} = \frac{1}{4p^2}$$

3. We have that $e(p,u) = g(u)r(p)$ for some strictly increasing g, r

   (a) From Shephard's Lemma, we have that $h_i(p,u) = g(u)\frac{\partial r(p)}{\partial p_i}$. Since e is two strictly increasing functions multiplied, we can say that $h_i(p, V(p,w)) = x_i(p,w)$ which means that $x_i(p,w) = g(V(p,w))\frac{\partial r(p)}{\partial p_i}$. It remains to find a form for $g(V(p,w))$. From the expenditure function we have that $e(p, V(p,w)) = w$, so $g(V(p,w))r(p) = w$ which implies that $V(p,w) = g^{-1}(w/r(p))$, where g^{-1} exists because g is strictly increasing. Thus, we have that $x_i^\star(p,w) = \frac{w}{r(p)}\frac{\partial r(p)}{\partial p_i}$.

   (b) If Walras' Law holds, we have that $p \cdot x = w$, which implies that

$$\sum_{i=1}^{L} p_i x_i(p,w) = w \implies \sum_{i=1}^{L} p_i \frac{w}{r(p)}\frac{\partial r(p)}{\partial p_i} = w$$

   Which means that

$$\sum_{i=1}^{L} p_i \frac{\partial r(p)}{\partial p_i} = r(p)$$

   We don't need to make any assumptions on $g(u)$ for this to hold, as it was eliminated before considering Walras' Law. For Walras' Law to hold, we need local non-satiation of the utility function itself.

   (c) The distribution of budgets does not matter for aggregate demand! Because $\sum_{i=1}^{I} x^i(p, w^i) = \sum_{i=1}^{I} \frac{w^i}{r(p)} r'(p) = \frac{\sum_{i=1}^{I} w^i}{r(p)} r'(p) = x(p, \sum_{i=1}^{I} w^i)$, we can construct a representative agent with total wealth who has the same preferences as all of the agents.

4. We know that the expenditure function of the consumer is $e(p,U) = Up_1^\alpha p_2^\beta$

   (a) We need to know that the expenditure function is (i) continuous, (ii) nondecreasing in each p_i, (iii) strictly increasing in U, (iv) homogeneous of degree 1 in p, and (v) concave in p. Parts (i) and (iii) are satisfied immediately. For e to be nondecraesing in each p_i, it must be the case that $\alpha, \beta \geq 0$. For them to be homogeneous of degree 1 in p, it must be the case that $e(\lambda p, U) = \lambda e(p,U)$, which requires that

$$e(\lambda p, U) = \lambda^{\alpha+\beta} e(p,U)$$

   be equal to $\lambda e(p,U)$, meaning that $\alpha + \beta = 1$. Finally, e must be concave in p, meaning that $e''(p,U) < 0$. This is satisfied as long as $\alpha, \beta \leq 1$.

   Thus, we must have that $\alpha, \beta \in [0,1]$ and $\alpha + \beta = 1$.

   (b) From Shephard's Lemma, we have that the Hicksian demand functions are

$$h_1(p,U) = \frac{\partial e(p,U)}{\partial p_1} = \alpha U p_1^{\alpha-1} p_2^\beta$$

   and

$$h_2(p,U) = \frac{\partial e(p,U)}{\partial p_2} = \beta U p_1^{\alpha} p_2^{\beta-1}$$

   To find the indirect utility function, we will use the identity that $e(p, V(p,w)) = w$, so we have that

$$w = V(p,w) p_1^\alpha p_2^\beta \implies V(p,w) = w p_1^{-\alpha} p_2^{-\beta}$$

   Finally, the uncompensated demand is found using Roy's Identity, where we have that

$$x_1(p,w) = -\frac{\frac{\partial V(p,w)}{\partial p_1}}{\frac{\partial V(p,w)}{\partial w}} = -\frac{-\alpha w p_1^{-\alpha-1} p_2^{-\beta}}{p_1^{-\alpha} p_2^{-\beta}} = \frac{\alpha w}{p_1}$$

and

$$x_2(p,w) = -\frac{\frac{\partial V(p,w)}{\partial p_2}}{\frac{\partial V(p,w)}{\partial w}} = -\frac{-\beta w p_1^{-\alpha} p_2^{-\beta-1}}{p_1^{-\alpha} p_2^{-\beta}} = \frac{\beta w}{p_2}$$

(c) From Corollary 2.57, we have that $h_i(p, V(p,w)) = x_i(p,w)$. From there, we have that

$$x_1(p,w) = h_1(p, V(p,w)) = \alpha V(p,w) p_1^{\alpha-1} p_2^{\beta} = \alpha w p_1^{-1} p_2^{0} = \frac{\alpha w}{p_1}$$

and

$$x_2(p,w) = h_2(p, V(p,w)) = \beta V(p,w) p_1^{\alpha} p_2^{\beta-1} = \beta w p_2^{-1} p_1^{0} = \frac{\beta w}{p_2}$$

(d) We have that $\alpha = \beta = \frac{1}{2}$, $w = 512$, and an increase in prices from $p = (1,1)$ to $p' = (16,16)$.

i. We have that the utility attained under the original prices is

$$V(p,w) = 512 \cdot 1^{-\alpha} \cdot 1^{-\beta} = 512$$

and that the utility attained under the new prices is

$$V(p',w) = 512 \cdot 16^{-\alpha} \cdot 16^{-\beta} = \frac{512}{16} = 32$$

We have that the compensating variation is

$$CV(p,p',w) = w - e(p', V(p,w)) = 512 - 512 \cdot 16^{\alpha} \cdot 16^{\beta} = -7,680$$

and that the equivalent variation is

$$EV(p,p',w) = e(p, V(p',w)) - w = 32 \cdot 1^{\alpha} 1^{\alpha} - 512 = -480$$

ii. The absolute value of the compensating variation is significantly higher than the absolute value of the equivalent variation, because the amount required to pay the consumer so that they will be able to afford their old consumption under the new prices is a lot higher than the amount their attained utility actually changes under the new prices.

It seems more reasonable to pay the consumer their compensating variation. The equivalent variation is the amount they would take from the consumer *instead* of changing prices, but in order for the consumer to agree to the price change, they would need to pay him the compensating variation.

5. Evaluate the following claims:

(a) We have that $u(x) = 2\ln(x_1) + 2\ln(x_2) = 2\ln(x_1x_2)$ and $u^\star(x) = x_1x_2$. We have that the first expenditure function is

$$e(u, p_1, p_2) := \min_{x \in \mathbb{R}_+} p_1x_1 + p_2x_2 \quad \text{s.t. } 2\ln(x_1x_2) \geq u$$

and the second expenditure function is

$$e(u^\star, p_1, p_2) := \min_{x \in \mathbb{R}_+} p_1x_1 + p_2x_2 \quad \text{s.t. } x_1x_2 \geq u^\star$$

If $u^\star = \exp(u/2)$, we have that the conditions here become

$$x_1x_2 \geq \exp\left(\frac{u}{2}\right) \implies 2\ln(x_1x_2) \geq u$$

So since the optimizing function and the feasible set are the same, we have that

$$e(u, p_1, p_2) = e(u^\star, p_1, p_2)$$

(b) We have that the price of good i changes from p_i to $p_i' > p_i$, and that the consumer's wealth increases from w to $w' = w + (p_i' - p_i)x_i^\star(p, w)$. First, note that the consumer will always attain weakly higher utility under the new prices and wealth. Considering their old optimal bundle $x^\star$, because of local non-satiation we have that $p \cdot x^\star = w$. This means that, since no other prices changed,

$$p' \cdot x^\star = p \cdot x^\star + (p_i' - p_i)x_i^\star(p, w) = w + (p_i' - p_i)x_i^\star(p, w) = w'$$

Since the old bundle is attainable under the new prices and wealth, the consumer will always attain weakly higher utility, as $u(x^\star) \leq \max_{x \in \Gamma(X)} u(x)$ by definition.

From the Slutsky equation, we have that the change in demand for a change in price of the same good is

$$\frac{\partial x_i(p, w)}{\partial p_i} = \frac{\partial h_i(p, u)}{\partial p_i} - x_i(p, w)\frac{\partial x_i(p, w)}{\partial w}$$

We have, from our definitions of the properties of Hicksian and Walrasian demand, that the first partial on the right is (weakly) negative since the Hicksian demand is itself the partial of the expenditure function with respect to price, and the expenditure function is concave in prices. We do not know whether the demand for good i is increasing in wealth or not. That depends on whether good i is normal or inferior. If it is normal, the consumer will demand less good i under the new prices and wealth. If it is inferior, and sufficiently inferior, it may be the case that they will demand more – in that case, good i would be a Giffen good.

(c) This claim is false. To see why, we will solve the consumer's maximization problem. The KKT conditions hold, so we can solve it from the first order conditions. The Lagrangian is

$$\mathcal{L} = \sum_{t=1}^{T} \beta^t u(c_t) + \lambda \left(w - \sum_{t=1}^{T} c_t \right)$$

the first order conditions for arbitrary t, $t + 1$ are:

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t u'(c_t) - \lambda = 0 \implies \lambda = \beta^t u'(c_t)$$

$$\frac{\partial \mathcal{L}}{\partial c_{t+1}} = \beta^{t+1} u'(c_{t+1}) - \lambda = 0 \implies \lambda = \beta^{t+1} u'(c_{t+1})$$

These combine to get the Euler Equation

$$u'(c_t) = \beta u'(c_{t+1})$$

Since this applies for all t, and assuming arbitrary utility functions, we can say that the ratio of optimal consumption in period t and period $t + 1$ is constant for all utility functions. Since we also have that $\sum_{t=1}^{T} c_t = w$, it must be that optimal consumption does not depend on the utility function at all.

6. The paper I read was "Pricing Power in Advertising Markets: Theory and Evidence" by Matt Gentzkow, Jesse Shapiro, Frank Yang, and Ali Yurukoglu. In this paper, the authors generalize a model of advertising that implies that TV advertisements attain more money per person seeing each advertisement even when the number of people watching TV decreases, because the average income of each person watching TV increases. This relies on Walras' Law holding at least somewhat abstractly, as if the average viewer has higher income, they will spend more on consumption, and thus more per ad seen. They test this model empirically, and find that it holds.