Econ 6190 Mid Term Exam

© Chen Qiu. Do not reproduce or share with any third party

9:25 am - 10:55 am, 6 October 2022

Instructions

This exam consists of 5 questions, not of equal length or difficulty. Answer all questions. Remember to always explain your answer. Good luck!

1. **[20 pts]** Let $X \sim \mathrm{N}(\mu, \sigma^2)$ for some unknown μ and **known** σ^2. Furthermore, suppose I believe that μ can only take two values, $\frac{1}{2}$ or $-\frac{1}{2}$, and I believe $P\{\mu = \frac{1}{2}\} = \frac{1}{2}$, and $P\{\mu = -\frac{1}{2}\} = \frac{1}{2}$. Now, I draw a single observation X_1 from the distribution of X, and it turns out $X_1 < 0$. Given that I observe $X_1 < 0$, what is my updated probability that $\mu = \frac{1}{2}$? That is, find $P\{\mu = \frac{1}{2} | X_1 < 0\}$. The following notations can be useful: $\Phi(t)$ is the cdf of a standard normal, and $\phi(t)$ is the pdf of a standard normal.

2. **[10 pts]** Let X be a random variable with density $f(x)$. Show that if the density satisfies $f(x) = f(-x)$ for all $x \in \mathbb{R}$, then:

   (a) **[5 pts]** The distribution function satisfies $F(-x) = 1 - F(x)$ for all $x \in \mathbb{R}$.

   (b) **[5 pts]** $\mathbb{E}[X] = 0$.

3. **[30 pts]** A joint pdf is defined by

$$f(x, y) = \begin{cases} C(x + 2y), & \text{if } 0 < y < 1 \text{ and } 0 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) **[5 pts]** Find the value of C.

   (b) **[5 pts]** Find the marginal distribution of X.

   (c) **[10 pts]** Find the joint cdf of X and Y.

   (d) **[10 pts]** Find the pdf of the random variable $Z = \frac{4}{(X+1)^2}$.

4. **[15 pts]** Let $X_i, i = 1, 2, 3$ be independent with $\mathrm{N}(i, i^2)$ distributions. For each of the following situations, use X_1, X_2, X_3 to construct a statistic with the indicated distribution (an answer without any explanation will get a zero grade):

(a) **[5 pts]** Chi-square distribution of 3 degrees of freedom.

(b) **[5 pts]** t distribution with 2 degrees of freedom.

(c) **[5 pts]** F distribution with 1 and 2 degrees of freedom.

5. **[25 pts]** Suppose $X \sim \mathrm{N}(\mu, \sigma^2)$ with an unknown mean μ and **known** variance $\sigma^2 > 0$. We draw a random sample $\mathbf{X} := \{X_1, X_2, \dots X_n\}$ of size n from X. We are interested in estimating μ based on $\mathbf{X}$.

   (a) **[5 pts]** Find a minimal sufficient statistic for μ.

   (b) Suppose now $\sigma^2 = 1$ and $n = 1$. Consider the following estimator $\hat{\theta} = \frac{c^2}{c^2+1} X_1$ for some known $c > 0$.

      i. **[5 pts]** Find the MSE of $\hat{\theta}$. Is $\hat{\theta}$ unbiased?

      ii. **[10 pts]** Compare the MSE of $\hat{\theta}$ with the MSE of $\tilde{\theta} = X_1$. Which one is more efficient? (Hint: there is a range of values of μ for which $\hat{\theta}$ is more efficient).

      iii. **[5 pts]** Based on your answer to (ii), which of the two estimators, $\hat{\theta}$ or $\tilde{\theta}$, is more efficient when $\mu = c$?