Econ 6190 Problem Set 4

Fall 2024

1. Let $\{X_1, \ldots, X_n\}$ be a random sample from the uniform distribution on the interval $(\theta, \theta + 1)$, $-\infty < \theta < \infty$. Find a minimal sufficient statistic for θ. This question shows that the dimension of a minimal sufficient statistic does not necessarily match the dimension of the unknown parameter.

2. [Mid term, 2022] Suppose $X \sim \mathrm{N}(\mu, \sigma^2)$ with an unknown mean μ and **known** variance $\sigma^2 > 0$. We draw a random sample $\mathbf{X} := \{X_1, X_2, \ldots X_n\}$ of size n from X. We are interested in estimating μ based on $\mathbf{X}$.

   (a) Find a minimal sufficient statistic for μ.

   (b) Suppose now $\sigma^2 = 1$ and $n = 1$. Consider the following estimator $\hat{\theta} = \frac{c^2}{c^2+1} X_1$ for some known $c > 0$.

      i. Find the MSE of $\hat{\theta}$. Is $\hat{\theta}$ unbiased?

      ii. Compare the MSE of $\hat{\theta}$ with the MSE of $\tilde{\theta} = X_1$. Which one is more efficient? (Hint: there is a range of values of μ for which $\hat{\theta}$ is more efficient).

      iii. Based on your answer to (ii), which of the two estimators, $\hat{\theta}$ or $\tilde{\theta}$, is more efficient when $\mu = c$?

3. Let $\{X_1, \ldots, X_n\}$ be a random sample from finite second moment, and let $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$ be an estimator for $\sigma^2 = \mathrm{var}(X)$. Show $\mathbb{E}[\hat{\sigma}^2] = (1 - \frac{1}{n})\sigma^2$ and thus find the bias of $\hat{\sigma}^2$.

4. [Hong] Suppose $\{X_1, X_2 \ldots X_n\}$ is iid $\mathrm{N}(0, \sigma^2)$. Consider the following estimator for σ^2:

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

   Find:

   (a) the sampling distribution of $n\hat{\sigma}^2/\sigma^2$.

   (b) $\mathbb{E}\hat{\sigma}^2$.

   (c) $\mathrm{var}(\hat{\sigma}^2)$.

   (d) $\mathrm{MSE}(\hat{\sigma}^2)$.

5. Let $\{X_1, \dots, X_n\}$ be a random sample from a Poisson distribution with parameter λ

$$P\{X_i = j\} = \frac{e^{-\lambda}\lambda^j}{j!}, j = 0, 1 \dots$$

(a) Find a minimal sufficient statistic for λ, say T.

(b) Suppose we are interested in estimating probability of a count of zero $\theta = P\{X = 0\} = e^{-\lambda}$. Find an unbiased estimator for θ, say $\hat{\theta}_1$. (Note $P\{X = 0\} = \mathbb{E}[\mathbf{1}\{X = 0\}]$.)

(c) Is the estimator in (b) a function of the minimal sufficient statistics T?

(d) Use the definition of a sufficient statistic and an unbiased estimator, show that the estimator $\hat{\theta}_2 = \mathbb{E}[\hat{\theta}_1 | T]$ is also unbiased and $\text{MSE}(\hat{\theta}_2) \le \text{MSE}(\hat{\theta}_1)$.

(e) Based on (d), find an analytic form of $\hat{\theta}_2$.