1. Real Sequences

Takuma Habu*
takumahabu@cornell.edu

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1 Real numbers

1.1 $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$

Definition 1. $\mathbb{N} := \{1, 2, \ldots\}$ is the set of *natural* numbers (sometimes 0 is included in $\mathbb{N}$). $\mathbb{Z} := \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is the set of *integers*. $\mathbb{Q} := \{a/b : a \in \mathbb{Z},\, b \in \mathbb{N}\}$ is the set of *rational* numbers. Finally, $\mathbb{R}$ is the set of *real* numbers.

Remark 1. $\mathbb{N}$ is closed under the operations of addition and multiplication; i.e., the sum and the product of any two natural numbers is a natural number. However , $\mathbb{N}$ is not closed under subtraction and division. $\mathbb{Z}$ (unlike the natural numbers) is closed under subtraction, but not division. Finally, the set of rational numbers, $\mathbb{Q}$ is closed under all four operations. However, the set of rational numbers is not *complete*, that is, the rational number line, $\mathbb{Q}$, has a "gap" at each irrational value. We have the following relationships:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

We will skip the axioms that defines these sets of numbers and instead take the following characterising property of $\mathbb{R}$ (completeness) as an axiom.

1.2 Completeness of $\mathbb{R}$

Definition 2. Let S be a subset of $\mathbb{R}$ (i.e., $S \subseteq \mathbb{R}$). If $b \in \mathbb{R}$ is such that $b \geq s$ for every s in S ($\forall s \in S$), then b is an *upper bound* of S. If such an upper bound for S exists, then we say S is *bounded (from) above*. *Lower bounds* are defined analogously. S is *bounded* if it is bounded above and below.

Definition 3. Let $S \subseteq \mathbb{R}$ be bounded above. Suppose there exists β such that:

(i) β is an upper bound of S

(ii) if $\gamma < \beta$, then γ is *not* an upper bound of S

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Then, β is called the *least upper bound* of S, or its *supremum*, written $\sup S$.

Symmetrically, suppose $S \subseteq \mathbb{R}$ is bounded below. Then, α is the *greatest lower bound* or *infimum* of S, written $\inf S$, if it is a lower bound of S and if every $\gamma > \alpha$ is not a lower bound of S.

Exercise 1. Requirement (ii) in the definition of $\sup S$ above can be written as: $\forall \epsilon > 0$, $\exists s \in S$ such that $s > \sup S - \epsilon$ (why?) What is the equivalent condition for the greatest lower bound?

Exercise 2. Why can we write "*the*" least upper bound? (Formally, prove that $\sup S$ is unique: if β and β' both satisfy the definition, then $\beta = \beta'$.)

Exercise 3. TFU (True, False, Uncertain): If $\sup S$ exists, then $\sup S \in S$.

Axiom 1 (Completeness Axiom)**.** *If S is a nonempty subset of $\mathbb{R}$ which is bounded above, then $\sup S$ exists (in $\mathbb{R}$).*

Remark 2. This is not true in, for example, $\mathbb{Q}$: the set $S = \{x \in \mathbb{Q} : x^2 < 2\}$ is bounded, but the only candidate for $\sup S$, $s = \sqrt{2}$, doesn't belong to $\mathbb{Q}$.

Exercise 4. Let $S \subset \mathbb{R}$ be nonempty and bounded. Prove that $\inf S \leq \sup S$. What can you say if $\inf S = \sup S$?

Exercise 5. Recall the formal definition of maximum and minimum of a set (don't look them up—model your definitions on those of supremum and infimum). TFU: Every set (in $\mathbb{R}$) has a maximum. Every *bounded* set has a maximum.

Exercise 6. TFU: If $S \subseteq \mathbb{R}$ has a maximum $\max S$, then $\max S = \sup S$.

Exercise 7 (PS1)**.** Let S and T be nonempty and bounded subsets of $\mathbb{R}$. TFU: $\sup(S \cup T) = \max\{\sup S, \sup T\}$.

1.3 Density of $\mathbb{Q}$ in $\mathbb{R}$

Proposition 1 (Archimedean property)**.** *If $a > 0$ and $b \in \mathbb{R}$, then there exists an $n \in \mathbb{N}$ such that $na > b$.*

Proof. Suppose instead that there exist $a > 0$ and $b \in \mathbb{R}$ such that $na \leq b$ for all $n \in \mathbb{N}$. In particular, this means that b is an upper bound for the set $S := \{na : n \in \mathbb{N}\}$. Since S is nonempty and $S \subseteq \mathbb{R}$, by the Completeness axiom, $s := \sup S$ exists. Since $a > 0$, $s - a < s$. Therefore $s - a$ in not an upper bound for S, and so $s - a < ma$ for some $m \in N$. Rearranging, $s < (m+1)a$: but this contradicts that s is an upper bound for s because $(m+1)a$ is also in S (since $m + 1 \in \mathbb{N}$). ■

Proposition 2 (Archimedean property)**.** *The set $\mathbb{N}$ of natural numbers is unbounded from above in $\mathbb{R}$.*

Exercise 8. Prove that Proposition 1 and Proposition 2 are equivalent: Proposition 1 follows from Proposition 2 and vice versa.

Exercise 9. TFU: If $\epsilon > 0$, then there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon < n$.

Proposition 3 (Density of $\mathbb{Q}$ in $\mathbb{R}$)**.** *For any $x, y \in \mathbb{R}$ with $y > x$, there exists $q \in \mathbb{Q}$ such that $x < q < y$.*

Proof. Fix $x, y \in \mathbb{R}$ such that $y > x$. By Proposition 1 (set $n = y - x$ and $b = 1$), there exists an $n \in \mathbb{N}$ such that $n(y-x) > 1 \Leftrightarrow ny > nx + 1$. Let $m := \min\{k \in \mathbb{Z} : k > na\}$.[1] By definition, $na < m$ and $na \geq m - 1$ (why?) and so $na < m \leq 1 + na < nb$. Letting $q := \frac{m}{n}$ and noting that q is rational completes the proof. ■

Exercise 10. TFU: If $a < b$, then there exist infinitely many rationals between a and b.

1.4 Extended real numbers

Definition 4. Let $+\infty$ (or just ∞) be a *symbol* that satisfies $a < +\infty$ for all $a \in \mathbb{R}$. Symmetrically, the symbol $-\infty$ satisfies $a > -\infty$, for all $a \in \mathbb{R}$. Finally, $-\infty < +\infty$. We call $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ the *extended real line*.

Remark 3. $+\infty$ and $-\infty$ are *not* real numbers, so statements on real numbers do not (automatically) extend to them. Plausible facts like $a + \infty = \infty$, $(-\infty) + (-\infty) = -\infty$, etc. are true in $\overline{\mathbb{R}}$. However, expressions like $+\infty + (-\infty)$, $\infty \cdot 0$, etc. are left undefined (just like $1/0$ is undefined in $\mathbb{R}$).

Definition 5. Let $S \subseteq \mathbb{R}$ be unbounded above. Then we define $\sup S := +\infty$. Analogously, if S is unbounded below, then $\inf S := -\infty$. (A strict reading of the definition of supremum and infimum, now in $\overline{\mathbb{R}}$, shows that these definitions are not new.)

Remark 4. With this last definition and the Completeness axiom, we can say that *all* subsets of $\mathbb{R}$ have a supremum and an infimum (possibly in $\overline{\mathbb{R}}$).

Exercise 11. According to a strict interpretation of the definition of supremum and infimum, what are $\sup \emptyset$ and $\inf \emptyset$ (where $\emptyset$ is the empty set)?

2 Sequences

Definition 6. A *sequence* (in $\mathbb{R}$) is a function $x : \mathbb{N} \to \mathbb{R}$. Instead of using the standard notation $x(n)$ for functions we use x_n. Some (equivalent) notations for a sequence x are:

$$(x_1, x_2, \ldots) \equiv (x_n)_{n=1}^{\infty} \equiv (x_n)_{n \in \mathbb{N}} \equiv (x_n)_n \equiv (x_n).$$

For brevity, we will generally adopt the notation $(x_n)_n$ if no confusion arise.

Remark 5. You will often see sequences denoted as $\{x_n\}_{n=1}^{\infty}$. Braces exclusively for *sets*, which are unordered: $\{2, 3\}$ is the same set as $\{3, 2\}$, which are both the same as $\{2, 3, 2, 2, 2, 3\}$ (with some abuse of notation), etc.

Example 1. Consider the sequence $(1, -1, 1, -1, \ldots) = ((-1)^n)_{n=1}^{\infty}$. (Make sure you understand the notation on the right hand side of the equality.) Its *set of values* is $\{(-1)^n : n \in \mathbb{N}\} = \{1, -1\}$. Seen as a function, $\{1, -1\}$ is the range and $\mathbb{N}$ is the domain (like it is for all sequences).

[1] One way to formally prove the existence of m is to prove that every nonempty subset of $\mathbb{Z}$ that is bounded from below has a minimum.

2.1 Convergence of a sequence

Definition 7. A sequence $(x_n)_n$ *converges* to $x \in \mathbb{R}$ if: for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n > N$ implies $|x_n - x| < \epsilon$. The point x is called the *limit* of $(x_n)_n$, and we write

$$\lim_{n\to\infty} x_n = x \text{ or } x_n \to x.$$

Exercise 12. TFU: If a sequence has a limit, then the limit is unique. **Hint:** recall the *triangle inequality*: $|a - b| \leq |a - c| + |c - b|$, for all $a, b, c \in \mathbb{R}$.

Proposition 4 (Sandwich rule)**.** *Suppose that a sequence $(x_n)_n$ converges to x and that $a \leq x_n \leq b$ for all $n \in \mathbb{N}$ for some $a, b \in \mathbb{R}$, $b > a$. Then, $a \leq x \leq b$.*

Proof. Suppose that $x_n \geq a$ for all $n \in \mathbb{N}$ but $a > x$. Define $\epsilon := a - x > 0$. Since $x_n \to x$, there exists $n \in \mathbb{N}$ sufficiently large such that

$$x_n - x \leq |x_n - x| < \epsilon = a - x \Rightarrow a > x_n,$$

which is a contradiction. Symmetric argument for $x_n \leq b$ for all $n \in \mathbb{N}$ shows that we must also have $x \leq b$. ■

Exercise 13. Find the limit (if they exist) of the following sequences, or show that they do not exist.

(i) $(a_n)_n = (\frac{1}{n})_n$

(ii) $(b_n)_n = ((-1)^n)_n$

(iii) $(c_n)_n = ((-1)^{2n})_n$

Exercise 14 (PS2)**.** TFU: Suppose (x_n) and (y_n) are Real sequences and that $x_n \to x$ and $y_n \to y$. Then, (i) $(x_n + y_n)_n \to x + y$, (ii) $x_n y_n \to xy$, (iii) $x_n - y_n \to x - y$, (iv) $\frac{1}{x_n} \to \frac{1}{x}$; (v) $\frac{x_n}{y_n} \to \frac{x}{y}$.

Exercise 15. TFU: a sequence $(x_n)_n$ converges to x if and only if there exists $\epsilon > 0$ such that all terms x_n are contained in $(x - \epsilon, x + \epsilon)$.

Exercise 16. TFU: a sequence $(x_n)_n$ converges to x if and only if for all $\epsilon > 0$ all but finitely many terms x_n's are contained in $(x - \epsilon, x + \epsilon)$.

Exercise 17. TFU: a sequence $(x_n)_n$ converges to x if and only if for all $\epsilon > 0$ infinitely many terms are contained in $(x - \epsilon, x + \epsilon)$.

Exercise 18 (PS2)**.** TFU: a sequence $(x_n)_n$ converges to x if and only if for all $\epsilon > 0$ infinitely many terms are contained in $(x - \epsilon, x + \epsilon)$, and x is the only number with this property.

2.1.1 Infinite limits

Definition 8. A sequence (x_n) *diverges to* (or converges to) $+\infty$ if for every $M \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $x_n \geq M$ for all $n \geq N$. We write $\lim x_n = +\infty$ or $x_n \to +\infty$ as before. *Divergence to* (convergence to) $-\infty$ is defined analogously.

Remark 6. Informally, a sequence diverges to $+\infty$ (resp. $-\infty$) if it has arbitrarily large (resp. small) elements in its tail.

Exercise 19. TFU: If a sequence does not converge, then it diverges to either $+\infty$ or $-\infty$.

Exercise 20. TFU: Let (x_n) diverge to $+\infty$ and $y_n \to y > 0$ (y can be finite or $+\infty$). Then, $\lim x_n y_n$ exists (and is ...?).

Exercise 21. TFU: Let (x_n) diverge to $+\infty$ and $y_n \to 0$. Then, $\lim x_n y_n$ exists (and is ...?).

2.1.2 Bounded sequences

Definition 9. A sequence (x_n) is *bounded* if its set of values $\{x_n : n \in \mathbb{N}\}$ is bounded. *Bounded above* and *bounded below* are defined in the same manner.

Exercise 22. TFU: Every bounded sequence is convergent.

Exercise 23 (PS2)**.** TFU: Every convergent sequence (with a finite limit) is bounded.

Exercise 24. TFU: A sequence diverges to $+\infty$ if and only if the sequence if unbounded.

2.1.3 Monotone sequences

Definition 10. A sequence $(x_n)_n$ is *nondecreasing* if $x_n \leq x_{n+1}$, for every $n \in \mathbb{N}$. It is *strictly increasing* if $x_n < x_{n+1}$ for every $n \in \mathbb{N}$. To avoid ambiguity, I will try not to use the term "increasing". Definitions of *nonincreasing* and *strictly decreasing* sequences are analogous. Finally, a sequence is *monotone* if it is either *nondecreasing* or *nonincreasing.*

Exercise 25. Complete the following: A sequence is both nondecreasing and nonincreasing if and only if it is ....

Proposition 5. *If $(x_n)_n$ is bounded and monotone, then it converges.*

Proof. We will prove the statement for a nondecreasing sequence $(x_n)_n$. The statement for nonincreasing sequences follow from the fact that $(x_n)_n$ is nonincreasing if and only if $(-x_n)$ is nondecreasing. So suppose $(x_n)_n$ is bounded and nondecreasing. By the Completeness axiom, $u := \sup\{x_n : n \in \mathbb{N}\} < +\infty$ exists. We want to show that $x_n \to u$. Fix any $\epsilon > 0$. Since $u - \epsilon$ is not an upper bound for $(x_n)_n$ (why?), there exists $N \in \mathbb{N}$ such that $x_N > u - \epsilon$. Since $(x_n)_n$ is nondecreasing, for all $n > N$, we also have $x_n > u - \epsilon$. By definition of u, $x_n \leq u$ for all $n \in \mathbb{N}$. Combining these, $u - \epsilon < x_n \leq u$ for all $n > N$ and hence $|x_n - u| < \epsilon$ for all $n > N$. This proves that $x_n \to u$. ■

Proposition 6. *If $(x_n)_n$ is unbounded and nondecreasing, then it diverges to $+\infty$. Similarly, if $(x_n)_n$ is unbounded and nonincreasing, then it diverges to $-\infty$.*

Proof. Let $M > 0$. Since $\{x_n : n \in \mathbb{N}\}$ is unbounded by hypothesis and it is bounded below by x_1 (why?), it must be unbounded above. Then, there exists $N \in \mathbb{N}$ such that $x_N > M$. Since $(x_n)_n$ is nondecreasing, $x_n \geq x_N > M$ for all $n > N$, which shows that $\lim x_n = +\infty$. The proof for the case in which $(x_n)_n$ is nonincreasing is analogous.. ■

Remark 7. Combining these two propositions gives the *Monotone Convergence Theorem* for Real sequences; i.e., all monotone sequences either converge to a finite $x \in \mathbb{R}$ (either the supremum or the infimum of $\{x_n : n \in \mathbb{N}\}$) or diverge to $\pm\infty$. Thus, if $(x_n)_n$ is a monotone sequence, $\lim x_n$ is always a meaningful expression. This is particularly useful because we did *not* compute the limit value. A similar thing will happen with Cauchy sequences.

Corollary 1. *A monotone sequence $(x_n)_n$ converges if and only if it is bounded.*

2.2 Subsequences

Definition 11. Let (x_n) be a sequence. A *subsequence* of (x_n) is a sequence obtained by (only) deleting elements of (x_n). More formally, a subsequence of (x_n) is any sequence $(x_{n_k})_k$ where $(n_k)_k$ is a strictly increasing sequence of non-negative integers.

Exercise 26 (PS2)**.** TFU: If a sequence converges, then every subsequence converges (to the same limit).

Exercise 27 (PS2)**.** TFU: If a sequence is bounded, then every subsequence is bounded.

Exercise 28 (PS2)**.** TFU: If a sequence is unbounded, then every subsequence is unbounded.

Exercise 29 (PS2)**.** TFU: If a sequence is unbounded, then it has a subsequence which is bounded.

2.3 The Bolzano-Weierstrass theorem

Proposition 7. *Every sequence $(x_n)_n$ has a monotonic subsequence.*

Proof. For each $n \in \mathbb{N}$ define the set $S_n := \{x_n, x_{n+1}, \ldots\}$.

If S_1 has no maximum element,[2] then construct a subsequence $(x_{n_k})_k$ as follows.

$$\begin{aligned} n_1 &:= 1 \\ n_2 &:= n_1 + \min\{k' \in \mathbb{N} : x_{n_1+k'} \geq x_{n_1}\} \\ &:= \vdots \\ n_k &:= n_{k-1} + \min\left\{k' \in \mathbb{N} : x_{n_{k-1}+k'} \geq x_{n_{k-1}}\right\} \ \forall k = 3, 4 \ldots. \end{aligned}$$

Observe that x_{n_2} is the first term in $(x_{n_1+1}, x_{n_1+2}, \ldots)$ that is greater than $x_{n_1} = x_1$. Moreover, x_{n_2} is well-defined because S_1 has no maximum—if there weren't such a term, then $x_{n_1} = x_1$ would be the maximum of S_1. Similarly, x_{n_3} is well-defined as the first term in $(x_{n_2+1}, x_{n_2+2}, \ldots)$ that is greater than x_{n_2}. If there weren't such a term, then $x_{n_2} > x_m$ for all $m \geq n_2$; also, $x_{n_2} \geq x_m$ for all $m < n_2$ by construction; so x_{n_3}would be the maximum of S_1. Observe that, by construction, $(x_{n_k})_k$ is nondecreasing.

Suppose that S_1 has the maximum element but there exists S_n (for some $n > 1$) that has no maximum element. Then, we could reapply the same argument from above to construct a nondecreasing subsequence by taking letting $x_{n_1} := x_n$.

[2]Since I only left this definition as an exercise, let me give it formally: $b \in \mathbb{R}$ is the maximum of set $S \subset \mathbb{R}$ if $b \in S$ and $b \geq s$ for all $s \in S$. The minimum is defined analogously. Note that unbounded sets do not have maximum or minimum.

The only remaining case is if $\max S_n$ exists for all $n \in \mathbb{N}$. Consider the following recursively defined sequence of indices:

$$n_1 = \min \{m \in \mathbb{N} : x_m = \max S_1\}$$
$$n_{k+1} = \min \{m \in \mathbb{N} : x_m = \max S_{n_k+1}\} \ \forall k \in \mathbb{N}.$$

(Note that S_n is a set, hence $\max S_n$ is just a number, and not a set of maximisers.) The subsequence $(x_{n_k})_k$ is nonincreasing because the sets S_n are nested appropriately. ∎

Exercise 30. TFU: Referring to the previous proof, if $\max S_1$ does not exist then neither do $\max S_n$, for all $n = 2, 3 \ldots$..

Exercise 31 (PS2)**.** In the second part of the proof of Proposition 7, can you replace $\min\{m \in \mathbb{N} : x_m = \max S_{n_k+1}\}$ with $\max\{m \in \mathbb{N} : x_m = \max S_{n_k+1}\}$?

Theorem 1 (Bolzano-Weierstrass)**.** *Every bounded sequence has a convergent subsequence.*

Proof. Let $(x_n)_n$ be a bounded sequence. By Proposition 7, it has a monotonic subsequence $(x_{n_k})_k$. By Exercise 27, $(x_{n_k})_k$ is bounded. By Proposition 5, monotone and bounded sequences converge. ∎

2.4 lim sup and lim inf

Definition 12. The *limit superior* (read "lim sup") of a sequence (x_n) is

$$\limsup_{n\to\infty} x_n := \lim_{m\to\infty} \sup \{x_n : n \geq m\}.$$

The *limit inferior* ("lim inf") is

$$\liminf_{n\to\infty} x_n := \lim_{m\to\infty} \inf \{x_n : n \geq m\}.$$

Proposition 8. *Limit superior and limit inferior of a sequence always exist.*

Proof. We prove the case for lim sup. Define $a_n := \sup\{x_k : k \geq n,\ k \in \mathbb{N}\}$.

Suppose first that $a_n < \infty$ for all $n \in \mathbb{N}$. Then, we must have $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$ since a_{n+1} is a supremum over a smaller set than a_n. Thus, $(a_n)_n$ is monotone decreasing. If $(a_n)_n$ is unbounded, (a_n) diverges to $-\infty$ (Proposition 6); if, instead, $(a_n)_n$ is bounded, then (a_n) converges to a limit $a = \sup\{a_n : n \in \mathbb{N}\}$ (Proposition 5).

Suppose instead that $a_n = \infty$ for some $n \in \mathbb{N}$. If there are finitely many such n's, $a_n < \infty$ for all $n > N$ for some sufficiently large N. Applying the previous argument implies that $\limsup_{n\to\infty} x_n$ is well-defined. If, instead, $a_n = \infty$ for all $n \in \mathbb{N}$, then the limit of a_n is $+\infty$. ∎

Proposition 9. *Let (x_n) be a sequence. If $\liminf x_n = \limsup x_n$, then $\lim x_n$ is well-defined and $\lim x_n = \liminf x_n = \limsup x_n$.*

Proof. Suppose $\liminf x_n = \limsup x_n = x \in \mathbb{R}$. (The cases $\pm\infty$ are easier and left as an exercise.) Fix any $\epsilon > 0$. By definition of lim sup, there exists $N_0 \in \mathbb{N}$ such that $|x - \sup\{x_n : n \geq N_0\}| < \epsilon$ (why?). In particular, $\sup\{x_n : n \geq N_0\} < x + \epsilon$, so $x_n < x + \epsilon$ for all $n > N_0$. In the same fashion (how?), we can prove that there exists N_1 such that $x_n > x - \epsilon$ for all $n > N_1$. Putting these together, for all $n > \max\{N_0, N_1\}$, $x - \epsilon < x_n < x + \epsilon$; equivalently, $|x_n - x| < \epsilon$, which is what we wanted to prove. ■

Exercise 32 (PS2)**.** TFU: If $(x_n)_n$ is a sequence, there exists an $M \in \mathbb{N}$ such that $\limsup x_n = \sup\{x_n : n \geq M\}$.

Exercise 33 (PS2)**.** Replace $\star$ with an appropriate symbol, then prove: For any sequences (x_n), (y_n),

$$\limsup_{n\to\infty}(x_n + y_n) \;\star\; \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$$

provided the right hand side is not of the form $\infty + (-\infty)$ (which is undefined).

Exercise 34 (PS2)**.** Consider the following non-theorem: *Let $x_n \to x \geq 0$ and (y_n) be any sequence. Then* $\limsup x_n y_n = x \limsup y_n$. Disprove this, then identify a tiny change to the assumptions that makes it true (but don't prove it).

2.5 Cauchy Sequences

Definition 13. A sequence $(x_n)_n$ is *Cauchy* if, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_n - x_m| < \epsilon$ for all $n, m > N$.

In words, a sequence $(x_n)_n$ is *Cauchy* if the distance between two elements in the tail of the sequence can be made arbitrarily small. The crucial distinction between Cauchy sequences and a convergent sequence is that the former does not refer to the limit point the sequence whereas the latter requires the limit point to exist.

Proposition 10. *If $(x_n)_n$ converges to $x \in \mathbb{R}$, then $(x_n)_n$ is Cauchy.*

Proof. Fix $\epsilon > 0$. Since $x_n \to x$, there exists $N \in \mathbb{N}$ such that $|x_n - x| < \frac{\epsilon}{2}$ for all $n > N$. Since it is just a change of labels, it is also the case that for all $m > N$, $|x_m - x| < \frac{\epsilon}{2}$. Next, by the triangle inequality

$$|x_n - x_m| = |x_n - x + x - x_m| \leq |x_n - x| + |x - x_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence, $(x_n)_n$ is Cauchy. ■

Proposition 11. *If $(x_n)_n$ is Cauchy, then it is bounded.*

Proof. If $(x_n)_n$ is Cauchy, then, in particular, for $\epsilon = 1$, there exists $N \in \mathbb{N}$ such that $|x_n - x_m| < 1$ for all $n, m > N$. In particular, this holds fixing $m = N + 1$. The "reverse" triangle inequality[3] then gives $|x_n| < |x_{N+1}| + 1$, for all $n > N$. Now take $M = \max\{|x_{N+1}| + 1, |x_0|, \ldots, |x_N|\} < +\infty$ and note that $|x_n| \leq M$ for all $n \in \mathbb{N}$. Hence $(x_n)_n$ is bounded. ■

Proposition 12. *If $(x_n)_n$ is a Cauchy sequence and there is a subsequence $(x_{n_k})_k$ that converges to $x \in \mathbb{R}$, then $(x_n)_n$ converges to x as well.*

Exercise 35. Prove Proposition 12.

Theorem 2 (Cauchy criterion)**.** *A sequence $(x_n)_n$ is convergent if and only if it is Cauchy.*

[3] That is, $|x - y| \geq ||x| - |y||$.

Proof. There are two implications to prove. The "only if" was Proposition 10. Let us prove the "if" part. Suppose that (x_n) is a Cauchy sequence. By Proposition 11, $(x_n)_n$ is bounded. Now since $(x_n)_n$ is a bounded sequence, by the Bolzano-Weierstrass Theorem there is a subsequence $(x_{n_k})_k$ which converges. Then, by Proposition 12, we know that $(x_n)_n$ must converge as well. ■

Remark 8. A (metric) space is called *complete* if every Cauchy sequence is convergent. Thus, the previous result establishes that $\mathbb{R}$ is *complete.* . Completeness is the idea that the set has no "holes". For example, $\mathbb{Q}$ is not complete because there are Cauchy sequences that are not convergent (e.g., take a sequence that converges to $\sqrt{2} \notin \mathbb{Q}$). We like to work with complete spaces because it ensures that solutions exist; e.g., we want to be able to solve $x^2 = 2$!

2.6 Sequences in $\mathbb{R}^d$

So far, we have only considered sequences in $\mathbb{R}$; i.e., $(x_n)_n$ such that $x_n \in \mathbb{R}$ for all $n \in \mathbb{N}$. All the results we discussed above can be extended to the case when $\mathbf{x}_n \in \mathbb{R}^d$ (i.e., a product space of $\mathbb{R}$) for all $n \in \mathbb{N}$ for any $d \in \mathbb{N}$. Recall that we measured "distance" between two real numbers using the absolute value of the different ($|\cdot|$).

Definition 14. If $\mathbf{x} \in \mathbb{R}^d$, write $\mathbf{x} = (x_1, \dots, x_k)$. The *Euclidean distance* between $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ is given by

$$\|\mathbf{x} - \mathbf{y}\|_d = \sqrt{\sum_{i=1}^{d} (x_i - y_i)^2}.$$

We often simply write $\|\cdot\|$ (without the subscript d).

We now define $\mathbf{x}_n \to \mathbf{x}$ if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|\mathbf{x}_n - \mathbf{x}\| < \epsilon$ for all $n > N$. To extend the previous results, one can use the fact that a sequence $(\mathbf{x}_n)$ in $\mathbb{R}^d$ converging to a limit $\mathbf{x}$ is equivalent to convergence in each coordinate. Let $x_{n,i}$ denote the ith element of $\mathbf{x}_n \in \mathbb{R}^d$.

Proposition 13. *A sequence $(\mathbf{x}_n)_n$ in $\mathbb{R}^d$ converges to a limit $\mathbf{x}$ if and only if $x_{n,i} \to x_i$ for all $i \in \{1, \dots, d\}$.*

Proof. First, suppose that $\mathbf{x}_n \to \mathbf{x}$. We wish to show that $x_{n,i} \to x_i$ for all $i \in \{1, \dots, d\}$; i.e., for each $i \in \{1, \dots, d\}$, and for any $\epsilon_i > 0$, there exists $N_{\epsilon_i} \in \mathbb{N}$ such that $|x_{n,i} - x_i| < \epsilon_i$ for all $n > N_{\epsilon_i}$. Let $\epsilon_i = \epsilon > 0$ for all $i \in \{1, \dots, d\}$. By definition of $\mathbf{x}_n \to \mathbf{x}$, we know that there exists $N_\epsilon \in \mathbb{N}$ such that, for all $n > N_\epsilon$,

$$\epsilon > \sqrt{\sum_{i=1}^{d} |x_{n,i} - x_i|^2} \geq \sqrt{|x_{n,j} - x_j|^2} = |x_{n,j} - x_j|\,,$$

for any $j \in \{1, \dots, d\}$. For each $i \in \{1, \dots, d\}$, by setting $N_{\epsilon_i} = N_\epsilon$, we have shown that $x_{n,i} \to x_i$.

Next, suppose that $x_{n,i} \to x_i$ for all $i \in \{1, \dots, d\}$. We wish to show that $\mathbf{x}_n \to \mathbf{x}$; i.e., for any $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that $\|\mathbf{x}_n - \mathbf{x}\| < \epsilon$ for all $n > N_\epsilon$. Define $\eta \coloneqq \epsilon/\sqrt{d}$. For each $i \in \{1, \dots, d\}$, by definition of $x_{n,i} \to x_i$, there exists $N_i^\eta \in \mathbb{N}$ such that $|x_{n,i} - x_i| < \eta$ for all $n > N_i^\eta$. Define $N_\epsilon \coloneqq \max\{N_1^\eta, \dots, N_d^\eta\}$ which is well defined since d is finite. Then, for any

$n > N_e$, we have

$$|x_{n,i} - x_i| < \eta = \frac{\epsilon}{\sqrt{d}} \ \forall i \in \{1, \ldots, d\}$$
$$\Leftrightarrow \|\mathbf{x}_n - \mathbf{x}\| = \sqrt{\sum_{i=1}^{d} |x_{n,i} - x_i|^2} < \sqrt{\sum_{i=1}^{d} \left(\frac{\epsilon}{\sqrt{d}}\right)^2} = \epsilon. \qquad \blacksquare$$