ECON 6090

Problem Set 3

Gabe Sekeres

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Worked with Sara Yoo

1. Consider the Production possibilities set

$$Y = \left\{(q, -z) \in \mathbb{R}^2_+ \times \mathbb{R}^2_- : z_1^\alpha z_2^\beta \geq \frac{q_1^2 + q_2^2}{2}\right\}$$

where $\alpha, \beta > 0$.

(a) We have that the cost minimization problem is

$$\min_{z \in \mathbb{R}^2_+} w_1 z_1 + w_2 z_2 \text{ s.t. } z_1^\alpha z_2^\beta \geq \frac{q_1^2 + q_2^2}{2}$$

Which has the Lagrangian

$$\mathcal{L} = w_1 z_1 + w_2 z_2 + \lambda\left(\frac{q_1^2 + q_2^2}{2} - z_1^\alpha z_2^\beta\right)$$

Taking first order conditions, we get

$$\frac{\partial \mathcal{L}}{\partial z_1} = w_1 - \lambda \alpha z_2^\beta z_1^{\alpha - 1} = 0 \implies \lambda = \frac{w_1}{\alpha z_2^\beta z_1^{\alpha-1}}$$

$$\frac{\partial \mathcal{L}}{\partial z_2} = w_2 - \lambda \beta z_1^\alpha z_2^{\beta - 1} = 0 \implies \lambda = \frac{w_2}{\beta z_1^\alpha z_2^{\beta-1}}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \frac{q_1^2 + q_2^2}{2} - z_1^\alpha z_2^\beta = 0 \implies \frac{q_1^2 + q_2^2}{2} = z_1^\alpha z_2^\beta$$

Combining, we get that

$$\frac{w_1}{\alpha z_2^\beta z_1^{\alpha-1}} = \frac{w_2}{\beta z_1^\alpha z_2^{\beta-1}} \implies \frac{w_1 z_1}{\alpha} = \frac{w_2 z_2}{\beta} \implies z_1 = \frac{w_2 \alpha}{w_1 \beta} z_2$$

Substituting into the constraint, we get that

$$\left(\frac{w_2 \alpha}{w_1 \beta} z_2\right)^\alpha z_2^\beta = \frac{q_1^2 + q_2^2}{2}$$

$$z_2^{\alpha+\beta} = \left(\frac{w_1 \beta}{w_2 \alpha}\right)^\alpha \frac{q_1^2 + q_2^2}{2}$$

$$z_2^\star(w_1, w_2, q_1, q_2) = \left(\frac{w_1 \beta}{w_2 \alpha}\right)^{\frac{\alpha}{\alpha+\beta}} \left(\frac{q_1^2 + q_2^2}{2}\right)^{\frac{1}{\alpha+\beta}}$$

Substituting back into the equation for z_1, we get that

$$z_1 = \frac{w_2 \alpha}{w_1 \beta}\left(\frac{w_1 \beta}{w_2 \alpha}\right)^{\frac{\alpha}{\alpha+\beta}} \left(\frac{q_1^2 + q_2^2}{2}\right)^{\frac{1}{\alpha+\beta}}$$

$$z_1^\star(w_1, w_2, q_1, q_2) = \left(\frac{w_2 \alpha}{w_1 \beta}\right)^{\frac{\beta}{\alpha+\beta}} \left(\frac{q_1^2 + q_2^2}{2}\right)^{\frac{1}{\alpha+\beta}}$$

(b) We can find the Marginal Rate of Transformation by implicitly differentiating the border of the production possibilities set. More specifically, since we have that price must remain constant, we have that

$$0 = 2q_1 \partial q_1 + 2q_2 \partial q_2 \implies MRT_{q_1,q_2} = \frac{\partial q_1}{\partial q_2} = -\frac{q_2}{q_1}$$

2. Consider a single-output firm with technology that can transform inputs $z \in \mathbb{R}^3_+$ into output according to the production function

$$f(z) = z_1^{\frac{1}{2}} z_2^{\frac{1}{4}} z_3^{\frac{1}{8}}$$

(a) We have that the production function is homogeneous of degree α, meaning that $f(\beta z) = \beta^\alpha f(z)$. We have that

$$f(\beta z) = (\beta z_1)^{\frac{1}{2}} (\beta z_2)^{\frac{1}{4}} (\beta z_3)^{\frac{1}{8}} = \beta^{\frac{7}{8}} z_1^{\frac{1}{2}} z_2^{\frac{1}{4}} z_3^{\frac{1}{8}} = \beta^{\frac{7}{8}} f(z)$$

so $\alpha = \frac{7}{8}$. This implies that the firm's cost function is homogeneous of degree $\frac{8}{7}$ in q, which implies that the firm faces increasing marginal cost of production in q.

(b) We have that the cost minimization problem is

$$\min_{z \in \mathbb{R}^3_+} w \cdot z \text{ s.t. } z_1^{\frac{1}{2}} z_2^{\frac{1}{4}} z_3^{\frac{1}{8}} \geq q$$

which admits the Lagrangian

$$\mathcal{L} = w \cdot z + \lambda \left(q - z_1^{\frac{1}{2}} z_2^{\frac{1}{4}} z_3^{\frac{1}{8}} \right)$$

Our first order conditions are

$$\frac{\partial \mathcal{L}}{\partial z_1} = w_1 - \frac{\lambda}{2} z_1^{-\frac{1}{2}} z_2^{\frac{1}{4}} z_3^{\frac{1}{8}} = 0 \implies \lambda = \frac{2w_1}{z_1^{-\frac{1}{2}} z_2^{\frac{1}{4}} z_3^{\frac{1}{8}}}$$

$$\frac{\partial \mathcal{L}}{\partial z_2} = w_2 - \frac{\lambda}{4} z_1^{\frac{1}{2}} z_2^{-\frac{3}{4}} z_3^{\frac{1}{8}} = 0 \implies \lambda = \frac{4w_2}{z_1^{\frac{1}{2}} z_2^{-\frac{3}{4}} z_3^{\frac{1}{8}}}$$

$$\frac{\partial \mathcal{L}}{\partial z_3} = w_3 - \frac{\lambda}{8} z_1^{\frac{1}{2}} z_2^{\frac{1}{4}} z_3^{-\frac{7}{8}} = 0 \implies \lambda = \frac{8w_3}{z_1^{\frac{1}{2}} z_2^{\frac{1}{4}} z_3^{-\frac{7}{8}}}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = q - z_1^{\frac{1}{2}} z_2^{\frac{1}{4}} z_3^{\frac{1}{8}} = 0 \implies q = z_1^{\frac{1}{2}} z_2^{\frac{1}{4}} z_3^{\frac{1}{8}}$$

Equating the first two conditions, we get that

$$\frac{2w_1}{z_1^{-\frac{1}{2}} z_2^{\frac{1}{4}} z_3^{\frac{1}{8}}} = \frac{4w_2}{z_1^{\frac{1}{2}} z_2^{-\frac{3}{4}} z_3^{\frac{1}{8}}} \implies z_1 w_1 = 2w_2 z_2 \implies z_2 = \frac{w_1}{2w_2} z_1$$

Equating the first and third conditions, we get that

$$\frac{2w_1}{z_1^{-\frac{1}{2}} z_2^{\frac{1}{4}} z_3^{\frac{1}{8}}} = \frac{8w_3}{z_1^{\frac{1}{2}} z_2^{\frac{1}{4}} z_3^{-\frac{7}{8}}} \implies z_1 w_1 = 4w_3 z_3 \implies z_3 = \frac{w_1}{4w_3} z_1$$

Combining into the constraint, we get that

$$q = z_1^{\frac{1}{2}} \left(\frac{w_1}{2w_2} z_1 \right)^{\frac{1}{4}} \left(\frac{w_1}{4w_3} z_1 \right)^{\frac{1}{8}} \implies z_1^{\frac{7}{8}} = q \left(\frac{2w_2}{w_1} \right)^{\frac{1}{4}} \left(\frac{4w_3}{w_1} \right)^{\frac{1}{8}}$$

Which implies that

$$z_1^\star(w, q) = q^{\frac{8}{7}} \left(\frac{2w_2}{w_1} \right)^{\frac{2}{7}} \left(\frac{4w_3}{w_1} \right)^{\frac{1}{7}}$$

Substituting back, we get that

$$z_2 = \frac{w_1}{2w_2} q^{\frac{8}{7}} \left(\frac{2w_2}{w_1}\right)^{\frac{2}{7}} \left(\frac{4w_3}{w_1}\right)^{\frac{1}{7}}$$

and

$$z_3 = \frac{w_1}{4w_3} q^{\frac{8}{7}} \left(\frac{2w_2}{w_1}\right)^{\frac{2}{7}} \left(\frac{4w_3}{w_1}\right)^{\frac{1}{7}}$$

which imply that

$$z_2^\star(w,q) = q^{\frac{8}{7}} \left(\frac{w_1}{2w_2}\right)^{\frac{5}{7}} \left(\frac{4w_3}{w_1}\right)^{\frac{1}{7}}$$

$$z_3^\star(w,q) = q^{\frac{8}{7}} \left(\frac{2w_2}{w_1}\right)^{\frac{2}{7}} \left(\frac{w_1}{4w_3}\right)^{\frac{6}{7}}$$

(c) We have that the cost function of the firm is $C(w,q) = w_1 z_1^\star + w_2 z_2^\star + w_3 z_3^\star$, so substituting:

$$C(w,q) = w_1 q^{\frac{8}{7}} \left(\frac{2w_2}{w_1}\right)^{\frac{2}{7}} \left(\frac{4w_3}{w_1}\right)^{\frac{1}{7}} + w_2 q^{\frac{8}{7}} \left(\frac{w_1}{2w_2}\right)^{\frac{5}{7}} \left(\frac{4w_3}{w_1}\right)^{\frac{1}{7}} + w_3 q^{\frac{8}{7}} \left(\frac{2w_2}{w_1}\right)^{\frac{2}{7}} \left(\frac{w_1}{4w_3}\right)^{\frac{6}{7}}$$

which equals

$$C(w,q) = q^{\frac{8}{7}} \left[w_1 \left(\frac{2w_2}{w_1}\right)^{\frac{2}{7}} \left(\frac{4w_3}{w_1}\right)^{\frac{1}{7}} + w_2 \left(\frac{w_1}{2w_2}\right)^{\frac{5}{7}} \left(\frac{4w_3}{w_1}\right)^{\frac{1}{7}} + w_3 \left(\frac{2w_2}{w_1}\right)^{\frac{2}{7}} \left(\frac{w_1}{4w_3}\right)^{\frac{6}{7}} \right]$$

so we have that the marginal cost of production is

$$\frac{\partial C(w,q)}{\partial q} = \frac{8}{7} q^{\frac{1}{7}} \left[w_1 \left(\frac{2w_2}{w_1}\right)^{\frac{2}{7}} \left(\frac{4w_3}{w_1}\right)^{\frac{1}{7}} + w_2 \left(\frac{w_1}{2w_2}\right)^{\frac{5}{7}} \left(\frac{4w_3}{w_1}\right)^{\frac{1}{7}} + w_3 \left(\frac{2w_2}{w_1}\right)^{\frac{2}{7}} \left(\frac{w_1}{4w_3}\right)^{\frac{6}{7}} \right]$$

3. Consider a single-output firm which takes a continuum of inputs. The production function is

$$f(z) = \left[\int_0^1 a(j) z(j)^{\frac{\sigma-1}{\sigma}} dj\right]^{\frac{\sigma}{\sigma-1}}$$

where $a(j)$ is a continuous function (and thus integrable on $[0,1]$) that reflects the relative productivities of various inputs.

(a) We have that the continuous cost minimization problem is

$$\min_{z(j)} \int_0^1 w(j) z(j) dj \text{ s.t. } q = \left[\int_0^1 a(j) z(j)^{\frac{\sigma-1}{\sigma}} dj\right]^{\frac{\sigma}{\sigma-1}}$$

which admits the Lagrangian

$$\mathcal{L} = \int_0^1 w(j) z(j) dj + \lambda \left(q - \left[\int_0^1 a(j) z(j)^{\frac{\sigma-1}{\sigma}} dj\right]^{\frac{\sigma}{\sigma-1}} \right)$$

The first order condition with respect to some $z(j)$ is

$$\frac{\partial \mathcal{L}}{\partial z(j)} = w(j) - \lambda \frac{\sigma}{\sigma-1} \left(\int_0^1 a(i) z(i)^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{1}{\sigma-1}} \frac{\sigma-1}{\sigma} a(j) z(j)^{-\frac{1}{\sigma}} = 0$$

and using the fact that $q = \left[\int_0^1 a(j)z(j)^{\frac{\sigma-1}{\sigma}}dj\right]^{\frac{\sigma}{\sigma-1}}$ in equilibrium, we get that this simplifies to

$$w(j) - \lambda a(j)z(j)^{-\frac{1}{\sigma}}q^{\frac{1}{\sigma}} = 0 \Longrightarrow z(j) = \left(\frac{\lambda a(j)q^{\frac{1}{\sigma}}}{w(j)}\right)^{\sigma}$$

We can find λ by substituting back into the budget constraint:

$$q = \left[\int_0^1 a(j)\left(\frac{\lambda a(j)q^{\frac{1}{\sigma}}}{w(j)}\right)^{\sigma-1}dj\right]^{\frac{\sigma}{\sigma-1}} \Longrightarrow q = \left[\lambda^{\sigma-1}q^{\frac{\sigma-1}{\sigma}}\int_0^1 a(j)^{\sigma}w(j)^{1-\sigma}dj\right]^{\frac{\sigma}{\sigma-1}}$$

and we get that

$$\lambda^{\star} = \left[\int_0^1 a(j)^{\sigma}w(j)^{1-\sigma}dj\right]^{-\frac{1}{\sigma-1}}$$

Thus, we have that

$$z^{\star}(j,w,q) = \left(\frac{a(j)}{w(j)}\right)^{\sigma} q \cdot \left[\int_0^1 a(i)^{\sigma}w(i)^{1-\sigma}di\right]^{-\frac{\sigma}{\sigma-1}}$$

(b) The conditional input demand for input j is increasing in the productivity of input j, as long as $\sigma \in (0,1)$:

$$\begin{aligned}\frac{\partial z^{\star}(j,w,q)}{\partial a(j)} &= \sigma a(j)^{\sigma-1}w(j)^{-\sigma}q\left[\int_0^1 a(i)^{\sigma}w(i)^{1-\sigma}di\right]^{-\frac{\sigma}{\sigma-1}} \\ &+ \frac{\sigma}{1-\sigma}\left(\frac{a(j)}{w(j)}\right)^{2\sigma-1} q \cdot \left[\int_0^1 a(i)^{\sigma}w(i)^{1-\sigma}di\right]^{\frac{1-2\sigma}{1-\sigma}} > 0\end{aligned}$$

(c) We have that the new cost minimization problem is

$$\min_{z(j)} \int_0^1 \frac{1}{2}z(j)^2 dj \text{ s.t. } 1 = \left[\int_0^1 a(j)z(j)^{\frac{\sigma-1}{\sigma}}dj\right]^{\frac{\sigma}{\sigma-1}}$$

which admits the Lagrangian

$$\mathcal{L} = \int_0^1 \frac{1}{2}z(j)^2 dj + \lambda\left(1 - \left[\int_0^1 a(j)z(j)^{\frac{\sigma-1}{\sigma}}dj\right]^{\frac{\sigma}{\sigma-1}}\right)$$

The first order condition with respect to some $z(j)$ is

$$\frac{\partial \mathcal{L}}{\partial z(j)} = z(j) - \lambda\frac{\sigma}{\sigma-1}\left(\int_0^1 a(i)z(i)^{\frac{\sigma-1}{\sigma}}di\right)^{\frac{1}{\sigma-1}}\frac{\sigma-1}{\sigma}a(j)z(j)^{-\frac{1}{\sigma}} = 0$$

and again using the fact that $q = 1 = \left[\int_0^1 a(j)z(j)^{\frac{\sigma-1}{\sigma}}dj\right]^{\frac{\sigma}{\sigma-1}}$ in equilibrium, we get that

$$z(j) - \lambda a(j)z(j)^{-\frac{1}{\sigma}}q^{\frac{1}{\sigma}} = 0 \Longrightarrow z(j) = (\lambda a(j))^{\frac{\sigma}{\sigma+1}}$$

We substitute back into the budget constraint to get λ:

$$1 = \left[\int_0^1 a(j)\left(\lambda a(j)\right)^{\frac{\sigma-1}{\sigma+1}}dj\right]^{\frac{\sigma}{\sigma-1}} \Longrightarrow \lambda^{\frac{\sigma+1}{\sigma}} = \left[\int_0^1 a(j)^{\frac{2\sigma}{\sigma+1}}dj\right]^{\frac{\sigma}{\sigma-1}} \Longrightarrow \lambda^{\star} = \left[\int_0^1 a(j)^{\frac{2\sigma}{\sigma+1}}dj\right]^{\frac{\sigma^2}{\sigma^2-1}}$$

so we get that

$$z^\star(j,w,1) = a(j)^{\frac{\sigma}{\sigma+1}} \left[\int_0^1 a(i)^{\frac{2\sigma}{\sigma+1}}\, di\right]^{\frac{\sigma^3}{(\sigma^2-1)(\sigma+1)}}$$

4. Consider a single-output firm with technology that can transform inputs $z \in \mathbb{R}^N_+$ into output according to the production function

$$f(z) = 2\sqrt{\min\{z_1, 2z_2, 3z_3, \ldots, Nz_n\}}$$

(a) We have that the profit maximization problem is

$$\max_{z\in\mathbb{R}^N_+} 2p\sqrt{\min\{z_1, 2z_2, 3z_3, \ldots, Nz_N\}} - w\cdot z$$

Note that the goods in this case are (transformations of) perfect complements. Define a variable as follows: $\zeta_i = i \cdot z_i$, which admits the maximization problem

$$\max_{\zeta\in\mathbb{R}^N_+} 2p\sqrt{\min\{\zeta_1, \zeta_2, \ldots, \zeta_N\}} - \sum_{i=1}^N \zeta_i \frac{w_i}{i}$$

Since the inputs are now truly perfect complements, we can say that $\zeta_1 = \zeta_2 = \cdots = \zeta_N = \zeta$, and the maximization problem becomes

$$\max_{\zeta\in\mathbb{R}_+} 2p\sqrt{\zeta} - \zeta\sum_{i=1}^N \frac{w_i}{i}$$

The first order conditions are

$$\frac{p}{\sqrt{\zeta}} - \sum_{i=1}^N \frac{w_i}{i} = 0 \Longrightarrow \zeta^\star = \frac{p^2}{\left(\sum_{i=1}^N \frac{w_i}{i}\right)^2}$$

Converting back into our original variables, we get that the unconditional input demand function for input i is:

$$z_i^\star(p,w) = \frac{p^2}{i\left(\sum_{i=1}^N \frac{w_i}{i}\right)^2}$$

(b) We now assume that the firm has output market power, so $P(q) = q^{-\varepsilon}$. Our new maximization problem is

$$\max_{z\in\mathbb{R}^N_+} f(z)^{-\varepsilon} f(z) - w\cdot z \equiv \max_{z\in\mathbb{R}^N_+} f(z)^{1-\varepsilon} - w\cdot z$$

Using the same change of variable as in part (a), and recalling that goods are still perfect complements, we get that with $\zeta_i = iz_i$, the maximization problem is

$$\max_{\zeta\in\mathbb{R}^N_+} \left(2\sqrt{\min\{\zeta_1, \zeta_2, \ldots, \zeta_N\}}\right)^{1-\varepsilon} - \sum_{i=1}^N \zeta_i \frac{w_i}{i} \equiv \max_{\zeta\in\mathbb{R}_+} \left(2\sqrt{\zeta}\right)^{1-\varepsilon} - \zeta\sum_{i=1}^N \frac{w_i}{i}$$

The first order conditions are

$$2^{-\varepsilon}(1-\varepsilon)\zeta^{-\frac{(\varepsilon+1)}{2}} - \sum_{i=1}^N \frac{w_i}{i} = 0 \Longrightarrow \zeta^\star = \left(\frac{1-\varepsilon}{2^\varepsilon \sum_{i=1}^N \frac{w_i}{i}}\right)^{\frac{2}{\varepsilon+1}}$$

So we get the unconditional input demand function given input prices w is

$$z_i^\star(w) = \frac{1}{i}\left(\frac{1-\varepsilon}{2^\varepsilon \sum_{i=1}^N \frac{w_i}{i}}\right)^{\frac{2}{\varepsilon+1}}$$

5. Producer theory in action (De Loecker, Eeckhout, and Unger 2020)

   (a) When the authors assert that "The Lagrange multiplier λ is a direct measure of marginal cost," they are using the implicit structure of the cost minimization problem. Formally, we have that the Lagrangian in the cost minimization problem is

   $$\mathcal{L} = w \cdot z - \lambda(q - f(z))$$

   Since the objective is minimized, the Lagrange multiplier measures how much the minimum (cost) increases for an increase in the output level. This is precisely the definition of marginal cost – when output increases by 1 unit, total cost (from the cost function) will increase by λ units.

   (b) The authors start with the cost minimization problem despite it not featuring price because they are not assuming that firms are price-takers. If they were, and firms were in perfect competition, then it would make a lot more sense to work under the profit maximization problem. However, since they are assuming that firms have output market power, they would need to estimate a demand function. By working with the cost minimization problem, they can simply estimate the cost function and then calculate the markup from the output price, which is public information. They are assuming that firms are profit maximizing, but since the profit maximization problem implies the cost minimization problem, they can reach the cost function by solving the cost minimization problem and then divide the price (observable) by the cost function (calculated) to get the value of the markup. In this way, they sidestep the (extremely hard, often intractable) problem of calculating the demand function.