ECON 6170 Problem Set 4 Solutions

Patrick Ferguson

Problem 1. Prove or disprove: $f : S \subseteq \mathbb{R} \to \mathbb{R}$ is continuous at x_0 iff

For every monotone sequence (x_n) in S converging to x_0, we have $\lim_n f(x_n) = f(x_0)$

One direction is trivial: if continuity means that $x_n \to x_0$ implies $f(x_n) \to f(x_0)$ for *any* (x_n) then it implies it in particular for monotone (x_n).

The other direction is quite challenging. Suppose $x_n \to x_0$. To prove continuity of f, we need to show that $f(x_n) \to f(x_0)$. To do this, we can restate the lemma from Section 2 as

$f(x_n) \to f(x_0)$ iff every subsequence $(f(x_{n_k}))$ contains a subsubsequence $f(x_{n_{k_i}}) \to f(x_0)$

Therefore, we want to prove

Every subsequence $(f(x_{n_k}))$ contains a subsubsequence $f(x_{n_{k_i}}) \to f(x_0)$

But consider the subsequence (x_{n_k}) of (x_n), consisting of those entries that are mapped to the corresponding entries of $(f(x_{n_k}))$. This has a monotone subsequence (Proposition 1.7), which we'll call $(x_{n_{k_i}})$. Given $x_n \to x_0$, we must also have $x_{n_{k_i}} \to x_0$. By hypothesis, this implies $f(x_{n_{k_i}}) \to f(x_0)$. But $(f(x_{n_{k_i}}))$ is a subsubsequence of $(f(x_{n_k}))$, so the latter has a subsubsequence converging to $f(x_0)$, as required.

Problem 2. Let $S \subseteq \mathbb{R}^d$ be open.[1] Prove: a function $f : S \to \mathbb{R}^k$ is continuous if and only if for every open set $A \subseteq \mathbb{R}^k$, $f^{-1}(A)$ is open.

The key to answering this problem is to recognise that in the "neighbourhood" definition of continuity $\|x - x_0\| < \delta$ is the same as $x \in B_\delta(x_0)$ and $\|f(x) - f(x_0)\| < \epsilon$ is the same as $f(x) \in B_\epsilon(f(x_0))$.

First, we suppose f is continuous. Let A be an arbitrary open subset of $\mathbb{R}^k$, and let x_0 be a point in the preimage of A. That is, $f(x_0) \in A$. Then, by openness of A, there exists an open ball centered at $f(x_0)$, $B_\epsilon(f(x_0))$, that is contained in A. Continuity of f implies that there exists a $\delta > 0$ such that, for $x \in S$, $\|x - x_0\| < \delta$ implies $\|f(x) - f(x_0)\| < \epsilon$. Without loss of generality, take δ small enough that if $\|x - x_0\| < \delta$ then $x \in S$. We have shown that the open ball $B_\delta(x_0)$ is contained in $f^{-1}(A)$. Since x_0 is an arbitrary point of A, this shows A is open.

Conversely, suppose f is a function such that the preimage of any open set under f is also open. In particular, the preimage of every open ball is open. In particular, for any $x_0 \in S$ and any

[1] The homework problem assumed $d = 1$. The proof is effectively the same.

$\epsilon > 0$, we know that $x_0 \in f^{-1}(B_\epsilon(f(x_0)))$. Since the latter is open, there is some $\delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(B_\epsilon(f(x_0)))$. Equivalently, there is some $\delta > 0$ such that for all $x \in S$, $\|x - x_0\| < \delta$ implies $\|f(x) - f(x_0)\| < \epsilon$. But this is just the ϵ-δ definition of continuity.