About TA sections:

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Our plan for today:[1]

- A brief overview of math you will need in this course (and most likely in other courses too): optimization, concavity, IFT
- Standard properties of utility functions
- Risk aversion
- Equilibrium concepts
- Euler equation
- Consumption Smoothing
- Transversality Condition

[1] Materials adapted from notes provided by a previous Teaching Assistant, Zhuoheng Xu.

1. Math Review

1.1 Optimization

Unconstrained optimization: Suppose $f(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ is C^2. Then:

- $\mathbf{x}^*$ - a **local maximum** (*necessary conditions*): 1) $\frac{\partial f(\mathbf{x}^*)}{\partial x_i} = 0 \quad \forall i \in \{1, ..., n\}$, 2) $H(\mathbf{x}^*)$ is negative semi-definite (NSD).

- $\mathbf{x}^*$ - a **strict local maximum** (*sufficient conditions*): 1) $\frac{\partial f(\mathbf{x}^*)}{\partial x_i} = 0 \quad \forall i \in \{1, ..., n\}$, 2) $H(\mathbf{x}^*)$ is negative definite (ND).

- If $f(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ is concave over its entire domain, $f(\mathbf{x})$ attains a **global maximum** at $\mathbf{x}^* \iff \frac{\partial f(\mathbf{x}^*)}{\partial x_i} = 0 \quad \forall i \in \{1, ..., n\}$ (*necessary and sufficient*).

Constrained optimization: We usually solve these problems using the Lagrangian function. It is important to remember that, in general, first-order conditions are only necessary but not sufficient. However, if the objective function and all the constraints are concave and smooth functions on a convex domain, necessary conditions are also sufficient. (Check the Kuhn-Tucker theorem for more details).

Remark: Lagrange multipliers indicate marginal values of the constraints, i.e., "how much the objective function would improve if the constraint was relaxed by one unit". In macroeconomics, the vector of Lagrange multipliers λ is often referred to as "the vector of shadow prices". We will talk about it in more detail in following sections.

1.2 Concavity

Some useful properties of concave functions:

- If $f(x)$ and $g(x)$ are concave (convex), then $af(x) + bg(x)$ is concave (convex) for $(a, b) > (0, 0)$.

- If $f(x)$ is concave (convex) and $g(x)$ is concave (convex) and increasing, then $g(f(x))$ is concave (convex).

- If $f(x)$ and $g(x)$ are concave (convex), then $\min\{f(x), g(x)\}$ $(\max\{f(x), g(x)\})$ is concave (convex).

1.3 Implicit Function Theorem

Let $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be continuously differentiable. Suppose $\mathbf{y} = \mathbf{g}(\mathbf{x})$ implicitly defines $\mathbf{y}$ as a function of $\mathbf{x}$ near $(\mathbf{x}_0, \mathbf{y}_0)$, where $F(\mathbf{x}_0, \mathbf{y}_0) = 0$.

If the Jacobian matrix $\frac{\partial F}{\partial \mathbf{y}}(\mathbf{x}_0, \mathbf{y}_0)$ is invertible, then there exists an open set $U \subseteq \mathbb{R}^n$ containing $\mathbf{x}_0$ and a unique continuously differentiable function $\mathbf{g} : U \to \mathbb{R}^m$ such that:

$$F(\mathbf{x}, \mathbf{g}(\mathbf{x})) = 0 \quad \forall \mathbf{x} \in U.$$

Additionally, the partial derivatives of $\mathbf{g}$ can be found using:

$$\frac{\partial \mathbf{g}}{\partial \mathbf{x}} = -\left(\frac{\partial F}{\partial \mathbf{y}}\right)^{-1} \frac{\partial F}{\partial \mathbf{x}}.$$

Intuition: The implicit function theorem helps us understand how one variable adjusts in response to changes in another while maintaining equilibrium.

Example: Consider a consumer optimizing their utility $U(x, y)$ subject to a budget constraint $px_1 + qx_2 = y$, where x_1 and x_2 are quantities of two goods (for example, a consumption good and an investment good), p and q are their respective prices, and y is the income.

The consumer's problem can be formulated as:

$$\max_{x,y} U(x, y)$$

subject to

$$px_1 + qx_2 - y = 0.$$

Let $F(x_1, x_2, \lambda) = \left(\frac{\partial \mathcal{L}}{\partial x_1}, \frac{\partial \mathcal{L}}{\partial x_2}, \frac{\partial \mathcal{L}}{\partial \lambda}\right)$, where $\mathcal{L}(x_1, x_2, \lambda) = U(x, y) + \lambda(y - px_1 - qx_2)$ is the Lagrangian.

At the optimum (x^*, y^*, λ^*), the first-order conditions are:

$$F(x^*, y^*, \lambda^*) = 0.$$

That is, at the optimum:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial U}{\partial x_1} - \lambda p = 0 \\ \frac{\partial \mathcal{L}}{\partial x_2} = \frac{\partial U}{\partial x_2} - \lambda q = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = y - px_1 - qx_2 = 0 \end{cases}$$

Suppose we find an optimal solution $(x_1^*, x_2^*, \lambda^*)$. The implicit function theorem tells us that if the Jacobian matrix of the partial derivatives of F with respect to (x_1, x_2, λ) is invertible at $(x_1^*, x_2^*, \lambda^*)$, then near this point, there exists a function $(x_1, x_2) = \mathbf{g}(p, q, y)$ that describes how the quantities of goods x_1 and x_2 adjust in response to changes in prices and income.

Intuition (what we usually call "comparative statics"): If the prices of the goods or the consumer's income changes slightly, the implicit function theorem guarantees that we can find new optimal quantities x and y that continue to maximize utility under the new conditions. This means the consumer adjusts their consumption smoothly in response to changes in prices and income, maintaining an optimal utility level.

Remark In Macro Qs, it is often the case that utility function is not given a specific functional form, so we cannot get a closed-form solution for consumption. If the question asks for how consumption changes with some parameter, consider using IFT.

2. Utility functions

2.1 Standard properties

We often assume utility functions to be:

1. Continuous
   Interpretation: goods are infinitely divisible, (rational) preferences are continuous.

2. Strictly increasing $U'(\cdot) > 0$
   Interpretation: consuming more is always more preferable then consuming less.

3. Twice continuously differentiable C^2
   Interpretation: for convenience because we want to check second order conditions.

4. Strictly concave $U''(\cdot) < 0$
   Interpretation: diminishing marginal utility + risk aversion (Jensen's inequality).

5. Inada conditions $\lim_{x\to 0} U'(x) = +\infty, \lim_{x\to\infty} U'(x) = 0$
   Interpretation: marginal utility is extremely high at very low levels of consumption, reflecting the critical importance of additional consumption when resources are scarce, and marginal utility approaches zero at very high levels of consumption, indicating that additional consumption provides little extra satisfaction when resources are abundant; practical implication - the utility function is well-behaved at the extreme points and rules out corner solutions.

2.2 Classical utility functions in macroeconomics

Constant Relative Risk Aversion (CRRA):

$$U(C) = \begin{cases} \frac{C^{1-\gamma}}{1-\gamma} & \text{if } \gamma \neq 1 \\ \ln(C) & \text{if } \gamma = 1 \end{cases}$$

where C is consumption and γ is the coefficient of relative risk aversion. It implies that the individual's relative risk aversion remains constant regardless of the level of consumption. You can check it yourself by computing the Arrow–Pratt measure of relative risk aversion:

$$r_R(w) = -\frac{w \cdot u''(w)}{u'(w)} = w \cdot r_A(w)$$

$$r_R(w) = -\frac{w \cdot u''(w)}{u'(w)} = \frac{\frac{du'(w)}{dw} \cdot w}{u'(w)} = -\frac{du'(w)}{u'(w)} \Big/ \frac{dw}{w} = -\frac{\%\Delta u'(w)}{\%\Delta(w)}$$

This measure allows us to compare attitudes towards risky situations whose outcomes are relative gains or losses from current wealth w.

Interpretation: The agent is willing to increase (decrease) the fraction of the portfolio held in the risky asset if the relative risk aversion is decreasing (increasing), given that there is an increase in wealth.

Constant Absolute Risk Aversion (CARA)

$$U(C) = -e^{-\alpha C}$$

where C is consumption and α is the coefficient of absolute risk aversion. It implies that the individual's absolute risk aversion remains constant regardless of the level of consumption.

$$r_A(w) = -\frac{u''(w)}{u'(w)}$$

This measure allows us to compare attitudes towards risky situations whose outcomes are absolute gains or losses from current wealth w. Note it is only a local measure of risk aversion - when comparing across agents, we say one is more/less risk averse than the other at a given wealth level w.

Interpretation: The agent is willing to invest more (less) in the risky asset if the absolute risk aversion is decreasing (increasing), given that there is an increase in wealth.

3. Equilibrium

3.1 Equilibrium concept

Most general definition of an equilibrium:

An equilibrium is an allocation (or a sequence of allocations) and a system of prices such that:

- Given prices, agents solve their optimization problems (e.g., households maximize their utility and firms maximize their profits);
- Markets clear.

Partial equilibrium analysis considers "some", not all markets, assuming other markets remain unchanged and there are no interactions with them. For example, in the consumption-saving from the lecture note, we consider an agent in isolation and assume that y_t, R_t are exogenous.

General equilibrium analysis considers the interactions effects across all markets, ensuring that all markets in the economy are in equilibrium simultaneously and capturing the interdependence of markets.

3.2 Euler equation

Recall the partial equilibrium model from class. The household takes income $\{y_t\}$ and interest rate $\{R_t\}$ as given. Assume standard properties for utility function (see above).

$$\max_{\{c_t, b_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to:

$$c_t + b_{t+1} \leq y_t + R_t b_t \quad (1)$$
$$b_{t+1} \geq -A \quad (2)$$
$$c_t \geq 0 \quad (3)$$
$$b_0 \text{ given} \quad (4)$$

Remark: In problems like this, the discount factor β is typically assumed to be between 0 and 1, capturing the idea that one unit of instantaneous utility tomorrow is less valuable than a unit of utility today.

Before solving, look at the constraints again:

- Notice that for any $\{c_t\}_{t=0}^{\infty}$, if there is any time t such that $c_t = 0$, the series is not a solution (recall utility function properties: strictly increasing, concave, satisfies the Inada condition). Hence, non-negativity of consumption constraints are not binding.
- A is assumed to be large, so the debt limit is never reached.

Using the Lagrange multiplier method:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[u(c_t) + \lambda_t \left(y_t + Rb_t - (c_t + b_{t+1}) \right) \right]$$

Necessary conditions for optimum:

$$\frac{\partial \mathcal{L}}{\partial c_t} = u'(c_t) - \lambda_t = 0$$
$$\frac{\partial \mathcal{L}}{\partial b_{t+1}} = -\lambda_t + \beta R_{t+1} \lambda_{t+1} = 0$$

Combine to get the **Euler Equation**:

$$u'(c_t) = \beta R_{t+1} u'(c_{t+1}) \quad \text{or equivalently,} \quad \frac{u'(c_t)}{\beta u'(c_{t+1})} = R_{t+1}$$

Interpretation: Euler equation describes optimal intertemporal choice:

- LHS = Marginal rate of substitution between c_t and c_{t+1}
- RHS = Gross interest rate at time $t+1$

- $u'(c_t) = \beta R_{t+1} u'(c_{t+1})$: in equilibrium, the marginal benefit of consuming one additional unit today is the same as the discounted marginal benefit of saving that one additional unit for tomorrow.

Remark on the Lagrangian:

- At optimum, the Lagrange multiplier equals marginal utility, which is basically the value of consumption perceived by the agent.

- There are alternative ways to write down the Lagrangian, and they will yield essentially the same equilibrium, but the interpretation of Lagrange multiplier will change. Compare:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[u(c_t) - \lambda_t \left(c_t + b_{t+1} - y_t - R_t b_t \right) \right]$$

Lagrange multiplier $\equiv$ value of consumption at period t perceived by the agent at period t

$$\mathcal{L} = \sum_{t=0}^{\infty} \left[\beta^t u(c_t) - \lambda_t \left(c_t + b_{t+1} - y_t - R_t b_t \right) \right]$$

Lagrange multiplier $\equiv$ value of consumption at period t perceived by the agent at period $0 \equiv$ "the present value" of consumption at period t

4. Interpreting Equilibrium

4.1 Consumption Smoothing

Consumption smoothing describes the desire of a stable path of consumption (an implication of concave utility; related to risk aversion). People desire to translate their consumption from periods of high income to periods of low income to obtain more stability and predictability. There exists many states of the world, which means there are many possible outcomes that can occur throughout an individual's life. Therefore, to reduce the uncertainty that occurs, people choose to give up some consumption today to prevent against an adverse outcome in the future. This can also be viewed as a result of the Permanent Income Hypothesis (PIH) in a deterministic context.

Intuition: Concave utility and Jensen's inequality (a sure amount would always be preferred over a risky bet with the same expected value).

Endowment stream y_t can be volatile (deterministically volatile for now) and we want to look at the conditions such that we have $c_t = c$. From Euler equation and strict concavity of $u(\cdot)$, consumption is constant if and only if $R_{t+1} = R = 1/\beta$.

From time t budget constraint

$$b_t = \frac{1}{R}(c + b_{t+1} - y_t)$$

Take b_0 as given. Iterate over b_t to get

$$b_0 = \frac{c - y_0}{R} + \frac{c - y_1}{R^2} + \frac{c - y_2}{R^3} + \frac{b_3}{R^3} = \sum_{t=0}^{\infty} \frac{c - y_t}{R^{t+1}} + \lim_{t\to\infty} \frac{b_{t+1}}{R^{t+1}}$$

Impose $\lim_{t\to\infty} \frac{b_{t+1}}{R^{t+1}} = 0$ (i.e., agents do not value bond holdings "after the world ends"). Rearrange the expression above to solve for c:

$$c = (1 - \beta) \times \left(Rb_0 + \sum_{t=0}^{\infty} \frac{y_t}{R^t} \right)$$

4.2 Understanding Transversality Condition (TVC)

Important to understand and remember: The TVC is an **optimality** condition of the problem. That is, every infinite horizon problem has a TVC condition associated to the optimal path. Without it, optimality conditions listed above are only necessary, but not sufficient.

Intuition: Consider the consumption-saving problem in the lecture, but in finite horizon. The maximization problem is then formulated as:

$$\max_{\{c_t, b_{t+1}\}} \sum_{t=0}^{T} \beta^t u(c_t)$$

Note that all the budget constraints hold as before, including the terminal one:

$$c_T + b_{T+1} = y_T + R_T b_T$$

There are only T periods in this economy, is there an incentive for us to save b_{T+1}? No. In other words, we need the present value of the "meaningless investment" b_{T+1} to be zero:

$$\beta^T u'(c_T) \cdot b_{T+1} = 0 \quad \text{- Present Value of Marginal Terminal Consumption}$$

Analogously, in an infinite-horizon problem, even though we do not have a "terminal period", the intuition is still the same. Take the limit:

$$\lim_{T\to\infty} \beta^T u'(c_T) b_{T+1} = 0 \quad \text{- TVC}$$

If $\beta = 1/R$, and $c_t = c$, then if we drop the constant $u'(c)$ and divide it by some finite constant R, the TVC becomes (as in the lecture slides):

$$\lim_{T\to\infty} \frac{b_{T+1}}{R^{T+1}} = 0$$