ECON 6170

Problem Set 1

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Exercise 7 from Notes The claim is true.

Proof. WLOG, assume that $\sup S \geq \sup T$. $\max\{\sup S, \sup T\} = \sup S$. By definition, $\sup S \geq s \;\forall\; s \in S$. Also since $\sup S \geq \sup T$, $\sup S \geq \sup T \geq t \;\forall\; t \in T$. Thus, since $\sup S \geq s \;\forall\; s \in S$ and $\sup S \geq t \;\forall\; t \in T$, $\sup S$ is an upper bound of $S \cup T$, and so $\sup S \geq \sup(S \cup T)$.

It remains to show that $\sup S$ is the least upper bound of $S \cup T$. This follows directly from the ε-ball definition of supremum. $\forall\, \varepsilon > 0, \exists\, s \in S$ s.t. $s + \varepsilon > \sup S$. If it were the case that $\sup S \geq \sup(S \cup T)$, then $\exists\, \varepsilon' > 0$ s.t. $\sup S = \sup(S \cup T) + \varepsilon$. However, choosing $\varepsilon < \varepsilon'$, $\exists\, s \in S$ s.t. $s + \varepsilon > \sup S \Rightarrow s > \sup S - \varepsilon > \sup(S \cup T)$. This is a contradiction, so $\sup S$ is the least upper bound of $S \cup T$, and since suprema are unique, $\sup(S \cup T) = \max\{\sup S, \sup T\}$. □

Exercise 1

(i) $\sup(A + B) = \sup A + \sup B$

Proof. Take some $a + b \in A + B$. Since $a \leq \sup A$ and $b \leq \sup B$, $a + b \leq \sup A + \sup B$. Thus, $\sup A + \sup B$ is an upper bound of $A + B$. It remains to show that $\sup A + \sup B$ is the least upper bound of $A + B$. FSOC, assume that $\sup(A + B) < \sup A + \sup B$. Choose $\varepsilon = (\sup A + \sup B - \sup(A + B))/3$. By the ε-ball definition of suprema, $\exists\, a \in A$ and $b \in B$ s.t. $a + \varepsilon > \sup A$ and $b + \varepsilon > \sup B$. $a + b \in A + B$ by definition, but since $\varepsilon = (\sup A + \sup B - \sup(A + B))/3$, $a + b > \sup A + \sup B - 2\varepsilon > \sup(A + B)$. This is a contradiction, so $\sup A + \sup B$ is the least upper bound of $A + B$, and since suprema are unique, $\sup(A + B) = \sup A + \sup B$. □

Alternative Topological Proof:[1]

Proof. Consider the closure of A, denoted $\overline{A}$, where $\overline{A} = A \cup \partial A$, the union of A and the boundary of A, as well as $\overline{B}$. Since the closure contains the union of all sequences in the set, $\sup(A) \in \overline{A}$, and $\sup(A) = \sup(\overline{A})$. Similarly, $\sup(B) = \sup(\overline{B}) \in \overline{B}$. Also note that $\sup(A + B) = \sup(\overline{A} + \overline{B}) \in \overline{A} + \overline{B}$. Also note that since $a \leq \sup(\overline{A}) \;\forall\; a \in \overline{A}$ and $b \leq \sup(\overline{B}) \;\forall\; b \in \overline{B}$, $a + b \leq \sup(\overline{A}) + \sup(\overline{B}) \;\forall\; a + b \in \overline{A} + \overline{B}$. Thus, since $\sup(\overline{A}) + \sup(\overline{B}) \in \overline{A} + \overline{B}$, $\sup(\overline{A}) + \sup(\overline{B}) = \sup(\overline{A} + \overline{B})$, since suprema are unique, and so $\sup A + \sup B = \sup(A + B)$. □

(ii) $\sup(A - B) = \sup A - \inf B$

Proof. Take some $a - b \in A - B$. Since $a \leq \sup A$ and $b \geq \inf B$, $a - b \leq \sup A - \inf B$. Thus, $\sup A - \inf B$ is an upper bound of $A - B$. It remains to show that $\sup A - \inf B$ is the least upper bound of $A - B$. FSOC, assume that $\sup A - \inf B > \sup(A - B)$. Choose $\varepsilon = (\sup A - \inf B > \sup(A - B))/3$. By the ε-ball definition of suprema and infima, $\exists\, a \in A$ and $b \in B$ s.t. $a + \varepsilon > \sup A$ and $b - \varepsilon < \inf B$. $a - b \in A - B$ by definition, but we have that $a - b > \sup A - \inf B - 2\varepsilon > \sup(A - B)$. This is a contradiction, so $\sup A - \inf B$ is the least upper bound of $A - B$, and since suprema are unique $\sup(A - B) = \sup A - \inf B$. □

[1] Because Topology is fun!

Exercise 2

(i) $\sup_{a\in A}\inf_{b\in B} f(a,b) \leq \inf_{b\in B}\sup_{a\in A} f(a,b)$

Proof. By the ε-ball definition of suprema, $\forall\, \varepsilon > 0$, $\exists\, a' \in A$ s.t. $\inf_{b\in B} f(a',b)+\varepsilon > \sup_{a\in A}\inf_{b\in B} f(a,b)$. Also, from the definition of infima, we have that $\inf_{b\in B} f(a',b) \leq \inf_{b\in B}\sup_{a\in A} f(a,b)$. Combining, we get that

$$\sup_{a\in A}\inf_{b\in B} f(a,b) - \varepsilon < \inf_{b\in B} f(a',b) \leq \inf_{b\in B}\sup_{a\in A} f(a,b)$$

and since this is true $\forall\, \varepsilon > 0$, we have that $\sup_{a\in A}\inf_{b\in B} f(a,b) \leq \inf_{b\in B}\sup_{a\in A} f(a,b)$. $\square$

(ii) Consider the function $f : [0,1]^2 \to \mathbb{R}$ where

$$f(a,b) = \begin{cases} 0 & a \neq 0, b = 0 \\ 0 & a \neq 1, b = 1 \\ 1 & \text{otherwise} \end{cases}$$

$\inf_{b\in B} f(a,b) = 0$ since given any a, either $b = 0$ or $b = 1$ will attain $f(a,b) = 0$, so the left side is $\sup_{a\in A} 0 = 0$. However, $\sup_{a\in A} f(a,b) = 1$, since given any b, a choice of $a = 0$ or $a = 1$ will attain $f(a,b) = 1$, so the left side is $\inf_{b\in B} 1 = 1$. Thus, $\sup_{a\in A}\inf_{b\in B} f(a,b) < \inf_{b\in B}\sup_{a\in A} f(a,b)$.