Problem Set 2

Due: TA Discussion, 6 September 2024.

1 Exercises from class notes

All from "1. Real Sequences.pdf".

Exercise 14. TFU: If $x_n \to x$ and $y_n \to y$, then (i) $(x_n + y_n)_n \to x + y$, (ii) $x_n y_n \to xy$, (iii) $x_n - y_n \to x - y$, (iv) $\frac{1}{x_n} \to \frac{1}{x}$; (v) $\frac{x_n}{y_n} \to \frac{x}{y}$.

Exercise 18. TFU: A sequence $(x_n)_n$ converges to x if and only if for all $\epsilon > 0$ infinitely many terms are contained in $(x - \epsilon, x + \epsilon)$, and x is the only number with this property. **Hint:** Take x to be a real number.

Exercise 23. TFU: Every convergent sequence (with a finite limit) is bounded.

Exercise 27. TFU: If a sequence is bounded, then every subsequence is bounded.

Exercise 28. TFU: If a sequence is unbounded, then every subsequence is unbounded.

Exercise 29. TFU: If a sequence is unbounded, then it has a subsequence which is bounded.

Exercise 31. In the second part of the proof of Proposition 7, can you replace $\min\{m \in \mathbb{N} : x_m = \max S_{n_k+1}\}$ with $\max\{m \in \mathbb{N} : x_m = \max S_{n_k+1}\}$?

Exercise 32. TFU: If $(x_n)_n$ is a sequence, there exists an $M \in \mathbb{N}$ such that $\limsup x_n = \sup\{x_n : n \geq M\}$.

Exercise 33. Consider the following non-theorem: Let $(x_n)_n$ be a sequence that converges to $x \geq 0$ and $(y_n)_n$ be any sequence. Then,

$$\limsup x_n y_n = x \limsup y_n.$$

Disprove this, then identify a tiny change to the assumptions that makes it true (but don't prove it).

2 Additional Exercises

Exercise 1. Consider the following, bizarre notion of convergence: say a sequence $x_n \to^* x$ (or $(x_n)_n$ $*$-*converges* to x) if there exists an $N \in \mathbb{N}$ such that for all $\epsilon > 0$, $n > N$ implies that $|x_n - x| < \epsilon$.
Characterise the set of all $\star$-convergent sequences. That is, describe the set of all real sequences, $(x_n)_n$, which $*$-converge to some $x \in \mathbb{R}$. Prove your characterisation.

Remark. This exercise should be a warning for how seemingly slight adjustments of a definition in mathematics can accidentally produce profoundly different concepts or objects.