ECON 6090-Microeconomic Theory. TA Section 4

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In Section notes

Assumptions

1. $u(.)$ represents $\succsim$ and is continuous.
2. $\succsim$ satisfies local non satiation. (LNS)
3. $\succsim$ is strictly convex.

Expenditure minimization problem (EMP)

$$\min_x p \cdot x \text{ such that } u(x) \geq \bar{u}$$

Hicksian demand: $h(p, \bar{u})$

1. If $\inf u(x) \leq \bar{u} \leq \sup u(x)$ then there exist h^* that solves the EMP. (Extreme Value Theorem)
2. $h(p, \bar{u})$ is homogeneous of degree 0 (HoD0) in price (p).
3. $u(h(p, \bar{u})) = \bar{u}$. (LNS)
4. $h(p, \bar{u})$ is a well-defined function and it is continuous. ($\succsim$ strictly convex + Berge's Theorem of the Maximum)

Expenditure function: $e(p, \bar{u})$

1. Continuous in $(p, \bar{u})$.
2. Nondecreasing in p and strinctly increasing in $\bar{u}$.
3. HoD 1 in p.
4. Concave in p.

Roadmap

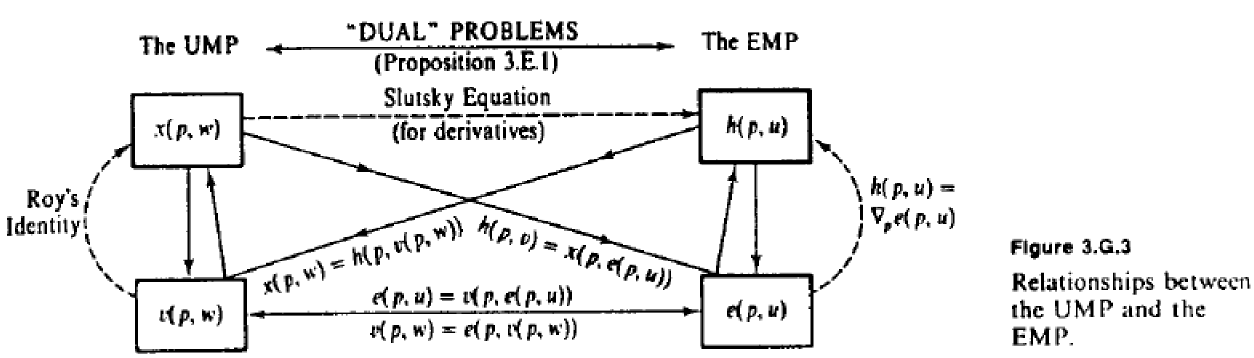

Figure 1: MWG chapter 3. Roadmap.

Exercises

(2009 Prelim 1)

(a)

$$\max_{x_1,x_2,x_3} x_1 x_2^{\frac{1}{2}} x_3^{\frac{1}{2}}$$

Subject to

$$p_1x_1 + p_2x_2 + p_3x_3 \leq w$$

(b) To find the consumer's demand functions we first notice that $u(.)$ is increasing in each good, so it satisfies LNS and therefore the constraint must be binding. Since all monotonic transformations preserve the order of $\succsim$, solving the problem in (a) is equivalent to,

$$\max_{x_1,x_2,x_3} log(x_1) + \frac{1}{2}log(x_2) + \frac{1}{2}log(x_3)$$

Subject to

$$p_1x_1 + p_2x_2 + p_3x_3 = w$$

Our Lagrangian is,

$$\mathcal{L}(x,\lambda) = log(x_1) + \frac{1}{2}log(x_2) + \frac{1}{2}log(x_3) + \lambda(w - p_1x_1 - p_2x_2 - p_3x_3)$$

And our first order conditions give,

$$\frac{\partial \mathcal{L}(x,\lambda)}{\partial x_1} = \frac{1}{x_1} - \lambda p_1 = 0$$

$$\frac{\partial \mathcal{L}(x,\lambda)}{\partial x_2} = \frac{1}{2x_2} - \lambda p_2 = 0$$

$$\frac{\partial \mathcal{L}(x,\lambda)}{\partial x_3} = \frac{1}{2x_3} - \lambda p_3 = 0$$

$$\frac{\partial \mathcal{L}(x,\lambda)}{\partial \lambda} = w - p_1x_1 - p_2x_2 - p_3x_3 = 0$$

From here we obtain,

$$x_2 = \frac{p_1}{p_2}\frac{x_1}{2}$$

$$x_3 = \frac{p_1}{p_3}\frac{x_1}{2}$$

Substituting in the budget constraint,

$$w = p_1x_1 + \frac{p_1x_1}{2} + \frac{p_1x_1}{2}$$

Solving the system of equations we get,

$$\implies x_1(p,w) = \frac{w}{2p_1}$$

$$\implies x_2(p,w) = \frac{w}{4p_2}$$

$$\implies x_1(p,w) = \frac{w}{4p_3}$$

To confirm that these are indeed our walrasian demand functions, we can check the corner solution or compute the Hessian of $u(x)$ and see if it is negative semidefinite.
Since neither of x_1, x_2, x_3 equals 0, then the answer above is the Walrasian Demand.

(c) With the addition of the coupon component, the problem becomes,

$$\max_{x_1,x_2,x_3} x_1 x_2^{\frac{1}{2}} x_3^{\frac{1}{2}}$$

Subject to

$$p_1x_1 + p_2x_2 + p_3x_3 \leq w \text{ (Budget constraint)}$$

$$x_1 + x_2 + x_3 \leq c \text{ (Coupon constraint)}$$

(d) Yes, for c big enough. Assume $p = (1, 1, 1)$, and $c > w$, then the problem becomes

$$\max_{x_1,x_2,x_3} x_1 x_2^{\frac{1}{2}} x_3^{\frac{1}{2}}$$

Subject to

$$x_1 + x_2 + x_3 \leq w \text{ (Budget constraint)}$$

$$x_1 + x_2 + x_3 < c \text{ (Coupon constraint)}$$

The leftover coupons will be $c - w$.

(e) Since the budget constraint and coupon constraint are "parallel", if $c > w$, then we only need to use the budget constraint. Otherwise, if $c \leq w$, we use the coupon constraint.
For example, if $c > w$, we just need to replace $p = (1, 1, 1)$ in the Walrasian demand we found in (a).

(2023 Prelim 1)

(a) The problem is

$$V(T) = \max_e B(e) \text{ subject to } \sum_{i=1}^{n} e_i = T$$

Let $T_2 > T_1$. Denote $e(T_1)$ as the maximizer under T_1. Then there exist $0 < \epsilon < \frac{T_2 - T_1}{n}$ such that $\sum_{i=1}^{n}(e_i + \epsilon) = T_1 + n\epsilon < T_2$. Since B is strictly increasing,

$$B(e + \epsilon) > B(e(T_1))$$

Also since $\sum_{i=1}^{n}(e_i + \epsilon) < T_2$,

$$V(T_2) \geq B(e + \epsilon) > B(e(T_1)) = V(T_1)$$

(b) The problem is

$$V(T) = \max_e B(e) \text{ subject to } \sum_{i=1}^{n} e_i = T$$

Since all the conditions are met, we can use the lagrangian method to solve this problem. The lagrangian is,

$$\mathcal{L} = B(e) + \lambda(T - \sum_{i=1}^{n} e_i)$$

And the first order condition is,

$$\frac{\partial \mathcal{L}}{\partial e_i} = \frac{\partial B(e)}{\partial e_i} - \lambda = 0 \implies \frac{\partial B(e)}{\partial e_i} = \lambda^*$$

In optimal, by the Envelope Theorem,

$$\frac{dV(T)}{dT} = \frac{d\mathcal{L}(e^*(T))}{dT} = \lambda^* = \frac{\partial B(e)}{\partial e_i}$$

(c) The problem is

$$V(T) = \max_{e} b_1(\alpha e_1) + b_2(e_2) \text{ subject to } e_1 + e_2 = T$$

$$\implies \max_{e_1 \geq 0} b_1(\alpha e_1) + b_2(T - e_1)$$

The first order condition gives,

$$\alpha b_1'(\alpha e_1^*(\alpha)) - b_2'(T - e_1^*(\alpha)) = 0$$

We want to know how does e_1^* changes when a decrease from 1 to α happens. For this we compute the derivative with respect to α on the FOC,

$$b_1'(\alpha e_1^*(\alpha)) + \alpha b_1''(\alpha e_1^*(\alpha))(e_1^*(\alpha) + \alpha e_1^{*'}(\alpha)) + b_2''(T - e_1^*(\alpha))e_1^{*'}(\alpha) = 0$$

We group terms and get,

$$\frac{\partial e_1^*(\alpha)}{\partial \alpha} = -\frac{b_1'(\alpha e_1^*(\alpha)) + \alpha b_1''(\alpha e_1^*(\alpha))e_1^*(\alpha)}{\alpha^2 b_1''(\alpha e_1^*(\alpha)) + b_2''(T - e_1^*(\alpha))}$$

Since each b_i is strictly increasing and strictly concave, we know $b'(.) > 0$ and $b''(.) < 0$. From here we obtain that the denominator of $\frac{\partial e_1^*(\alpha)}{\partial \alpha}$ must be negative, but the sign of the numerator, $b_1'(\alpha e_1^*(\alpha)) + \alpha b_1''(\alpha e_1^*(\alpha))e_1^*(\alpha)$, remains undetermined. Therefore the sign of $\frac{\partial e_1^*(\alpha)}{\partial \alpha}$ is undetermined.