Spring 2024

Suggested solutions

Final exam 2024

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May 9, 2024

- These are my suggested solutions to Ryan's 2024 Final
- Each question has a (subjective) level of difficulty associated with it on a scale of 1 (*) to 5 stars (*****)

(a) In the language of the course, list separately the exogenous state variables, the endogenous state variables, and the endogenous control variables in this model. Finally, make a list of all of the exogenous parameters of this economy (*)

- Exogenous state(s): A_t
- Endogenous state(s): C_{t-1}
- Endogenous jump(s) appearing as controls: C_t, N_t, D_t, Y_t, S_t
- Endogenous jump(s) that no one has control over (subtle category): W_t
- Parameters: α, β, ϕ, δ , χ, ε, ρ

(b) Write the household's Lagrangian optimization problem and find the first order necessary conditions for optimality of the household. Denote the multipliers on the household budget constraint and labor evolution constraint with $\lambda_{1,t}$ and $\lambda_{2,t}$ respectively (**)

The household solves the following problem (taking W_t as given):

$$\max_{C_t,\, S_t,\, N_t} E_0 \sum_{t=0}^{\infty} \beta^t \log(C_t) - \gamma(S_t + N_t)$$

subject to

$$W_t N_t + \pi_t = C_t$$

$$C_t = (1-\delta)C_{t-1} + P_t^h S_t$$

The Lagrangian of the problem writes:

$$\mathcal{L}(\cdot) = E_0 \sum_{t=0}^{\infty} \beta^t \left(\log(C_t) - \gamma(S_t + N_t) + \lambda_{1,t}(W_t N_t + \pi_t - C_t) + \lambda_{2,t}((1-\delta)C_{t-1} + P_t^h S_t - C_t)\right)$$

Take the FOCs of the above problem, and you get:

$$\frac{\partial \mathcal{L}(\cdot)}{\partial N_t} = 0 \iff \gamma = \lambda_{1,t} W_t$$

$$\frac{\partial \mathcal{L}(\cdot)}{\partial S_t} = 0 \iff \gamma = P_t^h \lambda_{2,t}$$

$$\frac{\partial \mathcal{L}(\cdot)}{\partial C_t} = 0 \iff \frac{1}{C_t} - \lambda_{1,t} - \lambda_{2,t} + \beta E_t(1-\delta)\lambda_{2,t+1} = 0$$

Combining the above, we get:

$$C_t^{-1} - \frac{\gamma}{W_t} - \frac{\gamma}{P_t^h} + \beta E_t(1-\delta)\frac{\gamma}{P_{t+1}^h} = 0$$

$$\iff \frac{\gamma}{P_t^h} = C_t^{-1} - \frac{\gamma}{W_t} + \beta E_t (1-\delta) \frac{\gamma}{P_{t+1}^h}$$

(c) Interpret equation (10) above in words using a marginal cost = marginal benefit intuition (**)

This equation states that a necessary condition for the household to be along an optimal decision path is that the marginal cost of matching with a consumption good be equal to the marginal benefit generated by the newly formed match.

LHS = to get an extra shopping match, I need to invest $\frac{1}{P_t^h}$ goods in shopping effort, which costs me $\frac{\gamma}{P_t^h}$ utils! (remark: we express it all in terms of the the numeraire, i.e. the consumption good here)

RHS = the right-hand side represents the marginal benefit of the match. First, I get the extra kick in utility immediately generated by the match, C_t^{-1}. But the match may or may not last and generate value in the future (and possibly spare me the effort of shopping if the match does not get destroyed so I may save up the opportunity cost of future searches). It thus embeds a continuation value adjusted for the probability of destruction δ. Also, gathering the resources necessary for my shopping endeavor requires additional work effort. One unit of labor gives me W_t goods so, to get one extra unit of the good, I need to work $1/W_t$ units more, and this costs me γ/W_t utils.

(d) Write the social planner's Lagrangian optimization problem and find the first order necessary conditions for optimality. Denote the multipliers on the relevant constraints $\theta_{i,t}$ for $i = 1, 2, ...$ (**)

Let us write the Lagrangian of the central planner, keeping in mind that the planner internalizes the externality in her solving the problem:

$$\mathcal{L}(\cdot) = E_0 \sum_{t=0}^{\infty} \beta^t \left[\log(C_t) - \gamma(S_t + N_t) + \theta_{1,t}(A_t N_t^{\alpha} - C_t - \mu D_t) + \theta_{2,t}((1-\delta)C_{t-1} + \chi D_t^{\varepsilon} S_t^{1-\varepsilon} - C_t) \right]$$

The FOCs write:

$$\frac{\partial \mathcal{L}}{\partial S_t} = 0 \iff \theta_{2,t}(1-\varepsilon)\chi D_t^{\varepsilon} S_t^{-\varepsilon} = \gamma$$

$$\frac{\partial \mathcal{L}}{\partial N_t} = 0 \iff \theta_{1,t}\alpha A_t N_t^{\alpha-1} = \gamma$$

$$\frac{\partial \mathcal{L}}{\partial D_t} = 0 \iff \theta_{1,t}\mu = \theta_{2,t}\chi\varepsilon D_t^{\varepsilon-1} S_t^{1-\varepsilon}$$

$$\frac{\partial \mathcal{L}}{\partial C_t} = 0 \iff \frac{1}{C_t} - \theta_{1,t} - \theta_{2,t} + \beta E_t[\theta_{2,t+1}(1-\delta)] = 0$$

We thus get:

$$\frac{\partial \mathcal{L}}{\partial C_t} = 0 \iff C_t^{-1} - \frac{\gamma}{\alpha A_t N_t^{\alpha-1}} + \beta E_t \left[\frac{\gamma(1-\delta)}{(1-\varepsilon)\chi D_{t+1}^{\varepsilon} S_{t+1}^{-\varepsilon}} \right] = \frac{\gamma}{(1-\varepsilon)\chi D_t^{\varepsilon} S_t^{-\varepsilon}}$$

(e) Now write the Bellman equation that corresponds to the social planner's optimization problem in this economy and find the first-order necessary conditions for optimality using the envelope theorem. Show that the conditions from (1.c) and (1.d) are equivalent (***)

The planner's Bellman equation writes:

$$V(A_t, C_{t-1}) = \max_{N_t, D_t, C_t, Y_t, S_t} \log(C_t) - \gamma(S_t + N_t) + \beta E_t V(A_{t+1}, C_t)$$

subject to

$$Y_t = A_t N_t^{\alpha}$$

$$C_t = (1-\delta)C_{t-1} + \chi D_t^{\varepsilon} S_t^{1-\varepsilon}$$

$$Y_t = C_t + \mu D_t$$

Before moving on, one may want to perform a couple of adjustments:

$$N_t = \left[\frac{C_t + \mu D_t}{A_t}\right]^{\frac{1}{\alpha}}$$

$$S_t = \left[\frac{C_t - (1-\delta)C_{t-1}}{\chi D_t^{\varepsilon}}\right]^{\frac{1}{1-\varepsilon}}$$

Therefore, taking the FOC wrt C_t, one gets:

$$C_t^{-1} - \frac{\gamma}{(1-\varepsilon)\chi D_t^{\varepsilon}} S_t^{\varepsilon} - \gamma \frac{N_t^{1-\alpha}}{\alpha A_t} + \beta E_t V_2(A_{t+1}, C_t) = 0$$

Now let us turn to the enveloppe condition:

$$V_2(A_t, C_{t-1}) = \frac{\gamma(1-\delta)}{(1-\varepsilon)\chi D_t^{\varepsilon}} S_t^{\varepsilon}$$

Combining the two above equations, we get:

$$C_t^{-1} - \frac{\gamma}{\alpha A_t N_t^{\alpha-1}} + \beta E_t \left[\frac{\gamma(1-\delta)}{(1-\varepsilon)\chi D_{t+1}^{\varepsilon}} S_{t+1}^{\varepsilon}\right] = \frac{\gamma}{(1-\varepsilon)\chi D_t^{\varepsilon}} S_t^{\varepsilon}$$

This is equivalent to (d).

Solution I.

(a) Log-linearize equation (10) from first principals. You should log-linearize the equation around the steady-state, but you may treat the steady-state values of endogenous variables as parameters (*)

We log-linearize the following:

$$\frac{\gamma}{P_t^h} = C_t^{-1} - \frac{\gamma}{W_t} + \beta E_t(1-\delta)\frac{\gamma}{P_{t+1}^h}$$

We are asked to log-linearized from first principles, so we rewrite the equation as:

$$\begin{aligned}
\frac{\gamma}{\chi}\left(\frac{S_t}{D_t}\right)^{\varepsilon} &= C_t^{-1} - \frac{\gamma}{W_t} + \beta E_t(1-\delta)\frac{\gamma}{\chi}\left(\frac{S_{t+1}}{D_{t+1}}\right)^{\varepsilon} \\
\Rightarrow \frac{\gamma}{\chi}\varepsilon\left(\frac{S}{D}\right)^{\varepsilon} s_t - \frac{\gamma}{\chi}\varepsilon\left(\frac{S}{D}\right)^{\varepsilon} d_t &= -\frac{c_t}{C} + \frac{\gamma}{W}w_t + \beta(1-\delta)\frac{\gamma}{\chi}\varepsilon\left(\frac{S}{D}\right)^{\varepsilon} E_t[s_{t+1} - d_{t+1}] \\
\iff \frac{\gamma}{\chi}\varepsilon\left(\frac{S}{D}\right)^{\varepsilon} s_t - \frac{\gamma}{\chi}\varepsilon\left(\frac{S}{D}\right)^{\varepsilon} d_t + \frac{c_t}{C} - \frac{\gamma}{W}w_t &- \beta(1-\delta)\frac{\gamma}{\chi}\varepsilon\left(\frac{S}{D}\right)^{\varepsilon} E_t[s_{t+1} - d_{t+1}] = 0
\end{aligned}$$

(b) Using the log-linearized equation, compute the corresponding rows of the F_x, F_y, F_{xp}, F_{yp} matrices that would be required for the log-linearization solution procedure we used in class (*)

$$E_t\left[\underbrace{\begin{pmatrix}0\\0\end{pmatrix}'}_{F_x}\begin{pmatrix}a_t\\c_{t-1}\end{pmatrix} + \underbrace{\begin{pmatrix}\alpha\\0\\-\alpha\\0\\-\frac{\gamma}{W}\end{pmatrix}'}_{F_y}\begin{pmatrix}s_t\\n_t\\d_t\\y_t\\w_t\end{pmatrix} + \underbrace{\begin{pmatrix}0\\\frac{1}{C}\end{pmatrix}'}_{F_{xp}}\begin{pmatrix}a_{t+1}\\c_t\end{pmatrix} + \underbrace{\begin{pmatrix}-\kappa\\0\\\kappa\\0\\0\end{pmatrix}'}_{F_{yp}}\begin{pmatrix}s_{t+1}\\n_{t+1}\\d_{t+1}\\y_{t+1}\\w_{t+1}\end{pmatrix}\right] = 0$$

Where $\alpha = \frac{\gamma}{\chi}\varepsilon\left(\frac{S}{D}\right)^{\varepsilon}$ and $\kappa = \beta(1-\delta)\alpha$

Solution II.

(a) Using pseudo-code, describe an algorithm that solves for the approximate numerical value function you found in (1.d) over a finite grid of points *cgrid* and *agrid*. Below, I proved some initial steps. Your code does not need to compile, but you should pay special attention to indexing so that a naive programmer could implement your algorithm. Also, be sure to test for convergence of your iterations (****)

```
[agrid, theta, theta_bar] = AR1_rouwen(na,rho,0,sigmaa);
agrid = exp(agrid);

%A/C combos as initial states
[aagr,ccgr] = ndgrid(agrid,cgrid);
aagr = aagr(:)' ;
ccgr = ccgr(:)';

%S/N combos to choose from
[ccgr2,ddgr2] = ndgrid(cgrid,dgrid);
ddgr2 = ddgr2(:)';
ccgr2 = ccgr2(:)';

%%

crit = 1;
vinit = zeros(na,nd);

%%
while crit > 10^(-7)
    vinit_old = vinit;

    EVp    = repmat(theta*vinit,1,nd);
    for a = 1:na
        for c = 1:nc

            at   = agrid(a);
            cm1   = cgrid(c);
            s = (((ccgr2-cm1*(1-deltan))./chi).*ddgr2.^(-eps)).^(1/(1-
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                eps));
            n = ((ccgr2+mu.*ddgr2)./at).^(1/alph);
            vv = log(ccgr2) - gamma*(s+n) + bet*EVp(a,:);

            [vinit(a,c),idx_tmp] = max(vv);
          %  idx(a,c) = idx_tmp; (optional)
        end
    end
    crit = norm(vinit-vinit_old);
    vinit_old = vinit;
end

% cpol = reshape(ccgr2(idx(:)),[na,nc]);
% dpol = reshape(ddgr2(idx(:)),[na,nd]);
```