Final

ECON 6170

December 9, 2022

Instructions: You have the full final exam time to complete the following problems. You are to work alone. This test is not open book. In your answers, you are free to cite results that you can recall from class or previous homeworks *unless explicitly stated otherwise*. The exam is out of 20 points, and there are extra credit questions. The highest possible score is 24/20.

1. (5pts) Let $f, g : [0,1] \to \mathbb{R}$. Let $h = \max\{f, g\}$. Let $k = f + g$.

   (a) Define quasiconvexity and quasiconcavity.

   (b) Prove or disprove: If f and g are quasiconcave, h is quasiconcave
   False. For example, $f(x) := x$ and $g(x) := -x$ are quasiconcave, but $h(x) = |x|$ is not.

   (c) Prove or disprove: If f and g are quasiconvex, h is quasiconvex.
   True.

$$\begin{aligned}
h(\alpha x + (1-\alpha)y) &= \max\{f(\alpha x + (1-\alpha)y), g(\alpha x + (1-\alpha)y)\} \\
&\leq \max\{\max\{f(x), f(y)\}, \max\{g(x), g(y)\}\} \\
&= \max\{f(x), f(y), g(x), g(y)\} \\
&= \max\{\max\{f(x), g(x)\}, \max\{f(y), g(y)\}\} \\
&= \max\{h(x), h(y)\}
\end{aligned}$$

   (d) Prove or disprove: If f and g are quasiconvex, k is quasiconvex
   False. For example, $f(x) := -\mathrm{e}^x$ and $g(x) := -\mathrm{e}^{-x}$ are both monotone (and hence quasiconvex). But $k(x) = -\mathrm{e}^x - \mathrm{e}^{-x}$ is not quasiconvex, as for $x = \log 2$, for example,

$$\begin{aligned}
k\left(\frac{1}{2}x + \frac{1}{2}(-x)\right) &= k(0) = -2 > -2 - \frac{1}{2} = -\mathrm{e}^{\log 2} - \frac{1}{\mathrm{e}^{\log 2}} = k(\log 2) = k(x) \\
&= \max\{k(x), k(-x)\}
\end{aligned}$$

2. (5pts)

   (a) State the intermediate value theorem.

   (b) State the mean value theorem.

   (c) State Taylor's theorem.

   (d) Let $f : (0,1) \to \mathbb{R}$ be differentiable with $f'(x) > 0$ for all $x \in (0,1)$. Prove that f is strictly increasing.

   Let $0 < x < y < 1$. We want to show that $f(x) < f(y)$. Because f is differentiable on $(0,1)$, it is continuous on $(0,1)$, and hence on $[x,y]$, and it is differentiable on (x,y). Therefore, we can apply the mean-value theorem to obtain

$$f(y) - f(x) = f'(p)(y - x)$$

   for some $p \in (x,y)$. But $f'(p) > 0$ and $y > x$, so $f(y) > f(x)$.

3. (5pts) Implicit function theorem.

(a) Let $F(x_1, x_2, y) = x_1 + x_2 + y - e^{x_1 x_2 y}$ and $(x_1^0, x_2^0, y^0) = (0, 0.5, 0.5)$. Show that the set of (x_1, x_2, y) that solve $F(x_1, x_2, y) = 0$ near (x_1^0, x_2^0, y^0) is the graph of some function $y = h(x_1, x_2)$. Compute Dh.

Note: If $g(x) = e^x$, $\frac{d}{dx}g(x) = g(x)$.

F is the sum of compositions of continuously differentiable functions, so F itself is continuously differentiable. We also have $\frac{\partial F}{\partial y} = 1 - \mathrm{e}^{x_1 x_2 y} x_1 x_2$, which evaluated at $(0, 0.5, 0.5)$ is $1 - \mathrm{e}^0 \cdot 0 \cdot 0.5 = 1 \neq 0$. Therefore, the hypotheses of the implicit function theorem are satisfied, and so the level set $F(x_1, x_2, y) = 0$ near $(0, 0.5, 0.5)$ is the graph of some function $y = h(x_1, x_2)$. The implicit function theorem also gives us the formula

$$
\begin{aligned}
Dh &= -\frac{\partial F}{\partial y}^{-1} \cdot \begin{bmatrix} \frac{\partial F}{\partial x_1} & \frac{\partial F}{\partial x_2} \end{bmatrix} \\
&= -\frac{1}{1 - \mathrm{e}^{x_1 x_2 y} x_1 x_2} \cdot \begin{bmatrix} 1 - \mathrm{e}^{x_1 x_2 y} x_2 y & 1 - \mathrm{e}^{x_1 x_2 y} x_1 y \end{bmatrix}
\end{aligned}
$$

(b) Let $F : \mathbb{R}^4 \to \mathbb{R}^2$ be defined as

$$F(x_1, x_2, y_1, y_2) = (x_1^2 - x_2^2 - y_1^3 + y_2^2 + 4,\ \ 2x_1 x_2 + x_2^2 - 2y_1^2 + 3y_2^4 + 8).$$

Let $(x_1^0, x_2^0, y_1^0, y_2^0) = (2, -1, 2, 1)$. Show that the set of (x_1, x_2, y_1, y_2) that solve $F(x_1, x_2, y_1, y_2) = 0$ near $(x_1^0, x_2^0, y_1^0, y_2^0)$ is the graph of some function $(y_1, y_2) = h(x_1, x_2)$. Compute Dh.

Again, F is continuously differentiable, as it is the sum and product of continuously differentiable functions. In addition,

$$D_y F = \begin{bmatrix} -3y_1^2 & 2y_2 \\ -4y_1 & 12y_2^3 \end{bmatrix}$$

which evaluated at $(2, -1, 2, 1)$ is

$$
\begin{aligned}
D_y F(2, -1, 2, 1) &= \begin{bmatrix} -3 \cdot 2^2 & 2 \cdot 1 \\ -4 \cdot 2 & 12 \cdot 1^3 \end{bmatrix} \\
&= \begin{bmatrix} -12 & 2 \\ -8 & 12 \end{bmatrix}
\end{aligned}
$$

This matrix has determinant $-144 - (-16) = -128 \neq 0$, so the matrix is invertible. The conditions for IFT are then satisfied, and Dh is given by

$$
\begin{aligned}
Dh &= -D_y F^{-1} D_x F \\
&= -\begin{bmatrix} -3y_1^2 & 2y_2 \\ -4y_1 & 12y_2^3 \end{bmatrix}^{-1} \begin{bmatrix} 2x_1 & -2x_2 \\ 2x_2 & 2x_1 + 2x_2 \end{bmatrix} \\
&= \frac{1}{36y_1^2 y_2^3 - 8y_1 y_2} \begin{bmatrix} 12y_2^3 & -2y_2 \\ 4y_1 & -3y_1^2 \end{bmatrix} \begin{bmatrix} 2x_1 & -2x_2 \\ 2x_2 & 2x_1 + 2x_2 \end{bmatrix}
\end{aligned}
$$

4. (5pts) Suppose that $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is continuously differentiable. Then f has increasing differences in $(x, t) \in \mathbb{R}^n \times \mathbb{R}^m$ if and only if $\frac{\partial^2 f}{\partial x_i \partial t_j} \geq 0$ for all $i \in \{1, \ldots n\}, j \in \{1, \ldots m\}$.

   Prove only one direction—that f having increasing differences implies $\frac{\partial^2 f}{\partial x_i \partial t_j} \geq 0$ for all $i \in \{1, \ldots n\}, j \in \{1, \ldots m\}$.

   Note: The other direction is also simple to prove and involves using the fundamental theorem of calculus. But you need not prove it for this problem.

   Increasing differences means that for all $x' > x$ and $t' > t$,

$$f(x', t') - f(x, t') \geq f(x', t) - f(x, t)$$

   In particular, if $h, k > 0$

$$f(x + he_i, t + ke_j) - f(x, t + ke_j) \geq f(x + he_i, t) - f(x, t)$$

   so

$$\frac{1}{hk}\left(f(x + he_i, t + ke_j) - f(x, t + ke_j)\right) \geq \frac{1}{hk}\left(f(x + he_i, t) - f(x, t)\right)$$

   Taking limits of both sides as $h \to 0$,

$$\frac{1}{k} \cdot \frac{\partial f(x, t + ke_j)}{\partial x_i} \geq \frac{1}{k} \cdot \frac{\partial f(x, t)}{\partial x_i}$$

   or

$$\frac{1}{k}\left(\frac{\partial f(x, t + ke_j)}{\partial x_i} - \frac{\partial f(x, t)}{\partial x_i}\right) \geq 0$$

   Taking limits as $k \to 0$,

$$\frac{\partial^2 f(x, t)}{\partial t_j \partial x_i} \geq 0$$

5. (Extra Credit: 4 pts)

(a) Just as in the last question, state a characterization for the *supermodularity* of f in $x \in \mathbb{R}^n$ in terms of conditions on the partials of f.

Note: State this for the case of general m and n. Be careful about which partials you place conditions on.

$$\frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0$$

for all $i \neq j$.

(b) State the KKT theorem, precisely defining the constraint qualification, first-order conditions and complementary-slackness conditions.

(c) A function $f : E \subset \mathbb{R} \to \mathbb{R}$ is *uniformly continuous* if for all $\epsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in E$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. Give an example of a function which is continuous but not uniformly continuous.

Define $f : \mathbb{R}_{++} \to \mathbb{R}$ by $f(x) := 1/x$.

(d) Prove your example in (c) works.

We know from the lectures that $1/x$ is continuous. Fix $\epsilon > 0$ and choose an arbitrary $\delta > 0$. Without loss of generality, we can let $\delta < \frac{1}{2\epsilon}$. Then let $0 < x < y < \delta$, so that $|x - y| < \delta$. Let $y = 2x$. Then

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{1}{x} - \frac{1}{2x}\right| = \frac{1}{2x} > \frac{1}{2\frac{1}{2\epsilon}} = \epsilon$$

Since δ is arbitrary, this is true for some x and y, for all possible δ's we could choose, so $1/x$ is not uniformly continuous.