Midterm 2

ECON 6170

October 25, 2022

Instructions: You have the full class time to complete the following problems. You are to work alone. This test is not open book. In your answers, you are free to cite results that you can recall from class or previous homeworks *unless explicitly stated otherwise*. The exam is out of 20 points, and there are extra credit questions. The highest possible score is 24/20.

1. (5pts) Prove or disprove.

   (a) Let ϕ be the *constant correspondence*, i.e., $\phi(x) = \phi(y)$ for all x and y in its domain. Then ϕ is continuous.

   (b) Let $\phi, \rho, \tau : [0,1] \rightrightarrows \mathbb{R}$, where $\tau(x) = \phi(x) \cup \rho(x)$ for all $x \in [0,1]$. Then if ϕ and ρ are continuous, τ is continuous.

2. (5pts)

   (a) State the mean value theorem. (No partial credit for incomplete or incorrect statements.)

   (b) State Taylor's theorem. (No partial credit for incomplete or incorrect statements.)

   (c) Suppose f is differentiable on some open set E and $|f'(x)| \leq M$ for all $x \in E$. Then prove that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in E$.

3. (5pts) Let $L : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. Prove or disprove: L is differentiable everywhere.

4. (5pts) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = \frac{x^2}{y^2+1}$.

   Is f twice differentiable at $(x,y) = (0,0)$? If yes, compute f'' at $(0,0)$. If no, prove it.

   Hint 1: First check if f is differentiable at $(x,y) = (0,0)$.

   Hint 2: If it exists, the derivative of f evaluated at a point (x,y) is some 1×2 matrix (or, equivalently, the linear transformation induced by that matrix). But viewed as a function, it maps an arbitrary (x,y) to a point in $\mathbb{R}^2$. The second derivative of f, if it exists, is simply the first derivative of that function.

5. (a) (Extra Credit: 1 pt) State the separating hyperplane theorem for nonempty, convex and disjoint sets in $\mathbb{R}^n$. (No partial credit for incomplete or incorrect statements.)

   (b) (Extra Credit: 1 pt) State Berge's theorem. (No partial credit for incomplete or incorrect statements.)

   (c) (Extra Credit: 2 pt): Prove Berge's theorem.