ECON 6090-Microeconomic Theory. TA Section 3

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In Section notes

Review

(Potential) Properties of $\succsim$:

1. Rational + continuous
2. Strong monotonicity

$$\implies \text{Weak version: local non-satiation (LNS)}$$

3. Convexity

(Potential) Properties of $u(.)$

1. Continuity
2. (Quasi-) Concave

Relationship between properties of $\succsim$ and $u(.)$

1. Continuous+Rational $\succsim \implies \exists$ continuous $u(.)$ representing $\succsim$.
2. Monotonic $\succsim \implies u(.)$ is nondecreasing[1].
3. Convex $\succsim$ (+ LNS) $\implies \forall u(.)$ representing $\succsim$, $u(.)$ is quasi-concave.

Let the hessian of u be $H_u(x)$. We also have,

$$\text{convex } u(.) \iff H_u(x) \text{ P.S.D } \forall x$$

$$\text{concave } u(.) \iff H_u(x) \text{ N.S.D } \forall x$$

Properties of indirect utility function

1. $V(p, w)$ is continuous in (p, w).
2. Non-increasing in p. Strictly increasing in w.
3. HoD 0.

The Bordered Hessian

The *bordered Hessian* is a determinant-based tool used to verify second-order conditions for constrained optimization problems. Specifically, it applies to problems of the form:

$$\begin{aligned} \max \quad & f(x_1, x_2, \dots, x_n) \\ \text{s.t.} \quad & g(x_1, x_2, \dots, x_n) = 0, \end{aligned}$$

where f is the objective function, and g is the constraint.
To construct the bordered Hessian, follow these steps:

[1] Strong Monotonicity $\to$ Strictly Increasing

1. Compute the Lagrangian:

$$\mathcal{L}(x_1, x_2, \ldots, x_n, \lambda) = f(x_1, x_2, \ldots, x_n) + \lambda g(x_1, x_2, \ldots, x_n).$$

2. Form the bordered Hessian matrix H, which has the following structure:

$$H = \begin{bmatrix} 0 & \nabla g^\top \\ \nabla g & \nabla^2 \mathcal{L} \end{bmatrix},$$

   where:

   - ∇g is the gradient of the constraint function g,
   - $\nabla^2 \mathcal{L}$ is the Hessian matrix of the Lagrangian with respect to $x_1, x_2, \ldots, x_n$.

The bordered Hessian is evaluated at the candidate solution (x^*, λ^*). For a maximization problem:

- The $(n+1)$-th leading principal minor of H (the determinant of the upper-left $(n+1) \times (n+1)$ submatrix) must alternate in sign:

$$(-1)^k \det(H_k) > 0, \quad \text{for } k = 2, 4, \ldots, n+1,$$

  where H_k is the k-th leading principal minor of H.

- For minimization problems, all even-order leading principal minors must be positive.

Example
Consider the problem:

$$\max \quad f(x_1, x_2) = x_1 x_2, \quad \text{s.t.} \quad g(x_1, x_2) = x_1 + x_2 - 1 = 0.$$

1. Compute the Lagrangian:

$$\mathcal{L}(x_1, x_2, \lambda) = x_1 x_2 + \lambda(x_1 + x_2 - 1).$$

2. Compute the gradients:

$$\nabla g = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \nabla^2 \mathcal{L} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

3. Form the bordered Hessian:

$$H = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

4. Check the minors for second-order conditions.

Conditions for Quasiconcavity and Concavity Based on the Bordered Hessian

1. Quasiconcavity

For a differentiable function $f(x_1, x_2, \ldots, x_n)$, quasiconcavity is determined by the signs of the determinants of the bordered Hessian minors:

- **Necessary condition:** The $(n+1)$-th bordered Hessian minor, denoted by H_{n+1}, alternates in sign:

$$(-1)^k \det(H_k) \geq 0, \quad \text{for } k = 2, 4, \ldots, n+1.$$

- **Sufficient condition:** The $(n+1)$-th bordered Hessian minor alternates in sign strictly:

$$(-1)^k \det(H_k) > 0, \quad \text{for } k = 2, 4, \ldots, n+1.$$

2. Concavity

For a twice-differentiable function $f(x_1, x_2, \ldots, x_n)$, concavity requires that the bordered Hessian determinants satisfy the following conditions:

- **Necessary condition:** The bordered Hessian determinants for all even k must be non-positive:

$$\det(H_k) \leq 0, \quad \text{for } k = 2, 4, \ldots, n+1.$$

- **Sufficient condition:** The bordered Hessian determinants for all even k must be strictly negative:

$$\det(H_k) < 0, \quad \text{for } k = 2, 4, \ldots, n+1.$$

Exercises

Preference and utility representation

$$u(x,y) = x^3y^2$$

The gradient of $u(x,y)$ is the vector of partial derivatives with respect to x and y. Compute:

$$\frac{\partial u}{\partial x} = 3x^2y^2, \quad \frac{\partial u}{\partial y} = 2x^3y$$

Thus, the gradient is:

$$\nabla u(x,y) = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = \begin{bmatrix} 3x^2y^2 \\ 2x^3y \end{bmatrix}$$

The Hessian of $u(x,y)$ is the matrix of second-order partial derivatives. Compute:

$$\frac{\partial^2 u}{\partial x^2} = 6xy^2, \quad \frac{\partial^2 u}{\partial y^2} = 2x^3, \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = 6x^2y$$

The Hessian matrix is:

$$H_u(x,y) = \begin{bmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial y \partial x} & \frac{\partial^2 u}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 6xy^2 & 6x^2y \\ 6x^2y & 2x^3 \end{bmatrix}$$

The bordered Hessian is constructed for a two-variable function as:

$$H_b = \begin{bmatrix} 0 & \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial x} & \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial u}{\partial y} & \frac{\partial^2 u}{\partial y \partial x} & \frac{\partial^2 u}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 0 & 3x^2y^2 & 2x^3y \\ 3x^2y^2 & 6xy^2 & 6x^2y \\ 2x^3y & 6x^2y & 2x^3 \end{bmatrix}$$

Compute the principal minors of H_b:

- First minor (H_1):

$$\det(H_1) = 0.$$

- Second minor (H_2):

$$\det(H_2) = \begin{vmatrix} 0 & 3x^2y^2 \\ 3x^2y^2 & 6xy^2 \end{vmatrix} = 0$$

- Third minor (H_3):

$$\det(H_3) = \begin{vmatrix} 0 & 3x^2y^2 & 2x^3y \\ 3x^2y^2 & 6xy^2 & 6x^2y \\ 2x^3y & 6x^2y & 2x^3 \end{vmatrix}$$

  Expanding along the first row:

$$\det(H_3) = -3x^2y^2 \begin{vmatrix} 6xy^2 & 6x^2y \\ 6x^2y & 2x^3 \end{vmatrix}$$

  Compute the determinant of the 2×2 matrix:

$$\det \begin{bmatrix} 6xy^2 & 6x^2y \\ 6x^2y & 2x^3 \end{bmatrix} = 6xy^2 \cdot 2x^3 - 6x^2y \cdot 6x^2y = 12x^4y^2 - 36x^4y^2 = -24x^4y^2$$

  Substituting back:

$$\det(H_3) = -3x^2y^2(-24x^4y^2) = 72x^6y^4$$

For concavity:

- $H_2 \leq 0$: Fails because $\det(H_2) = 0$.
- $H_3 \leq 0$: Fails because $\det(H_3) = 72x^6y^4 > 0$

Therefore, $u(x,y)$ is **not concave**.
For quasiconcavity:

- $H_2 \geq 0$: Holds because $\det(H_2) = 0$.
- $H_3 \geq 0$: Holds because $\det(H_3) = 72x^6y^4 > 0$

Thus, $u(x,y)$ is **quasiconcave**.

Optimization and Comparative Statics

(a)

$$\max_{x_1,x_2} \; u_1(x_1)+u_2(x_2) \quad \text{subject to:} \quad p_1x_1+p_2x_2 \leq w.$$

The Lagrangian for this problem is:

$$\mathcal{L} = u_1(x_1)+u_2(x_2) - \lambda\left(p_1x_1+p_2x_2-w\right).$$

The first-order conditions are:

$$\frac{\partial \mathcal{L}}{\partial x_1} : \; u_1'(x_1^*) - \lambda^* p_1 = 0, \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} : \; u_2'(x_2^*) - \lambda^* p_2 = 0, \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} : \; p_1x_1^* + p_2x_2^* = w \tag{3}$$

(b) We are interested in $\frac{dx_1^*}{dw}$. Differentiating the FOCs with respect to w, we get,

$$u''(x_1^*)\frac{dx_1^*}{dw} - \frac{d\lambda^*}{dw}p_1 = 0$$

$$u''(x_2^*)\frac{dx_2^*}{dw} - \frac{d\lambda^*}{dw}p_2 = 0$$

$$p_1\frac{dx_1^*}{dw} + p_2\frac{dx_2^*}{dw} = 1$$

In matrix form,

$$\begin{bmatrix} -p_1 & u_1'' & 0 \\ -p_2 & 0 & u_2'' \\ 0 & p_1 & p_2 \end{bmatrix} \begin{bmatrix} \frac{\partial \lambda^*}{\partial w} \\ \frac{\partial x_1^*}{\partial w} \\ \frac{\partial x_2^*}{\partial w} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Solving the system we get,

$$\frac{dx_1^*}{dw} = \frac{p_1u_2''}{p_1^2u_2''+p_2^2u_1''} > 0$$

Because,

$$p_1u_2'' < 0 \text{ and } p_1^2u_2''+p_2^2u_1'' < 0$$

(c) We are interested in $\frac{dx_1^*}{dp_1}$. We use a similar approach as before, and take derivative of the FOCs with respect to p_1.

$$u''(x_1^*)\frac{dx_1^*}{dp_1} - \frac{d\lambda^*}{dp_1}p_1 - \lambda^* = 0$$

$$u''(x_2^*)\frac{dx_2^*}{dp_1} - \frac{d\lambda^*}{dp_1}p_2 = 0$$

$$p_1\frac{dx_1^*}{dp_1} + x_1^* + p_2\frac{dx_2^*}{dp_1} = 1$$

In matrix form,

$$\begin{bmatrix} -p_1 & u_1'' & 0 \\ -p_2 & 0 & u_2'' \\ 0 & p_1 & p_2 \end{bmatrix} \begin{bmatrix} \frac{\partial \lambda^*}{\partial p_1} \\ \frac{\partial x_1^*}{\partial p_1} \\ \frac{\partial x_2^*}{\partial p_2} \end{bmatrix} = \begin{bmatrix} \lambda^* \\ 0 \\ -x_1^* \end{bmatrix}$$

Solving the system we get,

$$\frac{dx_1^*}{dp_1} = \frac{-u_2''x_1^*p_1 + \lambda^*p_2^2}{p_1^2u_2''+u_1''p_2^2} < 0$$

Because,

$$p_1^2u_2''+u_1''p_2^2 < 0 \text{ and } -u_2''x_1^*p_1 + \lambda^*p_2^2 > 0$$