ECON 6130

Problem Set 1

Problem 1 (Bob and his coconuts)

1. Bob's maximization problem is:

$$\max_{\{c_t\},\{b_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$\begin{aligned} \text{subject to} \qquad & c_t + b_t \le (1+r_t)b_{t-1} \\ & b_0 + c_0 \le e_0 \\ & c_t \ge 0 \\ & b_t \ge 0, \quad b_{-1} = 0 \\ & e_0 > 0, \{r_t\} \text{ given} \end{aligned}$$

   with $\beta \in (0,1), u' > 0, u'' < 0$.

   Where c_t is the number of coconuts eaten in period t, b_t is the number of coconuts planted in period t, r_t is the harvest rate of coconuts, and β is the discount factor.

2. Since utility is strictly increasing, only the budget constraint is relevant for this problem. The Lagrangian is

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \lambda_t((1+r_t)b_{t-1} - c_t - b_t)$$

   and the first order conditions imply that

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t u'(c_t) - \lambda_t = 0$$

   and

$$\frac{\partial \mathcal{L}}{\partial b_t} = -\lambda_t + \lambda_{t+1}(r_{t+1}+1) = 0$$

   combining, we get

$$-\beta^t u'(c_t) + \beta^{t+1}(r_{t+1}+1)u'(c_{t+1}) = 0$$

   and so

$$u'(c_t) = \beta(r_{t+1}+1)u'(c_{t+1})$$

3. From the first order conditions, we can say that consumption is constant in every period if and only if $\beta(r_{t+1}+1) = 1 \; \forall \, t$. This implies also that $r_1 + 1 = r_2 + 1 = \cdots = r + 1 = \frac{1}{\beta} \Rightarrow r = \frac{1-\beta}{\beta}$. If that is true, $u'(c_t) = u'(c_{t+1}) \; \forall \, t$, and thus the maxima are attained at the same point, meaning that $c_t = c_{t+1} \; \forall \, t$.

4. We are assuming that $c_t = c \; \forall \; t$, which means we are also assuming $r_1 = r_2 = \cdots = r$. First, note that this means that $c_1 = c$, and since $b_{-1} = 0$, this implies that $b_0 = e_0 - c$. Note that this assumption relies on the fact that $\beta \in (0, 1)$ and $r = \frac{1}{\beta} > 1$, meaning that planting coconuts is worth strictly more than saving them for the next period – for each coconut Bob plants, he will attain r coconuts, while he only attains 1 coconut for each coconut he saves. Thus, $e_t = 0 \; \forall \; t = 1, \ldots$. Since we have that $b_0 = e_0 - c$, we can solve for b_t recursively. We have that $b_t = (r + 1)b_{t-1} - c$, meaning that $c = (1 + r)b_{t-1} - b_t \; \forall \; t = 1, \ldots$. Solving for b_0, we get that

EZ: this is not completely true

$$
\begin{aligned}
b_0 &= \frac{c}{1+r} + \frac{b_1}{1+r} \\
b_0 &= \frac{c}{1+r} + \frac{c}{(1+r)^2} + \frac{b_2}{(1+r)^2} \\
&\vdots \\
b_0 &= c\sum_{t=1}^{\infty} \frac{1}{(1+r)^t} + \lim_{t\to\infty} \frac{b_t}{(1+r)^t}
\end{aligned}
$$

and because $(1 + r) > 1$, $\lim_{t\to\infty} \frac{b_t}{(1+r)^t} = 0$. Thus, from the definition of geometric series, $b_0 = \frac{c}{r}$, so $\frac{c}{r} = e_0 - c$, meaning that $c = \frac{r}{1+r} e_0$.

At the beginning of each period t, Bob will have $(1 + r)b_{t-1}$ coconuts. At time $t = 1$, he will have $(1 + r)b_0 = c\frac{1+r}{r}$ coconuts, and he will consume c of them, leaving $b_1 = \frac{c}{r}$. At time $t = 2$, he will have $(1 + r)b_1 = c\frac{1+r}{r}$ coconuts, and will be left with $b_2 = \frac{c}{r}$. This holds for all t, so at any time t, he will have $(1 + r)b_{t-1} = c\frac{1+r}{r} = e_0$ coconuts to start with.

5. We have that

$$\lim_{r\to\infty} c = e_0 \lim_{r\to\infty} \frac{r}{1+r} = e_0$$

This limit is finite, which does not make much sense – as Bob's rate of return increases, he will get more and more coconuts available to him each period, so he should be consuming more. However, we assumed above that $r + 1 = \frac{1}{\beta}$, meaning that as $r \to \infty$, $\beta \to 0$, meaning that as he can attain more coconuts in the future, future coconuts are worth less to him. We actually have that Bob consumes his entire endowment in the first period, and doesn't plant any coconut seeds. If we relaxed the assumption that $r + 1 = \frac{1}{\beta}$, Bob would consume more as r increases, and actually make use of his increasing rate of return.

Problem 2 (CRRA Utility)

1. We have that

$$\lim_{\sigma\to 1} \frac{c_t^0 - 1}{1 - 1} = \frac{0}{0}$$

We can use l'Hôpital's rule, and we get that

$$\lim_{\sigma\to 1} \frac{e^{(1-\sigma)\ln(c_t)}(-\ln(c_t))}{-1} = \ln(c_t)$$

2. First, we define the respective derivatives. We have that

$$U'(c) = c^{-\sigma}$$

and

$$U''(c) = -\sigma c^{-\sigma-1}$$

Thus, the Arrow-Pratt coefficient of relative risk aversion is

$$\frac{-cU''(c)}{U'(c)} = \frac{-c \cdot (-\sigma c^{-\sigma-1})}{c^{-\sigma}} = \sigma \frac{c^{\sigma}}{c^{\sigma}} = \sigma$$

3. Recall that $U'(c) = c^{-\sigma}$. Then the intertemporal elasticity of substitution is

$$-\frac{\partial \ln(c_{t+1}/c_t)}{\partial \ln(U'(c_{t+1})/U'(c_t))} = -\frac{\partial \ln(c_{t+1}/c_t)}{\partial \ln(c_t^{\sigma}/c_{t+1}^{\sigma})} = \frac{\partial(\ln(c_{t+1}) - \ln(c_t))}{\sigma \partial(\ln(c_{t+1}) - \ln(c_t))} = \frac{1}{\sigma}$$

4. First, note that U is not strictly increasing, strictly concave, nor do the Inada conditions hold if $\sigma = 0$. For the rest of this problem, I will assume that $\sigma > 0$.

   Assuming that $c > 0$ (which we can because the Inada conditions do hold), we have that $U'(c) = \frac{1}{c^{\sigma}} > 0$, so the CRRA utility function is strictly increasing, and that $U''(c) = -\sigma \frac{1}{c^{\sigma+1}} < 0$, so the CRRA utility function is strictly concave. Finally, we can see that $\lim_{c \to 0} \frac{1}{c^{\sigma}} = \infty$ and $\lim_{c \to \infty} \frac{1}{c^{\sigma}} = 0$, so U satisfies the Inada conditions.

5. We have that for any c_t, c_{t+s},

$$MRS(c_{t+s}, c_t) = \frac{\frac{\partial u(c)}{\partial c_{t+s}}}{\frac{\partial u(c)}{\partial c_t}} = \frac{\beta^{t+s} U'(c_{t+s})}{\beta^t U'(c_t)} = \beta^s \frac{c_t^{\sigma}}{c_{t+s}^{\sigma}}$$

   Then we have that fixing some $\lambda > 0$,

$$MRS(\lambda c_{t+s}, \lambda c_t) = \frac{\beta^{t+s} U'(\lambda c_{t+s})}{\beta^t U'(\lambda c_t)} = \beta^s \frac{(\lambda c_{t+s})^{-\sigma}}{(\lambda c_t)^{-\sigma}} = \beta^s \frac{c_t^{\sigma}}{c_{t+s}^{\sigma}}$$

   Thus, u is homothetic if U is of CRRA form.

6. We have that the Lagrangian for each consumer is

$$\mathcal{L}^1 = \sum_{t=0}^{\infty} \beta^t U(c_t) + \mu^1 (y - \sum_{t=0}^{\infty} p_t c_t)$$

$$\mathcal{L}^2 = \sum_{t=0}^{\infty} \beta^t U(c_t) + \mu^2 (\lambda y - \sum_{t=0}^{\infty} p_t c_t)$$

   Taking derivatives with respect to c_t and c_{t+1}, we get

$$\frac{\partial \mathcal{L}^1}{\partial c_t} = \frac{\partial \mathcal{L}^2}{\partial c_t} = \beta^t c_t^{-\sigma} - \mu^i p_t = 0$$

   and

$$\frac{\partial \mathcal{L}^1}{\partial c_{t+1}} = \frac{\partial \mathcal{L}^2}{\partial c_{t+1}} = \beta^{t+1} c_{t+1}^{-\sigma} - \mu^i p_{t+1} = 0$$

   which imply that

$$\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma} = \frac{p_{t+1}}{\beta p_t}$$

$$\Rightarrow \ln(c_t) - \ln(c_{t+1}) = \frac{1}{\sigma} \ln\left(\frac{p_{t+1}}{\beta p_t}\right)$$

$$\Rightarrow \ln(c_0) - \ln(c_t) = \frac{1}{\sigma} \sum_{i=0}^{t-1} \ln\left(\frac{p_{i+1}}{\beta p_i}\right)$$

$$\Rightarrow c_t = c_0 \prod_{i=0}^{t-1} \left(\frac{p_{i+1}}{p_i}\right)^{-\frac{1}{\beta\sigma}}$$

Since this is independent of the income, it holds for both consumers. Substituting into the budget constraint, which holds with equality because utility is strictly increasing, strictly concave, and the Inada conditions hold, we get

$$\sum_{t=0}^{\infty} c_t p_t = y \Longleftrightarrow \sum_{t=0}^{\infty} c_0 \prod_{i=0}^{t-1} \left(\frac{p_{i+1}}{p_i}\right)^{-\frac{1}{\beta\sigma}} p_t = y \Longleftrightarrow \hat{c}_0 = \left[\sum_{t=0}^{\infty} \prod_{i=0}^{t-1} \left(\frac{p_{i+1}}{p_i}\right)^{-\frac{1}{\beta\sigma}} p_t\right]^{-1} y$$

$$\sum_{t=0}^{\infty} c_t p_t = \lambda y \Longleftrightarrow \sum_{t=0}^{\infty} c_0 \prod_{i=0}^{t-1} \left(\frac{p_{i+1}}{p_i}\right)^{-\frac{1}{\beta\sigma}} p_t = \lambda y \Longleftrightarrow \tilde{c}_0 = \lambda \left[\sum_{t=0}^{\infty} \prod_{i=0}^{t-1} \left(\frac{p_{i+1}}{p_i}\right)^{-\frac{1}{\beta\sigma}} p_t\right]^{-1} y$$

and thus,

$$\frac{\hat{c}_t}{\tilde{c}_t} = \frac{y}{\lambda y} \Longrightarrow \tilde{c}_t = \lambda \hat{c}_t \ \forall \ t = 0, \ldots$$

7. We have that $c_t + \frac{a_{t+1}}{1+r_{t+1}} = e_t + a_t$. Our Lagrangian is

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t U(c_t) + \lambda_t \left(e_t + a_t - c_t - \frac{a_{t+1}}{1 + r_{t+1}}\right)$$

and the necessary first order conditions for optimum are

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t U'(c_t) - \lambda_t = 0 \Longrightarrow \lambda_t = \beta^t U'(c_t)$$

and

$$\frac{\partial \mathcal{L}}{\partial a_t} = \lambda_t - \frac{\lambda_{t-1}}{1 + r_t} = 0$$

Combining, we get

$$\beta^t U'(c_t) = \frac{\beta^{t-1}}{1 + r_t} U'(c_{t-1})$$

Thus, the Euler Equation is

$$U'(c_{t-1}) = \beta(1 + r_t)U'(c_t)$$

and the Euler equation relating consumption at time t and time $t+1$ is

$$U'(c_t) = \beta(1 + r_{t+1})U'(c_{t+1})$$

8. We have that $c_t = (1+g)^t c_0$. This means that at time t, the Euler Equation says that

$$U'((1+g)^{t-1} c_0) = \beta(1+r)U'((1+g)^t c_0)$$

which implies that

$$(1+g)^{-\sigma(t-1)} c_0^{-\sigma} = \beta(1+r)(1+g)^{-\sigma t} c_0^{-\sigma}$$

meaning that

$$\frac{(1+g)^{\sigma t}}{(1+g)^{\sigma t - \sigma}} = \beta(1+r) \Rightarrow (1+g)^{\sigma} = \beta(1+r)$$

9. We have that $r = \frac{1}{\beta}(1+g)^{\sigma} - 1$, meaning that

$$\frac{\partial r}{\partial g} = \frac{\sigma}{\beta}(1+g)^{\sigma-1} > 0$$

As g increases, r will also increase. The magnitude of that increase will be inversely proportional to the intertemporal elasticity of substitution. As $1/\sigma$ gets smaller, the increase in r as g increases becomes larger. Intuitively, think about this relationship as follows: as the intertemporal elasticity of substitution shrinks, the consumer is less responsive to changes in the real interest rate. This means that the rate of growth will more directly affect the real interest rate.

10. Suppose that u is homothetic, meaning that $MRS(c_{t+s}, c_t) = MRS(\lambda c_{t+s}, \lambda c_t)\ \forall\ \lambda > 0, c$. Fix a balanced growth path $(1+g)$, with $g > 0$, and assume that consumption is growing along this balanced growth path such that $c_t = (1+g)^t c_0$. It remains to show that this growth path is consistent with a constant interest rate. Take the Euler equation:

$$U'(c_{t-1}) = \beta(1+r_t)U'(c_t)$$

We have that

$$U'((1+g)^{t-1}c_0) = \beta(1+r_t)U'((1+g)^t c_0)$$

which means that

$$\frac{U'((1+g)^{t-1}c_0)}{U'((1+g)^t c_0)} = \beta(1+r_t)$$

Then, taking $\lambda = (1+g)^j$ for some $j \in \mathbb{N}$, we have that additionally

$$\frac{U'((1+g)^{t-1+j}c_0)}{U'((1+g)^{t+j} c_0)} = \beta(1+r_{t+j})$$

Thus, since the left side of both equations is equivalent to $MRS(c_{t-1}, c_t)$ and $MRS(\lambda c_{t-1}, \lambda c_t)$, the fact that u is homothetic implies that $\beta(1+r_t) = \beta(1+r_{t+j})$, meaning that $r_t = r_{t+j}$. This holds $\forall\ j$, so a homothetic utility function with a balanced growth path implies a constant interest rate.

Problem 3 (CARA Utility)

1. We have that $U'(c) = \gamma e^{-\gamma c}$ and $U''(c) = -\gamma^2 e^{-\gamma c}$. Thus, the coefficient of absolute risk aversion is

$$\gamma(c) = -\frac{-\gamma^2 e^{-\gamma c}}{\gamma e^{-\gamma c}} = \gamma$$

and is constant. However, the coefficient of relative risk aversion is

$$-c\frac{-\gamma^2 e^{-\gamma c}}{\gamma e^{-\gamma c}} = c\gamma$$

which is increasing in c.

2. u is not homothetic. We have that u is homothetic if $MRS(c_t, c_{t+s}) = MRS(\lambda c_t, \lambda c_{t+s})\ \forall\ \lambda > 0$ and c. With CARA utility, we have that

$$MRS(c_t, c_{t+s}) = \frac{\frac{\partial u(c)}{\partial c_t}}{\frac{\partial u(c)}{\partial c_{t+s}}} = \frac{\beta^t \gamma e^{-\gamma c_t}}{\beta^{t+s}\gamma e^{-\gamma c_{t+s}}} = \beta^{-s} e^{\gamma(c_{t+s}-c_t)}$$

However, we also have that

$$MRS(\lambda c_t, \lambda c_{t+s}) = \frac{\beta^t \gamma\lambda e^{-\gamma\lambda c_t}}{\beta^{t+s}\gamma\lambda e^{-\gamma\lambda c_{t+s}}} = \beta^{-s} e^{\lambda\gamma(c_{t+s}-c_t)}$$

since $MRS(c_t, c_{t+s}) \neq MRS(\lambda c_t, \lambda c_{t+s})$, u is not homothetic.

3. We follow the same logic as in Problem 2 Part 6. The two Lagrangians are

$$\mathcal{L}^1 = \sum_{t=0}^{\infty} \beta^t U(c_t) + \mu^1 \left(y - \sum_{t=0}^{\infty} p_t c_t \right)$$

and

$$\mathcal{L}^2 = \sum_{t=0}^{\infty} \beta^t U(c_t) + \mu^2 \left(\lambda y - \sum_{t=0}^{\infty} p_t c_t \right)$$

Taking the FOCs with respect to c_t and c_{t+1}, we get

$$\frac{\partial \mathcal{L}^1}{\partial c_t} = \frac{\partial \mathcal{L}^2}{\partial c_t} = \beta^t \gamma e^{-\gamma c_t} - \mu^i p_t = 0$$

and

$$\frac{\partial \mathcal{L}^1}{\partial c_{t+1}} = \frac{\partial \mathcal{L}^2}{\partial c_{t+1}} = \beta^{t+1} \gamma e^{-\gamma c_{t+1}} - \mu^i p_{t+1} = 0$$

Combining, we get

$$\frac{e^{-\gamma c_{t+1}}}{e^{-\gamma c_t}} = \frac{p_{t+1}}{\beta p_t}$$

$$-\gamma(c_{t+1} - c_t) = \ln\left(\frac{p_{t+1}}{\beta p_t}\right)$$

$$c_t - c_0 = -\frac{1}{\gamma} \sum_{i=1}^{t-1} \ln\left(\frac{p_{i+1}}{\beta p_i}\right)$$

and finally,

$$c_t = c_0 - \frac{1}{\gamma} \sum_{i=1}^{t-1} \ln\left(\frac{p_{i+1}}{\beta p_i}\right)$$

Combining with the original budget constraint, we get

$$\sum_{t=0}^{\infty} p_t \left[c_0 - \frac{1}{\gamma} \sum_{i=1}^{t-1} \ln\left(\frac{p_{i+1}}{\beta p_i}\right) \right] = y$$

$$c_0 \sum_{t=0}^{\infty} p_t = y + \left[\frac{1}{\gamma} \sum_{t=0}^{\infty} p_t \sum_{i=1}^{t-1} \ln\left(\frac{p_{i+1}}{\beta p_i}\right) \right]$$

$$\hat{c}_0 = \left(\sum_{t=0}^{\infty} p_t \right)^{-1} \left(y + \left[\frac{1}{\gamma} \sum_{t=0}^{\infty} p_t \sum_{i=1}^{t-1} \ln\left(\frac{p_{i+1}}{\beta p_i}\right) \right] \right)$$

and we also get that

$$\tilde{c}_0 = \left(\sum_{t=0}^{\infty} p_t \right)^{-1} \left(\lambda y + \left[\frac{1}{\gamma} \sum_{t=0}^{\infty} p_t \sum_{i=1}^{t-1} \ln\left(\frac{p_{i+1}}{\beta p_i}\right) \right] \right)$$

Thus, we have that

$$\hat{c}_t - \tilde{c}_t = (1-\lambda) y \left(\sum_{t=0}^{\infty} p_t \right)^{-1} \quad \forall\, t = 0, \ldots$$

so

$$\tilde{c}_t = \hat{c}_t - (1-\lambda) y \left(\sum_{t=0}^{\infty} p_t \right)^{-1} \quad \forall\, t = 0, \ldots$$