ECON 6090: Solutions for Problem Set 3

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Fall 2024

Due date: Monday, October 21.

1. (**Production possibilities set**) Consider the production possibilities set

$$Y = \left\{ (q, -z) \in \mathbb{R}^2_+ \times \mathbb{R}^2_- : z_1^\alpha z_2^\beta \geq \frac{{q_1}^2 + {q_2}^2}{2} \right\}$$

where $\alpha, \beta > 0$.

(a) Find the conditional input demand function $z(w_1, w_2, q_1, q_2)$.
The cost minimization problem is

$$\min_{z_1, z_2} w \cdot z \ \text{ subject to } \ z_1^\alpha z_2^\beta \geq \frac{{q_1}^2 + {q_2}^2}{2}$$

The corresponding Lagrangian minimization problem is

$$\min_{z_1, z_2} w \cdot z - \lambda \left[z_1^\alpha z_2^\beta - \frac{{q_1}^2 + {q_2}^2}{2} \right]$$

The first order conditions are

$$w_1 = \lambda \alpha \frac{{z_1}^\alpha {z_2}^\beta}{z_1}$$
$$w_2 = \lambda \beta \frac{{z_1}^\alpha {z_2}^\beta}{z_2}$$

Solving this system of equations for z_1, z_2 gives

$$z_1 = \lambda \alpha \frac{1}{w_1} \left(\frac{{q_1}^2 + {q_2}^2}{2} \right)$$
$$z_2 = \lambda \beta \frac{1}{w_2} \left(\frac{{q_1}^2 + {q_2}^2}{2} \right)$$

All that remains is to find the value of λ. We can do this by plugging the expressions for z_1, z_2 above into the inequality that defines Y. This equality will be binding, since we can assume that $w_1, w_2 > 0$.

$$\frac{{q_1}^2 + {q_2}^2}{2} = z_1^\alpha z_2^\beta = \lambda^{\alpha+\beta} \left(\frac{{q_1}^2 + {q_2}^2}{2}\right)^{\alpha+\beta} \alpha^\alpha \beta^\beta \frac{1}{{w_1}^\alpha {w_2}^\beta}$$

Solving for λ and then substituting back into the expressions for z_1, z_2 gives the conditional input demand function

$$z(w_1, w_2, q_1, q_2) = \left(\frac{{q_1}^2 + {q_2}^2}{2}\right) \begin{bmatrix} \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} \left(\frac{w_2}{w_1}\right)^{\frac{\beta}{\alpha+\beta}} \\ \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} \left(\frac{w_1}{w_2}\right)^{\frac{\alpha}{\alpha+\beta}} \end{bmatrix}$$

(b) What is the marginal rate of transformation between output 1 and output 2? That is, given w_1, w_2, q_1, q_2, what is the proportional decrease in q_1 required to marginally increase q_2 while holding cost constant?

The conditional input demand function, and therefore the cost function, are proportional to $\left(\frac{{q_1}^2+{q_2}^2}{2}\right)$. So varying q_1, q_2 while keeping cost unchanged means moving along the isocost curve in the (q_1, q_2) space defined by

$$\left(\frac{{q_1}^2 + {q_2}^2}{2}\right) = k$$

for some constant k.

Using the Implicit Function Theorem, we can show that the partial derivative of q_1 with respect to q_2 along the isocost curve is

$$\frac{dq_1}{dq_2} = -\frac{\frac{\partial}{\partial q_2}\left(\frac{{q_1}^2+{q_2}^2}{2}\right)}{\frac{\partial}{\partial q_1}\left(\frac{{q_1}^2+{q_2}^2}{2}\right)} = -\frac{q_2}{q_1}$$

That is, to keep cost constant, a marginal increase in q_2 must be accompanied by a decrease in q_1 that is $\frac{q_2}{q_1}$ as large.

2. (**Cost minimization**) Consider a single-output firm with technology that can transform inputs $z \in \mathbb{R}^3_+$ into output according to the production function

$$f(z) = {z_1}^{\frac{1}{2}} {z_2}^{\frac{1}{4}} {z_3}^{\frac{1}{8}}$$

(a) This production function is homogeneous degree α. Find α. What does this imply about the firm's cost function? Is the firm's marginal cost of production increasing or decreasing in q?

For any $\alpha \in \mathbb{R}_+$,

$$\begin{aligned} f(\alpha z) &= \alpha z_1{}^{\frac{1}{2}} \alpha z_2{}^{\frac{1}{4}} \alpha z_3{}^{\frac{1}{8}} \\ &= \alpha^{\frac{7}{8}} z_1{}^{\frac{1}{2}} z_2{}^{\frac{1}{4}} z_3{}^{\frac{1}{8}} \\ &= \alpha^{\frac{7}{8}} f(z) \end{aligned}$$

which shows that f is homogeneous of degree $\frac{7}{8}$.

From Proposition 3.10 in the lecture notes, we know this implies that the cost function C is monotone degree $\frac{8}{7}$ in q.

Then, using Proposition 3.12, we know that the marginal cost function $\frac{\partial}{\partial q} C(w, q)$ is homogeneous of degree $\frac{1}{7}$ in q. Thus,

$$\frac{\partial}{\partial q} C(w, \alpha q) = \alpha^{\frac{1}{7}} \frac{\partial}{\partial q} C(w, q)$$

For $\alpha > 1$ (which represents an increase in q), this means that

$$\frac{\partial}{\partial q} C(w, \alpha q) > \frac{\partial}{\partial q} C(w, q)$$

That is, the marginal cost of production is increasing in q.

(b) Derive the conditional input demand function $z(w, q)$.

The Lagrangian problem for the cost minimization problem is

$$\min_z w \cdot z - \lambda \left[z_1{}^{\frac{1}{2}} z_2{}^{\frac{1}{4}} z_3{}^{\frac{1}{8}} - q \right]$$

The first order conditions are

$$w = \lambda \nabla f(z) = \lambda f(z) \begin{bmatrix} \frac{1}{2} z_1{}^{-1} \\ \frac{1}{4} z_2{}^{-1} \\ \frac{1}{8} z_3{}^{-1} \end{bmatrix}$$

Solving this system for z gives

$$z = \lambda f(z) \begin{bmatrix} \frac{1}{2} w_1{}^{-1} \\ \frac{1}{4} w_2{}^{-1} \\ \frac{1}{8} w_3{}^{-1} \end{bmatrix}$$

All that remains is to find the value of λ. We can do this by plugging the expressions for z above into the production function:

$$q = \lambda^{\frac{7}{8}} q^{\frac{7}{8}} {w_1}^{-\frac{1}{2}} {w_2}^{-\frac{1}{4}} {w_3}^{-\frac{1}{8}} 2^{-\frac{1}{2}} 4^{-\frac{1}{4}} 8^{-\frac{1}{8}}$$

Solving this equation for λ gives

$$\lambda = 2^{\frac{11}{7}} \underbrace{\left({w_1}^{\frac{1}{2}} {w_2}^{\frac{1}{4}} {w_3}^{\frac{1}{8}}\right)^{\frac{8}{7}}}_{\equiv W} q^{\frac{1}{7}}$$

Now substituting this into the expression for z above gives the conditional input demand function

$$z(w,q) = 2^{\frac{11}{7}} q^{\frac{8}{7}} W^{\frac{8}{7}} \begin{bmatrix} 2^{-1} {w_1}^{-1} \\ 2^{-2} {w_2}^{-1} \\ 2^{-3} {w_3}^{-1} \end{bmatrix}$$

(c) Derive an expression for the firm's marginal cost of production, i.e., the derivative of the cost function with respect to q.

$$\begin{aligned} C(w,q) = w \cdot z(w,q) &= q^{\frac{8}{7}} W^{\frac{8}{7}} 2^{\frac{11}{7}} \left(2^{-1} + 2^{-2} + 2^{-3}\right) \\ &= \frac{7}{2^{\frac{10}{7}}} q^{\frac{8}{7}} W^{\frac{8}{7}} \end{aligned}$$

3. (**Cost minimization with a continuum of inputs**) Consider a single-output firm which takes as input a continuum of inputs rather than a discrete set of inputs. We now denote the quantity input of commodity j as $z(j)$ (rather than z_j as we did in the discrete-inputs cases). The production function is

$$f(z) = \left[\int_0^1 a(j) z(j)^{\frac{\sigma-1}{\sigma}} dj\right]^{\frac{\sigma}{\sigma-1}}$$

where $a(j)$ is a continuous function integrable on $[0,1]$ that reflects the relative productivities of the various inputs.[1]

(a) Derive the conditional input demand function $z(j,w,q)$. The price for input j is given by $w(j)$, where w is a continuous function integrable on $[0,1]$.

The Lagrangian minimization problem for the cost minimization

[1] While a continuum of inputs may not immediately seem empirically relevant, this assumption and the functional form imposed for f in this problem are commonly used in, for example, models of international trade.

problem is

$$\min_z \int_0^1 w(z)z(j)dj - \lambda\left[f(z) - q\right]$$

Note that the marginal product of good j is

$$\frac{\partial f(z)}{\partial z(j)} = a(j)f(z)^{1/\sigma}z(j)^{-1/\sigma}$$

So the FOC of the Lagrangian minimization problem with respect to $z(j)$ is

$$w(z) = \lambda a(j)f(z)^{1/\sigma}z(j)^{-1/\sigma}$$

Solving for $z(j)$ gives

$$z(j) \ = \lambda^{\sigma}a(j)^{\sigma}w(z)^{-\sigma}q$$

All that remains is to find the value of λ. We can do this by plugging the expressions for z above into the production function:

$$q = \lambda^{\sigma}q\left[\int_0^1 a(j)^{\sigma+1}w(j)^{-(\sigma-1)}dj\right]^{\frac{\sigma}{\sigma-1}}$$

Solving for λ gives

$$\lambda = \left[\int_0^1 a(j)^{\sigma+1}w(j)^{-(\sigma-1)}dj\right]^{\frac{-1}{\sigma-1}}$$

Plugging back into the expression for $z(j)$ gives the conditional input demand function

$$\begin{aligned}
z(j,w,q) &= \lambda^{\sigma}a(j)^{\sigma}w(z)^{-\sigma}q \\
&= a(j)^{\sigma}w(j)^{-\sigma}q\left[\int_0^1 a(j)^{\sigma+1}w(j)^{-(\sigma-1)}dj\right]^{\frac{-\sigma}{\sigma-1}} \\
&= qa(j)^{\sigma}\left(\frac{w(j)}{W}\right)^{-\sigma}
\end{aligned}$$

where

$$W = \left[\int_0^1 a(j)^{\sigma+1}w(j)^{-(\sigma-1)}dj\right]^{\frac{-1}{\sigma-1}}$$

(b) How is the conditional input demand for input j affected by $a(j)$, the productivity of input j?

$$\frac{\partial}{\partial a(j)} z(j, w, q) = \sigma q a(j)^{\sigma-1} \left(\frac{w(j)}{W}\right)^{-\sigma}$$

So $z(j)$ is increasing in $a(j)$ since $\sigma > 0$.

(c) Now suppose that the firm has market power in input markets. If the firm uses $z(j)$ units of input j, the per-unit input price is $w(j, z(j)) = \frac{1}{2}z(j)$. Find the cost-minimizing choice of inputs to produce $q = 1$ units of output.

The Lagrangian of the cost minimization problem is

$$\min_z \int_0^1 \frac{1}{2} z(j)^2 dj - \lambda \left[f(z) - 1\right]$$

The FOC with respect to $z(j)$ is

$$z(j) = \lambda \frac{\partial f(z)}{\partial z(j)} = \lambda a(j) f(z)^{1/\sigma} z(j)^{-1/\sigma} = \lambda a(j) z(j)^{-1/\sigma}$$

Solving for $z(j)$ gives

$$z(j) = \lambda^{\frac{\sigma}{\sigma+1}} a(j)^{\frac{\sigma}{\sigma+1}}$$

All that remains is to find the value of λ. We can do this by plugging the expressions for z above into the production function:

$$1 = \lambda^{\frac{\sigma}{\sigma+1}} \underbrace{\left[\int_0^1 a(j)^{\frac{2\sigma}{\sigma+1}} dj\right]^{\frac{\sigma}{\sigma-1}}}_{A}$$

Solving for λ and substituting back into the expression for $z(j)$ gives

$$z(j) = \frac{a(j)^{\frac{\sigma}{\sigma+1}}}{A}$$

4. (**Profit maximization with a non-smooth production function**) Consider a single-output firm with technology that can transform inputs $z \in \mathbb{R}_+^N$ into output according to the production function

$$f(z) = 2\sqrt{\min\{z_1, 2z_2, 3z_3, \ldots, Nz_N\}}$$

(a) Derive the unconditional input demand function.

Recall that profit maximization implies cost minimization. With

this Leontief production function, cost minimization will require that (assuming $w_i > 0$ for all $i = 1, \ldots, N$)

$$z_1 = 2z_2 = \ldots = Nz_N$$

If this were not the case, then it would be possible to decrease some z_i (and therefore decreasing cost) without any effect on the quantity produced. Denote

$$\bar{z} = z_1 = 2z_2 = \ldots = Nz_N$$

Then, the firm's expenditure can be written as

$$w \cdot z = \left(\sum_{j=1}^{N} \frac{1}{j} w_j \right) \bar{z}$$

We can rewrite the production function in terms of $\bar{z}$:

$$f(z) = 2\sqrt{\bar{z}}$$

Thus, the profit maximization problem can be rewritten as a choice of $\bar{z}$

$$\max_{\bar{z}} 2\sqrt{\bar{z}}p - \left(\sum_{j=1}^{N} \frac{1}{j} w_j \right) \bar{z}$$

The first order condition is

$$\frac{p}{\sqrt{\bar{z}}} = \left(\sum_{j=1}^{N} \frac{1}{j} w_j \right)$$

which implies that

$$\bar{z} = \left(\frac{p}{\sum_{j=1}^{N} \frac{1}{j} w_j} \right)^2$$

So the unconditional input demand function is

$$x(p, w) = \left(\frac{p}{\sum_{j=1}^{N} \frac{1}{j} w_j} \right)^2 \begin{bmatrix} 1 \\ \frac{1}{2} \\ \vdots \\ \frac{1}{N} \end{bmatrix}$$

(b) Now suppose that the firm has market power in the output market. If the firm produces quantity q, the per-unit price is $P(q) = q^{-\epsilon}$ where $\epsilon \in (1, \infty)$. Derive the firm's choice of inputs $z_1, \ldots, z_N$.

Whatever quantity of output the firm chooses, it will still produce that quantity in a cost-minimizing way. That is, the cost of producing $2\sqrt{\bar{z}}$ units of output is still is still

$$\left(\sum_{j=1}^{N} \frac{1}{j} w_j \right) \bar{z}$$

Therefore, the profit-maximizing $\bar{z}$ is the solution to

$$\max \left(2\sqrt{\bar{z}}\right)^{1-\epsilon} - \left(\sum_{j=1}^{N} \frac{1}{j} w_j \right) \bar{z}$$

Taking the first order condition and solving for $\bar{z}$ gives

$$\bar{z} = (1-\epsilon)^{\frac{2}{1-\epsilon}} \, 2^{-\frac{2\epsilon}{1+\epsilon}} \left(\sum_{j=1}^{N} \frac{1}{j} w_j \right)^{-\frac{2}{1+\epsilon}}$$

And the inputs used are

$$z_j = (1-\epsilon)^{\frac{2}{1-\epsilon}} \, 2^{-\frac{2\epsilon}{1+\epsilon}} \left(\sum_{j=1}^{N} \frac{1}{j} w_j \right)^{-\frac{2}{1+\epsilon}} \begin{bmatrix} 1 \\ \frac{1}{2} \\ \vdots \\ \frac{1}{N} \end{bmatrix}$$

5. (**Producer theory in action**) De Loecker, Eeckhout, and Unger (QJE, 2020) is an influential paper on measuring market power. The approach described in this paper takes the cost minimization problem as a starting point. Read the first 11 pages of this article (through the end of Section II.B) paying particular attention to Sections II.A and II.B.

   (a) In going from equation (6) to (7), the authors assert that "The Lagrange multiplier λ is a direct measure of marginal cost." Give a justification for this assertion.

   (b) The authors' starting point in Section II.B is the cost minimization problem. However, the output price (the key component of the markup) does not feature in the CMP (recall that the only arguments of the cost function and conditional input demand function are w and q). Given this, why can the authors claim that this starting point leads to some insight about markups? Wouldn't it be more natural to use the profit maximization problem as a starting point?