ECON 6170 Problem Set 6 Answers

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Exercise 1. Let f be a function with a discontinuity at x such that there is a sequence $x_n \to x$ with $f(x_n) \to y \neq f(x)$.[1] Then $F(x) := \{f(x)\}$ is a valid example. F is singleton-valued and thus closed-valued. And there exists $x_n \to x$ and $y_n \to y$ such that $y_n \in \{f(x_n)\}$ for all n and $y \notin \{f(x)\}$, so F is not closed at x.

For example, $F : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by $F(x) := \{1\{x > 0\}\}$ is equal to $\{1\}$ for positive x and $\{0\}$ elsewhere. It is singleton-valued and thereby closed-valued. But the sequence $\frac{1}{n} \to 0$ and $1 \in F(\frac{1}{n})$ for all n but the constant sequence $1, 1, 1, \ldots$ converges to $1 \notin F(0)$, so F is not closed at 0.

Exercise 2. False. By Proposition 5 (ii), we know that our counterexample cannot be closed-valued. Let $F : \mathbb{R} \rightrightarrows \mathbb{R}$ be a correspondence defined by $F(x) := (0, 1]$ for all $x \in \mathbb{R}$. Let U be an open subset of $\mathbb{R}$. Then

$$F^{-1}(U) = \begin{cases} \mathbb{R} & \text{if } (0,1] \subseteq U \\ \emptyset & \text{otherwise} \end{cases}$$

$\mathbb{R}$ and $\emptyset$ are both open, so F is upper hemicontinuous. But $(x, \frac{1}{n}) \to (x, 0) \notin \operatorname{Gr} F$, so $F(x)$ is not closed.

Exercise 3. F is not lower hemicontinuous. To see this, consider the sequence $(2 - \frac{1}{n}) \to 2$ in X and the point $1 \in F(2) = [0, 2]$. Proposition 2 (ii) tells us that for F to be lower hemicontinuous, there must exist a sequence $y_n \in F(2 - \frac{1}{n})$ such that $y_n \to 1$. But $F(2 - \frac{1}{n}) = \{\frac{1}{n}, 2 + \frac{1}{n}\}$, so any such y_n sequence can only converge to 0 or 2.

G is not lower hemicontinuous either, by the exact same argument.

G is not upper hemicontinuous by Proposition 5 (ii), because it is closed-valued but doesn't have a closed graph. To see that it doesn't have a closed graph, consider the sequence in $\mathbb{R}^2$, $(2 - \frac{1}{n}, \frac{1}{n})$, which consists of elements of $\operatorname{Gr} G$. This sequence converges to $(2, 0)$ but $0 \notin G(2) = [1, 3]$, so $(2, 0) \notin \operatorname{Gr} G$.

F is upper hemicontinuous. By Proposition 2 (i), it is sufficient to show that if $x_n \to x$ and $y_n \in F(x_n)$ for all n then some subsequence $y_{n_k} \to y \in F(x)$. We proceed by cases.

Note that (x_n) has a subsequence entirely contained in either $(-\infty, 2)$, $[2, 3]$, or $(3, \infty)$. It is WLOG, then, to only consider sequences that are entirely contained in exactly one of these three intervals.

First, suppose $x_n < 2$ for all n. Then $x \leq 2$, and $y_n = 2 - x_n$ or $y_n = 4 - x_n$ for all n. This implies that there is either a subsequence $y_{n_k} = 2 - x_{n_k} \to 2 - x \in F(x)$ or a subsequence

[1] That is, the discontinuity shouldn't be oscillating or infinite from all directions.

$y_{n_k} = 4 - n_k \to 4 - x \in F(x)$.

Second, suppose $x_n > 3$ for all n. Then $x \geq 3$. Then $y_n = x_n - 3 \to x - 3 \in F(x)$.

Third, suppose $2 \leq x_n \leq 3$ for all n. Then $2 \leq x \leq 3$. Moreover, $y_n \in [2 - x_n, 4 - x_n] \subseteq [-1, 2]$ for all n. So y_n must have a convergent subsequence, $y_{n_k} \to y$. But $2 - x_{n_k} \leq y_{n_k} \leq 4 - x_{n_k}$ so, by the squeezing theorem, $2 - x \leq y \leq 4 - x$.

Exercise 4.

(i) Upper semicontinuous. The budget correspondence is compact-valued, so we can use Proposition 2 (i) to show that it is upper hemicontinuous. Let $(p^i, w^i) \to (p, w)$ be a convergent sequence in $\mathbb{R}^{n+1}_{++}$ and (x^i) be a sequence in $\mathbb{R}^n_+$ such that $x^i \in B(p^i, w^i)$ for all i. We use superscripts for sequence indexation, and subscripts for vector indexation. Because (p^i, w^i) converges, it must be bounded. It follows that (x^i) must also be bounded, and hence must have a convergent subsequence $x^h \to x$.

Then
$$p^h \cdot x^h \leq w^h$$
for all h. Note that $p^h \cdot x^h \to p \cdot x$ by continuity of $(x, y) \mapsto x \cdot y$. Weak inequalities hold in the limit, so
$$p \cdot x \leq w$$
and therefore, $x \in B(p, w)$.

(ii) Lower semicontinuous. Take an arbitrary $x \in B(p, w)$. Let $(p^i, w^i) \to (p, w)$. We want to construct a sequence of points in $\mathbb{R}^n_+$ that converges to x, such that the points lie in the corresponding $B(p^i, w^i)$. We have
$$p \cdot x \leq w$$
which implies
$$(1 + \delta^i)p \cdot (1 - \delta^i)x \leq (1 + \delta^i)(1 - \delta^i)w = (1 - (\delta^i)^2)w$$
for any $0 < \delta^i < 1$. Choose $(\delta^i)_{i=1}^{\infty}$ such that $\delta^i \to 0$; and for large i, $p^i_j \leq (1 + \delta^i)p_j$ for $j = 1, 2, \ldots, n$, and $w^i \geq (1 - (\delta^i)^2)w$. Such a (δ^i) exists because $p^i \to p$ and $w^i \to w$. Define (x^i) by $x^i_j := (1 - \delta^i)x_j$ for all i, j. Then $x^i \to x$ and
$$\begin{aligned} p^i \cdot x^i &= p^i \cdot (1 - \delta^i)x \\ &\leq (1 + \delta^i)p \cdot (1 - \delta^i)x \\ &\leq (1 - (\delta^i)^2)w \\ &\leq w^i \end{aligned}$$
so $x^i \in B(p^i, w^i)$ for large i.

Exercise 5 (Additional exercise)**.**

(a) Graph is closed.

Upper hemicontinuous as $F(X)$ is bounded so we can apply Proposition 5 (i).

Not lower hemicontinuous as we can take a sequence $x_n \to x$ in the domain at which $F(x_n)$ is a singleton and $F(x)$ is the interval. Take $y \in F(x)$ such that y lies in the interior of that interval. Then no $y_n \to y$ for $y_n \in F(x_n)$.

(b) Graph is not closed due to open endpoints in the center.

Because F is closed-valued ($F(x) = \{y\}$ or $F(x) = \emptyset$), this implies that F is not upper hemicontinuous, by Proposition 5 (ii). We can also disprove upper hemicontinuity using Proposition 2 (i) and taking a sequence $x_n \to x^*$, where x^* is the point such that $F(x) = \emptyset$. Clearly there is no $y \in \emptyset$ such that $y_{n_k} \to y$. Alternatively, we can also simply note that $F^{-1}(\emptyset) = \{x^*\}$ which is closed.

F is lower hemicontinuous, because if $y \in F(x)$ then $y \in \{a, b\}$, the range of F. If $y = a$, then there exists $\varepsilon > 0$ such that $y_n = a$ for all $y_n \in F(x_n)$ with $x_n \in (x - \varepsilon, x + \varepsilon)$. We can also use the inverse image characterisation of lower hemicontinuity. If $U \subseteq Y$ is open then

$$F_{-1}(U) = \begin{cases} \mathbb{R} \setminus \{x^*\} & \text{if } a, b \in U \\ \emptyset & \text{if } a, b \notin U \\ (-\infty, x^*) & \text{if } a \in U, b \notin U \\ (x^*, \infty) & \text{if } a \notin U, b \in U \end{cases}$$

where x^* is the empty-valued point, $a < b$ are the two values in $F(X)$. In all cases, $F_{-1}(U)$ is open.

(c) The graph is closed.

Because $F(X)$ is bounded, we can apply Proposition 5 (i) to infer that F is upper hemicontinuous. We can also use Proposition 2 (i), noting that if y_n is in $F(x_n)$ then $F(x_n)$ is nonempty, so (x_n, y_n) must lie on one of the curves in the graph. These are closed and bounded sets, so (sequentially) compact. Therefore, (x_n, y_n) must have a convergent subsequence, (x_{n_k}, y_{n_k}) converging to a point on the same curve. But $x_{n_k} \to x$, so $(x_{n_k}, y_{n_k}) \to (x, y)$ with $y \in F(x)$.

F is not lower hemicontinuous. This follows from Proposition 2 (ii). Take one of the endpoints of the curves, denoted by the solid circles, and denote this (x, y). Take a sequence $x_n \to x$, where $F(x_n) = \emptyset$ for all n. Then there is no y_n sequence in a sequence of empty sets, so we cannot construct $y_n \to y$ satisfying $y_n \in F(x_n)$ for large n.

(d) Graph is not closed as we can take a sequence on the graph approaching the open point.

To determine upper hemicontinuity, it will be helpful to write the correspondence down:

$$F(x) := \begin{cases} a & \text{if } x < x^* \\ [a, b) & \text{if } x = x^* \\ b & \text{if } x > x^* \end{cases}$$

Note that $F^{-1}(-\infty, b) = (-\infty, x^*]$ which is not closed. Thus, F is not upper hemicontinuous. Note also that because F is not closed-valued we cannot use Propositions 2 or 5 to prove that F is *not* upper hemicontinuous.

F is not lower hemicontinuous by the exact same reasoning as in (a).

(e) Graph of F is the union of two curves, so is closed.

We can disprove upper hemicontinuity by taking a sequence $x_n \to x^*$, where x^* is the point at the vertical asymptote, such that $x_n > x^*$ for all n. Then $y_n \in F(x_n)$ implies $y_n \to \infty \notin F(x^*)$.

F is not lower hemicontinuous. To see this, consider (x^*, y) at the right endpoint of the left arm. Let $x_n \to x^*$ be such that $x_n > x^*$ for all n. Then there is no y_n with $y_n \in F(x_n)$ for large n such that $y_n \to y$.

Part 2:

Γ_1 is compact-valued and we can construct a sequence $(x^n, y^n) \in \operatorname{Gr}\Gamma_1$ that converges to $(x_1, y_1) \notin \operatorname{Gr}\Gamma_1$ or $(x_2, y_2) \notin \operatorname{Gr}\Gamma_1$. Therefore, Γ_1 is not upper hemicontinuous at either x_1 or x_2.

A sequence in the graph of Γ_2, $(x^n, y^n) \to (x_1, y_1)$ will satisfy $y_1 \in \Gamma_2(x_1)$, so Γ_2 is upper hemicontinuous at x_1. $\Gamma_2(x_2) \subseteq U$, U open, implies $\Gamma_2(B_\varepsilon(x_2)) \subseteq U$ for sufficiently small ε, so Γ_2 is upper hemicontinuous at x_2 also. However, Γ_2 is not upper hemicontinuous at at least some of the points $x > x_2$. This is because $\Gamma_2(x)$ is open but $\Gamma_2(B_\varepsilon(x)) \nsubseteq \Gamma_2(x)$ for any $\varepsilon > 0$.

Γ_3 is not upper hemicontinuous at x_1. For if we let $\Gamma_3(x_1) = [a, b]$, then $\Gamma_3(x_1) \subseteq (a - \delta, b + \delta)$ but $\Gamma_3(x_1 + \varepsilon) \nsubseteq (a - \delta, b + \delta)$ for sufficiently small δ and arbitrarily small ε. An analogous argument tells us that Γ_3 is not upper hemicontinuous at x_2 either.

Γ_2 and Γ_3 aren't lower hemicontinuous at either x_1 or x_2, by the same argument used in (a) above.

If $y_1 \in \Gamma_1(x_1)$, then (x_1, y_1) must be at the black dot. Clearly, any $x^n \to x_1$ will have $y^n \in \Gamma_1(x^n)$ such that $y^n \to y_1$. If $y_2 \in \Gamma_1(x_2)$ then (x_2, y_2) must lie along the solid line, including the black dots. Again it is clear that $x^n \to x_2$ implies that there exists $y^n \in \Gamma_1(x^n)$ such that $y^n \to y_2$. Therefore, Γ_1 is lower hemicontinuous at both x_1 and x_2. It is visually clear that it is lower hemicontinuous everywhere else too.