ECON 6170

Problem Set 4

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Exercise 6 Every real-valued function $f : S \to \mathbb{R}$ is continuous at every isolated point $x \in S$.

Proof. We have that x is an isolated point, meaning that $\exists\ \varepsilon > 0$ s.t. $B_\varepsilon(x) \cap S = \{x\}$. Fix some $\varepsilon' > 0$. We have that f is continuous at x if $\exists\ \delta$ s.t. $|f(x) - f(y)| < \varepsilon'$ whenever $|x - y| < \delta\ \forall\ y \in S$. We can take $\delta < \varepsilon$. Then, since $B_\varepsilon(x) \cap S = \{x\}$, the set $\{y \in S \mid |x - y| < \delta\}$ is a singleton that contains only x. Thus, since $|f(x) - f(x)| = 0 < \varepsilon'$ for each $\varepsilon' > 0$, f is trivially continuous at x. This holds for any f where $f(x)$ is well-defined. □

Exercise 7 Prove the following using the $\varepsilon - \delta$ definition of continuity.

Proposition 7. If $f : S \to \mathbb{R}$ is continuous at x_0, $f(x_0) \in T \subseteq \mathbb{R}$, and $g : T \to \mathbb{R}$ is continuous at $f(x_0)$, then the composite function $g \circ f$ is continuous at x_0.

Proof. Fix $\varepsilon > 0$. Since g is continuous at $f(x_0)$, $\exists\ \delta_g > 0$ s.t. $|f(x_0) - f(y)| < \delta_g \implies |g(f(x_0)) - g(f(y))| < \varepsilon\ \forall\ f(y) \in T$. Then take $\varepsilon_f = \delta_g$. Since f is continuous at x_0, $\exists\ \delta_f > 0$ s.t. $|x_0 - y| < \delta_f \implies |f(x_0) - f(y)| < \varepsilon_f\ \forall\ y \in S$. Then, we have that the composition works as follows: For $\varepsilon > 0$, $\exists\ \delta_f$ s.t. $\forall\ y \in S, |x_0 - y| < \delta_f \implies |f(x_0) - f(y)| < \delta_g \implies |g(f(x_0)) - g(f(y))| < \varepsilon$. Thus, $g \circ f$ is continuous. □

Exercise 8 True!

Proof. Note that $\max\{f, g\} = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|$. We will use many elements of Proposition 6. Since g is continuous, $-g$ is continuous from (ii), taking $k = -1 \in \mathbb{R}$. Then $f - g$ is continuous from (iii), and $|f - g|$ is continuous from (i). Additionally, $(f + g)$ is continuous from (iii), and $\frac{1}{2}|f - g|$ and $\frac{1}{2}(f + g)$ are both continuous from (ii), taking $k = \frac{1}{2} \in \mathbb{R}$. Finally, $\max\{f, g\} = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|$ is continuous from (iii). □

Exercise 9 This statement is true! First, we will prove the useful lemma indicated.
Lemma 1. *A sequence $\{x_n\}$ converges to x if and only if for every subsequence $\{x_{n_k}\}$ there exists sub-subsequence $\{x_{n_{k_l}}\}$ that converges to x.*

Proof. ($\Rightarrow$) $x_n \to x \Rightarrow \{x_n\}$ Cauchy, meaning that $\forall\ \varepsilon > 0, \exists\ N \in \mathbb{N}$ s.t. $|x_n - x_m| < \varepsilon\ \forall\ n, m > N$. Taking some subsequence $\{x_{n_k}\}$, we have that $|x_{n_i} - x_{n_j}| < \varepsilon$ if $n_i, n_j > N$ where N is from the initial sequence. Thus, $\{x_{n_k}\}$ is Cauchy. Taking a sub-subsequence $\{x_{n_{k_l}}\}$, we have that $|x_{n_{k_i}} - x_{n_{k_j}}| < \varepsilon$ as long as $n_{k_i}, n_{k_j} > N$, where N is again from the initial sequence. Thus, $\{x_{n_{k_l}}\}$ is Cauchy, so it converges by Theorem 2. It remains to show that $\{x_{n_{k_l}}\}$ converges to x. FSOC, assume that $x_{n_{k_l}} \to y \neq x$. Then $|y - x| = \delta > 0$. Taking $\varepsilon = \delta/3$, we have that $\exists\ N \in \mathbb{N}$ s.t. $x_{n_{k_l}} \in B_\varepsilon(y) \Rightarrow x_{n_{k_l}} \notin B_\varepsilon(x)\ \forall\ n_{k_l} > N$, which implies that $x_n \not\to x$, which is a contradiction. Thus, $x_{n_{k_l}} \to x$.

($\Leftarrow$) Proof by contrapositive. Assume that there exists a subsequence $\{x_{n_k}\}$ such that all sub-subsequences $\{x_{n_{k_l}}\}$ do not converge to x. Consider two cases. First, assume that there exists some $\{x_{n_{k_l}}\}$ such that $x_{n_{k_l}} \to y \neq x$. This is the exact same case as the assumed contradiction above, where we showed that $x_n \not\to x$. Second, assume that all $\{x_{n_{k_l}}\}$ do not converge. This means that $\forall\ y \in \mathbb{R}, \exists\ \varepsilon > 0$ s.t. $\forall\ N \in$

$\mathbb{N}, \exists\, n > N$ s.t. $|x_{n_{k_l}} - y| > \varepsilon$. Taking $y = x$, and recalling that $x_{n_{k_l}} \in \{x_n\}\ \forall\, n_{k_l}$, this is a direct negation of the definition of convergence, so $x_n \not\to x$. $\square$

Now we move on to the main result:

Proposition 1. *$f : S \to \mathbb{R}$ is continuous at x_0 if and only if for every monotonic sequence $\{x_n\}$ converging to x_0, $f(x_n) \to f(x_0)$.*

Proof. ($\Rightarrow$) If f is continuous, then $x_n \to x \Rightarrow f(x_n) \to f(x)$. This holds also for monotone $x_n \to x$.

($\Leftarrow$) We have that for all monotone $\{y_n\}$ where $y_n \to y$, $f(y_n) \to f(y)$. Take some $\{x_n\}$ not necessarily monotone, where $x_n \to x$. It suffices to show that $f(x_n) \to f(x)$, from the sequential definition of continuity. Take any subsequence $\{x_{n_k}\}$. From Proposition 7, it has a monotone sub-subsequence $\{x_{n_{k_l}}\}$, and from Exercise 26, $x_{n_{k_l}} \to x$. By assumption, $f(x_{n_{k_l}}) \to f(x)$. Thus, since we have that for the sequence $f(x_n)$, every subsequence $f(x_{n_k})$ has a sub-subsequence $f(x_{n_{k_l}})$ that converges to $f(x)$, $f(x_n) \to f(x)$ by Lemma 1. $\square$

Exercise 1 Let $S \subset \mathbb{R}$ be open. Prove that a function $f : S \to \mathbb{R}^s$ is continuous if and only if for every open set $A \subset \mathbb{R}^d, f^{-1}(A)$ is open.

Proof. ($\Rightarrow$) We have that f is continuous. FSOC, assume that $f^{-1}(A)$ is not open, meaning that there exists $x \in f^{-1}(A)$ s.t. $\forall\, \varepsilon > 0, B_\varepsilon(x) \not\subseteq f^{-1}(A)$. Fix some $\delta > 0$. Then $\exists\, y_1 \in B_\delta(x) \subseteq S$ s.t. $y_1 \notin f^{-1}(A)$. Consider the sequence defined by $y_n = \{y \in S : y \in B_{\frac{1}{n}\delta}(x),\ y \notin f^{-1}(A)\}$. Definitionally, $\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N}$ s.t. $\forall\, n > N, |y_n - x| < \varepsilon$, because $\frac{1}{n}\delta \to 0$. Thus, $y_n \to x$. However, $f(y_n) \notin A\ \forall\, y_n$. Since A is open, $\exists\, \varepsilon' > 0$ s.t. $B_{\varepsilon'}(f(x)) \subseteq A$. $f(y_n) \notin B_{\varepsilon'}(f(x))$, so taking $\varepsilon = \varepsilon'$, $\nexists\, n \in \mathbb{N}$ s.t. $|f(y_n) - f(x)| < \varepsilon$. This contradicts the assumption that f is continuous, since $y_n \to x$ but $f(y_n) \not\to f(x)$.

($\Leftarrow$) We have that for every open $A \subset \mathbb{R}^d$, $f^{-1}(A)$ is open. Fix some $x \in f^{-1}(A)$, and some $\varepsilon > 0$. $B_\varepsilon(f(x))$ is an open subset of S by definition, so $f^{-1}(B_\varepsilon(f(x)))$ is open by assumption. Since $x \in f^{-1}(B_\varepsilon(f(x)))$, $\exists\, \delta > 0$ s.t. $B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x)))$. Thus, we have shown that for every $\varepsilon > 0, x, y \in S, \exists\, \delta > 0$ s.t. $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$. Thus, f is continuous. $\square$