Microeconomics

Qualifying Exam Solutions

Gabe Sekeres

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Part I

Question

Consider the production possibilities set

$$Y = \left\{(q, -z) \in \mathbb{R}^2_+ \times \mathbb{R}^2_- : z_1^\alpha z_2^\beta \geq [q_1^\sigma + q_2^\sigma]^{\frac{1}{\sigma}}\right\}$$

where $\alpha = \beta = \frac{1}{3}$ and $\sigma > 0$.

(a) Derive the conditional input demand function $z(w, q)$. (Note: Here we are considering a two-output technology, so $q \in \mathbb{R}^2_+$.)

(b) Derive the cost function $C(w, q)$.

(c) Now suppose $\sigma = \frac{3}{2}$. Derive the unconditional input demand function $x(p, w)$.

(d) Give an expression for the derivative of the profit function with respect to w_1 and explain how you arrived at this expression. You should **not** accomplish this by first finding an expression for the profit function and then differentiating.

Solution

(a) The conditional input demand function solves the cost minimization problem subject to a certain level of output, so

$$z^\star(w, q) \equiv \underset{z\in\mathbb{R}^2_+}{\operatorname{argmin}}\, w_1 \cdot z_1 + w_2 \cdot z_2 \text{ s.t. } z_1^\alpha z_2^\beta \geq [q_1^\sigma + q_2^\sigma]^{\frac{1}{\sigma}}$$

Recall that we can take logs of the condition and retain the same feasible set:

$$z^\star(w, q) \equiv \underset{z\in\mathbb{R}^2_+}{\operatorname{argmin}}\, w_1 \cdot z_1 + w_2 \cdot z_2 \text{ s.t. } \alpha \ln z_1 + \beta \ln z_2 \geq \frac{1}{\sigma} \ln [q_1^\sigma + q_2^\sigma]$$

Since this is convex, and the structure of the feasible set guarantees an interior solution (this could also be verified later), we solve this with a Lagrangian:

$$\mathcal{L} = w_1 \cdot z_1 + w_2 \cdot z_2 - \lambda \left(\alpha \ln z_1 + \beta \ln z_2 - \frac{1}{\sigma} \ln \left[q_1^\sigma + q_2^\sigma \right] \right)$$

which admits first order conditions

$$\frac{\partial \mathcal{L}}{\partial z_1} : w_1 - \frac{\lambda \alpha}{z_1} = 0 \Longrightarrow \lambda = \frac{z_1 w_1}{\alpha}$$
$$\frac{\partial \mathcal{L}}{\partial z_2} : w_2 - \frac{\lambda \beta}{z_2} = 0 \Longrightarrow \lambda = \frac{z_2 w_2}{\beta}$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} : \alpha \ln z_1 + \beta \ln z_2 = \frac{1}{\sigma} \ln \left[q_1^\sigma + q_2^\sigma \right]$$

So we have that

$$z_1 = \frac{\alpha}{\beta} \cdot \frac{w_2}{w_1} \cdot z_2 = \frac{w_2}{w_1} z_2 \Longrightarrow \ln \frac{w_2}{w_1} z_2^2 = 3 \cdot \frac{1}{\sigma} \ln \left[q_1^\sigma + q_2^\sigma \right] \Longrightarrow \ln \sqrt{\frac{w_2}{w_1}} z_2 = \frac{3}{2\sigma} \ln \left[q_1^\sigma + q_2^\sigma \right]$$

So

$$z_2^\star = \sqrt{\frac{w_1}{w_2}} \cdot \left[q_1^\sigma + q_2^\sigma \right]^{\frac{3}{2\sigma}}$$

and

$$z_1^\star = \frac{w_2}{w_1} \cdot z_2^\star = \sqrt{\frac{w_2}{w_1}} \left[q_1^\sigma + q_2^\sigma \right]^{\frac{3}{2\sigma}}$$

So formally,

$$z^\star(w, q) = \begin{bmatrix} \sqrt{\frac{w_2}{w_1}} \cdot \left[q_1^\sigma + q_2^\sigma \right]^{\frac{3}{2\sigma}} \\ \\ \sqrt{\frac{w_1}{w_2}} \cdot \left[q_1^\sigma + q_2^\sigma \right]^{\frac{3}{2\sigma}} \end{bmatrix}$$

(b) The cost function is the value function of the cost minimization problem subject to a certain level of output, so

$$C(w, q) \equiv \min_{z \in \mathbb{R}_+^2} w_1 \cdot z_1 + w_2 \cdot z_2 \text{ s.t. } z_1^\alpha z_2^\beta \geq \left[q_1^\sigma + q_2^\sigma \right]^{\frac{1}{\sigma}}$$

Because this is a value function and we've already found the optimal conditional inputs, we have that

$$C(w, q) = w_1 \cdot z_1^\star(w, q) + w_2 \cdot z_2^\star(w, q) = 2\sqrt{w_1 w_2} \cdot \left[q_1^\sigma + q_2^\sigma \right]^{\frac{3}{2\sigma}}$$

(c) The unconditional input demand function is the set of inputs that solves the full profit maximization problem, which implies the cost minimization problem. Note that with $\sigma = \frac{3}{2}$, our cost function becomes $C(w,q) = 2\sqrt{w_1 w_2} \cdot [q_1^\sigma + q_2^\sigma]$. The profit maximization problem is

$$\pi(p,w) = \max_{q \in \mathbb{R}_+^2} p_1 \cdot q_1 + p_2 \cdot q_2 - C(w,q) \equiv \max_{q \in \mathbb{R}_+^2} \underbrace{p_1 \cdot q_1 + p_2 \cdot q_2 - 2\sqrt{w_1 w_2} \cdot [q_1^\sigma + q_2^\sigma]}_{\mathcal{L}}$$

Again this is concave and has an interior solution, so we find the optimal outputs using first order conditions:

$$\frac{\partial \mathcal{L}}{\partial q_1} : p_1 - 2\sigma\sqrt{w_1 w_2} q_1^{\sigma-1} = 0 \Longrightarrow \sqrt{q_1} = \frac{p_1}{3\sqrt{w_1 w_2}} \Longrightarrow q_1^\star(p,w) = \frac{p_1^2}{9w_1 w_2}$$
$$\frac{\partial \mathcal{L}}{\partial q_2} : p_2 - 2\sigma\sqrt{w_1 w_2} q_2^{\sigma-1} = 0 \Longrightarrow \sqrt{q_2} = \frac{p_2}{3\sqrt{w_1 w_2}} \Longrightarrow q_2^\star(p,w) = \frac{p_2^2}{9w_1 w_2}$$

To find the unconditional input demand function, we substitute the optimal quantities into the conditional input demand functions, since $x(p,w) = z(w, q^\star(p,w))$. We get:

$$x_1^\star(p,w) = \sqrt{\frac{w_2}{w_1}} \left[\left(\frac{p_1^2}{9w_1 w_2} \right)^{\frac{3}{2}} + \left(\frac{p_2^2}{9w_1 w_2} \right)^{\frac{3}{2}} \right] = \frac{p_1^3 + p_2^3}{27 w_1^2 w_2}$$
$$x_2^\star(p,w) = \sqrt{\frac{w_1}{w_2}} \left[\left(\frac{p_1^2}{9w_1 w_2} \right)^{\frac{3}{2}} + \left(\frac{p_2^2}{9w_1 w_2} \right)^{\frac{3}{2}} \right] = \frac{p_1^3 + p_2^3}{27 w_1 w_2^2}$$

(d) The way we're meant to do this problem is to recall that from Shepherd's Lemma, $\frac{\partial \pi(p,w)}{\partial w_i} = -x_i^\star(p,w)$, so we have that

$$\frac{\partial \pi(p,w)}{\partial w_1} = -x_1^\star(p,w) = -\frac{p_1^3 + p_2^3}{27 w_1^2 w_2}$$

The way I did it was to construct the profit function, take the derivative with respect to w_1, and then re-construct Shepherd's Lemma from there because I didn't perfectly remember it. This requires substituting the optimal outputs found in part (c) into the profit maximization problem, so:

$$\begin{aligned} \pi(p,w) &= p_1 \cdot \frac{p_1^2}{9w_1 w_2} + p_2 \cdot \frac{p_2^2}{9w_1 w_2} - 2\sqrt{w_1 w_2} \left[\left(\frac{p_1^2}{9w_1 w_2} \right)^{\frac{3}{2}} + \left(\frac{p_2^2}{9w_1 w_2} \right)^{\frac{3}{2}} \right] \\ &= \frac{p_1^3 + p_2^3}{9w_1 w_2} - 2 \cdot \frac{p_1^3 + p_2^3}{27 w_1 w_2} \\ &= \frac{p_1^3 + p_2^3}{27 w_1 w_2} \end{aligned}$$

So

$$\frac{\partial \pi(p,w)}{\partial w_1} = -\frac{p_1^3 + p_2^3}{27 w_1^2 w_2} = -x_1^\star(p,w)$$

Part II

Question

Consider a consumer who has utility $u(x_1, \ldots, x_L, h)$ over the amounts of consumption $(x_1, \ldots, x_L)$ of goods $1, \ldots, L$, and hours worked h. The utility function $u(\cdot)$ is continuous and strictly increasing in each x_ℓ, and is weakly decreasing in h. Prices are given by $p = (p_1, \ldots, p_L)$. The wage rate is normalized to 1 throughout, so that the consumer can spend amount h to purchase consumption goods. Consumption and labor hours have to be non-negative, but assume that there is no upper bound on labor hours h.

(a) Write down the "work-minimization problem": the problem of minimizing the hours of work subject to achieving a given level of utility and subject to the budget constraint.

Call the value $h^\star$ of this problem the "minimal-work function". Does it have similar properties to the expenditure function? In particular:

(b) Is the "minimal-work function" non-decreasing in p_i for $i \in \{1, \ldots, L\}$? (Hint: If $(x_1, \ldots, x_L, h)$ satisfies all constraints under higher prices, would it still satisfy all constraints under lower prices?).

(c) Is the "minimal-work function" homogeneous of degree 1 in p? (Hint: if you doubled the prices of all goods, would the minimal-work-hours double? Try e.g. with $u(x_1, h) = x_1^2/h$.)

(d) What would your answer to (b) and (c) be if you knew that the utility function is constant in work hours h?

Prove your answers. For negative answers you can provide a counter-example. You do not need to re-prove known properties of the standard expenditure function.

Solution

(a) The work minimization problem takes some $\bar{u}$ as given, so the consumer solves

$$\begin{aligned} & \min_{h \in \mathbb{R}_+} h \\ \text{s.t.} \quad & u(x_1, \ldots, x_L, h) \geq \bar{u} \\ & p \cdot x \equiv \sum_{i=1}^{L} p_i \cdot x_i \leq h \end{aligned}$$

Since utility is strictly increasing in consumption, we can reduce this to:

$$\min_{x \in \mathbb{R}_+^L} p \cdot x \text{ s.t. } u(x_1, \ldots, x_L, p \cdot x) \geq \bar{u}$$

(b) Yes.

Proof. Note that a function being non-decreasing in an input is equivalent to it being non-increasing in the negative of that input. It therefore suffices to show that $h^\star$ is non-increasing in $-p_i$ for arbitrary p_i. Take some p and p' so that $p_i > p'_i$, and $p_j = p'_j$ for all $j \neq i$. We want to show that $h^\star(p, \bar{u}) \geq h^\star(p', \bar{u})$. Call $x^\star(p, \bar{u})$ the optimal output under p. Since utility is weakly decreasing in h, we have that $u(x^\star(p, \bar{u}), p' \cdot x^\star(p, \bar{u})) \geq \bar{u}$ since $p' \cdot x^\star(p, \bar{u}) \leq p \cdot x^\star(p, \bar{u})$. Thus, $x^\star(p, \bar{u})$ is in the feasible set under prices p'. From the properties of optimization problems, it must be the case that $h^\star(p', \bar{u}) \leq h^\star(p, \bar{u})$ since the minimal bundle under prices p is feasible under prices p'. Thus, $h^\star$ is non-increasing as p_i decreases, meaning that it is non-decreasing in p_i. □

n.b. This is a really messy and confusing proof, there's got to be a better way to do this, but I can't think of one.

(c) No. For a counterexample, consider the given utility function $u(x, h) = \frac{x^2}{h}$, and consider $\bar{u} = 1$, with the price change $p = 1$ to $p' = 2$. (Note $x, p \in \mathbb{R}_+$). We have that

$$h^\star(p, \bar{u}) = \min_{x \in \mathbb{R}_+} 1 \cdot x \text{ s.t. } \frac{x^2}{1 \cdot x} \geq 1 = \min x \text{ s.t. } x \geq 1 = 1$$
$$h^\star(p', \bar{u}) = \min_{x \in \mathbb{R}_+} 2 \cdot x \text{ s.t. } \frac{x^2}{2 \cdot x} \geq 1 = \min 2x \text{ s.t. } x \geq 2 = 4$$

So since $h^\star(1, \bar{u}) = 1$ but $h^\star(2 \cdot 1, \bar{u}) = 4 > 2 = 2 \cdot h^\star(1, \bar{u})$, $h^\star$ is not homogeneous of degree 1.

(d) My answer to (b) would not change, in the sense that the same proof works as we do not differentiate between non-increasing in h and decreasing in h there. However, note that an increase in p_i would now lead to a strict decrease in x_i, meaning a strict decrease in $h^\star$, as long as the original demand for x_i was positive. My answer to (d) would now change. In the case that the utility function is constant in work hours h, the minimal work function would now be homogeneous of degree 1 in p.

Proof. Take some p and $\alpha > 0$, and fix $\bar{u}$. We have that

$$h^\star(p, \bar{u}) = \min_{x \in \mathbb{R}^L_+} p \cdot x \text{ s.t. } u(x, p \cdot x) \geq \bar{u}$$

and

$$h^\star(\alpha \cdot p, \bar{u}) = \min_{x \in \mathbb{R}^L_+} \alpha \cdot p \cdot x \text{ s.t. } u(x, \alpha \cdot p \cdot x) \geq \bar{u}$$

However, since $u(\cdot)$ is constant in h, we have that $u(x, p \cdot x) = u(x, \alpha \cdot p \cdot x)\ \forall\ \alpha$. Thus,

$$h^\star(\alpha \cdot p, \bar{u}) = \min_{x \in \mathbb{R}^L_+} \alpha \cdot p \cdot x \text{ s.t. } u(x, p \cdot x) \geq \bar{u} = \alpha \cdot \min_{x \in \mathbb{R}^L_+} p \cdot x \text{ s.t. } u(x, p \cdot x) \geq \bar{u}$$

So $h^\star(\alpha \cdot p, \bar{u}) = \alpha \cdot h^\star(p, \bar{u})$, so $h^\star$ is now homogeneous of degree 1. □

Part III

Question

Two firms have commonly known marginal costs c_1 and c_2, where $0 < c_1 < c_2$. Both firms produce homogeneous goods and face a market demand function $D : \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$D(p) \coloneqq \max\{1 - p, 0\}.$$

(a) Suppose firm 1 is a monopoly (i.e. firm 2 does not exist). What price would it be willing to charge as c_1 varies?
Hint: Be sure to consider the case when $c_1 \leq 1$ as well as $c_1 > 1$.

(b) Suppose that consumers buy from firm 1 with probability $\alpha_1 \in [0, 1]$ when indifferent between buying from the two firms. Let p_1^M denote the price that firm 1 operating as a monopoly would charge and suppose that $c_2 < p_1^M$.

(1) Argue that $p_1 = p_2$ in any pure-strategy Nash equilibrium.

(2) What is the unique pure-strategy Nash equilibrium when $\alpha_1 = 1$? Justify your answer.
n.b. Tak clarified in an email that this question should specify "unique pure-strategy Nash equilibrium in undominated strategies". This is the only part that difference affects.

(3) Is there a pure-strategy Nash equilibrium when $\alpha_1 \in [0, 1)$? Justify your answer.

(4) Fix a finite $p_2 > c_2$. Does the firm 1's profit maximization problem always have a solution when $\alpha_1 \in [0, 1)$? Why or why not?

(c) Maintain that $c_2 < p_1^M$ and fix some $\eta > 0$ and consider the following strategy profile: firm 1 posts a price equal to c_2 and firm 2 randomizes uniformly over $[c_2, c_2 + \eta]$.
Hint: CDF of a random variable distributed uniformly over $[a, b]$ is $F(x) = \frac{x-a}{b-a}$.

(1) Verify that firm 2's strategy is a best response to firm 1's price.

(2) Show that firm 1 does not prefer to charge prices outside the interval $[c_2, c_2 + \eta]$.

(3) Show that firm 1 does not prefer to charge some $p_1 \in (c_2, c_2 + \eta]$ over charging $p_1 = c_2$ for sufficiently low $\eta > 0$.

(4) Conclude whether the proposed strategy profile is a Nash equilibrium. How does the way in which consumers break ties between the two firms (i.e., α_1) matter?

Solution

n.b. Basically all of this is significantly more formal than it needs to be. However, it's a Tak question.

(a) Formally, the firm's full profit maximization problem is

$$\pi(p, c_1) = \max_{p \in \mathbb{R}_+} D(p)(p - c_1) \equiv \max_{p \in \mathbb{R}_+} \mathbb{1}\{p \leq 1\}(1-p)(p-c_1)$$

If $c_1 \geq 1$, the best the firm can do is charging $p = 1$ and attaining $\pi(p, c_1) = 0$. If $c_1 < 1$, there always exists some $\varepsilon > 0$ such that $p_\varepsilon = c_1 + \varepsilon < 1$, and $(1 - p_\varepsilon)(p_\varepsilon - c_1) > 0$, so the firm attains strictly positive profit by charging a price in the interval $(c_1, 1)$. It is without loss to consider the closure of that interval, as the boundaries will always attain profit of 0. The profit maximization problem becomes

$$\max_{p \in [c_1, 1]} (1-p)(p-c_1) = \max_{p \in [c_1, 1]} p - p^2 + c_1 \cdot p$$

and since this is a concave function over a compact interval, we attain a solution. Furthermore, since the boundaries are 0 and there exists a point with positive profit in the interval, we attain an interior solution. We can get that solution with first order conditions, by KKT. Thus:

$$\frac{\partial}{\partial p} : 1 - 2p + c_1 = 0 \Longrightarrow p^\star = \frac{1 + c_1}{2}$$

Note that when $c_1 = 1$, this equals the earlier solution where $p^\star = 1$, so we face no discontinuities. Formally:

$$p^\star(c_1) = \begin{cases} \frac{1+c_1}{2} & c_1 \leq 1 \\ 1 & c_1 > 1 \end{cases}$$

(b) (1) ***Proof.*** Argue towards a contradiction. First, assume that there is a Nash equilibrium where $p_1 > p_2$. Then either $p_2 \geq c_2$, in which case firm 1 could strictly improve by charging anything in (c_1, p_2), or $p_2 < c_2$ in which case firm 2 could strictly improve by charging $p_2 = c_2$ to attain 0 profit. Next, assume that there is a Nash equilibrium where $p_1 < p_2$. Either $p_1 \geq c_2$, in which case firm 2 could strictly improve by charging anything in (c_2, p_1), or $p_1 < c_2$ in which case firm 1 could improve by charging anything in $(p_1, \min\{p_2, c_2\})$, since $c_2 < p_1^M$ and profit is strictly increasing below the monopoly price. □

(2) The unique pure-strategy Nash equilibrium[1] is $p_1 = p_2 = p = c_2$. First, we will show that it is a Nash equilibrium. For firm 2, they attain 0 profit in equilibrium, but deviating to any $p' > p$ would also attain 0 profit, so there is no strict

[1] In weakly undominated strategies. Note that there exists a Nash equilibrium with strategies $p_1 = p_2 = c'$ for any $c' \in [c_1, c_2]$, but firm 2 is playing a weakly dominated strategy if $c' \in [c_1, c_2)$. It is still a Nash equilibrium, because there are no profitable deviations, but a clearly suboptimal equilibrium.

improvement, and deviating to any $p' < p$ would attain strictly negative profit, so $p_2 = c_2$ is a best response to $p_1 = c_2$. For firm 1, since $c_2 < p_1^M$, we will use the fact that profit is strictly increasing for the monopolist up to the monopoly price. Charging any $p' < p$ would attain strictly less profit, and charging any $p' > p$ would attain 0 profit. Since $\alpha = 1$ and $c_2 > c_1$, firm 1's equilibrium profit is $(c_2 - c_1)(1 - c_2) > 0$, so $p_1 = c_2$ is a best response. Thus, this is a pure-strategy Nash equilibrium.

Next, we will show that this equilibrium is unique. Towards a contradiction, assume there is another equilibrium at $p' > c_2$. Then firm 2 could strictly improve by charging some $p_2 \in (c_2, p')$ which would attain strictly positive profit. Contradiction. Assume there is another equilibrium $p' < c_2$. Then firm 2 is playing a weakly dominated strategy, because $u_2(p_1, p') \le u_2(p_1, c_2)$ for all p_1, where the inequality holds strictly for any $p_1 > p'$. Thus, this is the unique pure-strategy Nash equilibrium in weakly undominated strategies.

(3) There is not. Argue towards a contradiction. Fix some $\alpha_1 \in [0, 1)$. Assume that there exists some pure-strategy Nash equilibrium where $p_1 = p_2 = p$ (which is without loss due to (1)), and further assume that $p \in [c_2, p_1^M]$. This second assumption is without loss because for firm 2, playing $p_2 < c_2$ is weakly dominated, as shown in (2), and for firm 1 getting an α_1 share of the market at price $p > p_1^M$ is worth strictly less profit than getting a full share of the market at the optimal monopoly price. Recall that in this region, profit given a full share of the market is strictly increasing in price. Under this equilibrium, firm 1's profit is

$$\pi_1(p, p) = \alpha_1 \cdot (p - c_1) \cdot (1 - p)$$

Consider a deviation to $p - \varepsilon$ for some small ε. Then firm 1's profit would be $\pi_1(p - \varepsilon, p)$. This is strictly decreasing in ε (as long as ε is sufficiently small that $p - \varepsilon \in [c_2, p_1^M]$, which we can freely assume), and as $\varepsilon \to 0$, $\pi_1(p - \varepsilon, p) \to \pi_1(p, p + \delta)$ for any $\delta > 0$, and since $\alpha_1 < 1$, $\pi_1(p, p + \delta) > \pi_1(p, p)$. Thus, there exists ε sufficiently small that $\pi_1(p - \varepsilon, p) > \pi_1(p, p)$, so firm 1 would always have a profitable deviation and this is not an equilibrium.

(4) No. Firm 1's profit maximization problem is essentially discontinuous, we could write it as:

$$\max_{p \in \mathbb{R}_+} \mathbb{1}\{p < p_2\} \cdot (p - c_1) \cdot (1 - p) + \mathbb{1}\{p = p_2\} \cdot \alpha_1 \cdot (p - c_1) \cdot (1 - p) + \mathbb{1}\{p > p_2\} \cdot 0$$

If $p_2 > p_1^M$, this problem is solved by choosing the monopoly price. Otherwise, however, since $\alpha_1 < 1$, as we saw in part (3) there exists ε sufficiently small that choosing $p_2 - \varepsilon$ attains strictly higher profit than choosing p_2. Thus, this maximization problem is

$$\max_{p \in [c_1, p_2)} (p - c_1) \cdot (1 - p)$$

which is the maximization of a strictly increasing function over an interval that is open above, so the optimum is not attained. Thus, Firm 1's profit maximization problem may not have a solution depending on the value of p_2.

(c) (1) Firm 2 choosing any price lower than c_2 will lead to strictly negative profit. Firm 2 choosing any price greater than or equal to c_2 will lead to zero profit, which is the same expected profit attained under this strategy. Thus, this strategy is a (weak) best response.

(2) Since we assume that $c_2 < p_1^M$, we have that monopoly profit is strictly increasing in price on the interval $[c_1, c_2]$. Thus, comparing profit from some $p_1 < c_2$ to $p_1 = c_2$ we have that

$$\pi_1(p_1, p_2) = (p_1 - c_1)(1 - p_1) < \underbrace{\mathbb{P}\{p_1 < p_2\}}_{=1} \cdot (c_2 - c_1)(1 - c_2) + \mathbb{P}\{p_1 \geq p_2\} \cdot 0 = \pi_1(c_2, p_2)$$

so any price below c_2 is strictly dominated by choosing $p_1 = c_2$ for firm 1. Additionally, since we assume that $c_1 < c_2$, choosing c_2 guarantees strictly positive profit for firm 1, where choosing some $p_1 > c_2 + \eta$ guarantees zero profit. Thus, firm 1 does not prefer to charge prices outside the interval $[c_2, c_2 + \eta]$.

(3) We are assuming that firm 2 is randomizing uniformly over the interval $[c_2, c_2 + \eta]$. The expected utility of choosing $p_1 = c_2$ is

$$\mathbb{E}\pi_1(c_2, p_2) = \underbrace{\mathbb{P}\{p_2 < c_2\}}_{=0} \cdot 0 + \underbrace{\mathbb{P}\{p_2 = c_2\}}_{=0} \cdot \alpha_1 (c_2 - c_1)(1 - c_2) + \underbrace{\mathbb{P}\{p_2 > c_2\}}_{=1} \cdot (c_2 - c_1)(1 - c_2)$$

so $\mathbb{E}\pi_1(c_2, p_2) = (c_2 - c_1)(1 - c_2)$. The expected utility of choosing some $p' \in (c_2, c_2 + \eta]$ is

$$\mathbb{E}\pi_1(p', p_2) = \underbrace{\mathbb{P}\{p_2 < p'\}}_{=\frac{p' - c_2}{\eta}} \cdot 0 + \underbrace{\mathbb{P}\{p_2 = p'\}}_{=0} \cdot \alpha_1 (p' - c_1)(1 - p') + \underbrace{\mathbb{P}\{p_2 > p'\}}_{=1 - \frac{p' - c_2}{\eta}} \cdot (p' - c_1)(1 - p')$$

So firm 1 does not prefer to charge $p' \in (c_2, c_2 + \eta]$ to charging c_2 as long as

$$(c_2 - c_1)(1 - c_2) \geq \frac{c_2 + \eta - p'}{\eta} (p' - c_1)(1 - p')$$

$$\iff \eta \Big((c_2 - c_1)(1 - c_2) - (p' - c_1)(1 - p') \Big) \geq \underbrace{(c_2 - p')(p' - c_1)(1 - p')}_{<0}$$

and since the left hand side is increasing to 0 as η decreases, there exists η sufficiently small that the inequality holds, so $c_2 \succ_1 p'$.

(4) The proposed strategy is a Nash equilibrium for sufficiently small η, as neither side can profitably deviate. The tiebreaking mechanism does not matter as a tie happening is a measure zero event with precisely zero probability associated with it.

Part IV

Question

Consider a simple 2×2 payoff matrix as follows. As usual, Player 1 is the row player and Player 2 is the column player.

	S	C
S	$5, 2$	$3, 1$
C	$6, 3$	$4, 4$

(a) First, consider the one-period simultaneous-move game with this payoff matrix. Find all Nash equilibria.

(b) Consider the game where Player 1 moves first, and then Player 2 moves after observing Player 1's action. Find the subgame perfect equilibrium of this game, a.k.a. the "Stackelberg equilibrium".

(c) For the rest of the problem, consider a noisy-leader game with pure actions. Player 1 moves first and chooses a pure action $a_1 \in \{S, C\}$. Instead of observing the action, Player 2 receives a signal $\phi \in \{s, c\}$ about Player 1's action. The signal strategy works as follows:

$$\Pr[\phi = s \mid a_1 = S] = \Pr[\phi = c \mid a_1 = C] = 1 - \varepsilon$$

for some small $\varepsilon \in (0, 1)$. In other words, Player 2 observes the signal corresponding to the true action of Player 1 with probability $1 - \varepsilon$. Player 2's pure strategy is thus a function $a_2(\phi)$ that maps each signal realization to a pure action $a_2 \in \{S, C\}$.

Model this as an extensive game with imperfect information. Draw the game tree, the information sets, and specify which player (or Nature) moves at each history and the payoffs at the terminal nodes. **Hint:** Nature moves after Player 1 with actions $\{s, c\}$.

(d) For which values of ε is the strategy profile $(a_1 = S, a_2(\phi) \equiv S)$, i.e., Player 1 plays S and Player 2 plays S, a Perfect Bayesian Equilibrium? Explain.

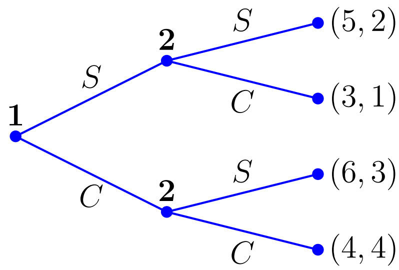
(e) Find a Perfect Bayesian Equilibrium where Player 2 mixes when the signal is c and plays S for sure when the signal is s, or prove there is none. You can assume the noise ε is small, say $\varepsilon < \frac{1}{4}$.

(f) Find a Perfect Bayesian Equilibrium where Player 1 mixes and Player 2 chooses a pure strategy ($a_2(\phi) \in \{S, C\}$, $\forall\, \phi$), or prove there is none. You can assume the noise ε is small, say $\varepsilon < \frac{1}{4}$.

Solution

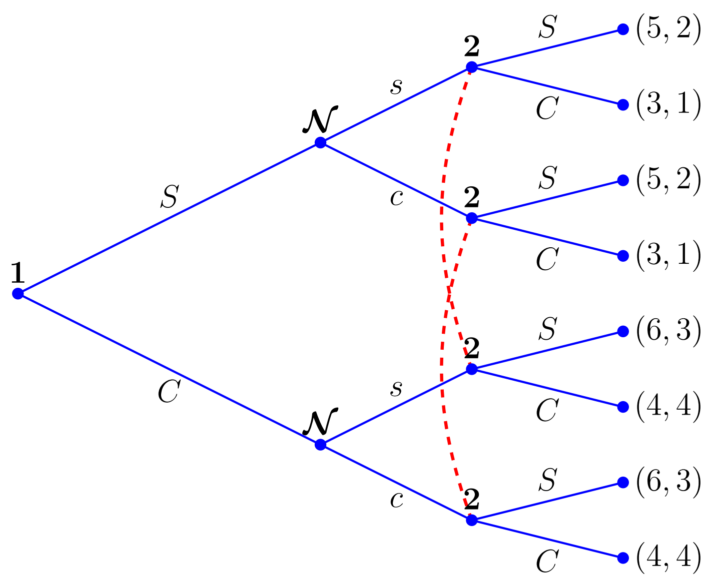
(a) For Player 1, C strictly dominates S. After S is eliminated for Player 1, C strictly dominates S for Player 2. Thus, by iterated deletion of strictly dominated strategies, the only Nash equilibrium is (C, C), where each player attains 4.

(b) This is the described game in extensive form:



There are two proper subgames: the subgame following Player 1 choosing S, and the subgame following Player 1 choosing C. In the first, S strictly dominates C for Player 2, so they will always choose it, attaining payoffs of $(5, 2)$. In the second, C strictly dominates S for Player 2, so they will always choose it, attaining payoffs of $(4, 4)$. Since $5 > 4$, Player 1 will prefer the first subgame, so will choose S, and Player 2 will also choose S. This is the unique subgame-perfect equilibrium, where the players play (S, S) and attain payoffs of $(5, 2)$.

(c) Conceptually, think of Player 1 making a choice, then Nature (randomly) sending a signal of c or s, and then (viewing only the signal), Player 2 making a choice. In extensive form, we have:



(d) This strategy is never a Perfect Bayesian Equilibrium. If Player 2's beliefs are that Player 1 plays S regardless of what signal they receive, they do optimally by choosing $a_2(\phi) = S \ \forall \ \phi$. However, knowing that Player 2 is using that strategy, Player 1 can strictly increase their own payoff by deviating to C. This holds for any $\varepsilon \in (0,1)$, since Player 2 ignores the signal.[2]

(e) ***n.b.*** This is excessively formal, and on an exam infeasible. What steps can be feasibly be removed depends on the structure of your argument, so I've included them all.

Assume that Player 1 chooses S with probability $p \in [0,1]$. Given a certain signal, Player 2's expectations on the 'state' (in this case, which action Player 1 chose) are constructed using Bayesian inference:

$$\begin{aligned}
\mathbb{P}\{a_1 = S \mid \phi = s\} &= \frac{\mathbb{P}\{\phi = s \mid a_1 = S\}\mathbb{P}\{a_1 = S\}}{\mathbb{P}\{\phi = s \mid a_1 = S\}\mathbb{P}\{a_1 = S\} + \mathbb{P}\{\phi = s \mid a_1 = C\}\mathbb{P}\{C\}} \\
&= \frac{(1-\varepsilon)\cdot p}{(1-\varepsilon)\cdot p + \varepsilon \cdot (1-p)} \\
\mathbb{P}\{a_1 = S \mid \phi = c\} &= \frac{\mathbb{P}\{\phi = c \mid a_1 = S\}\mathbb{P}\{a_1 = S\}}{\mathbb{P}\{\phi = c \mid a_1 = S\}\mathbb{P}\{a_1 = S\} + \mathbb{P}\{\phi = c \mid a_1 = C\}\mathbb{P}\{C\}} \\
&= \frac{\varepsilon \cdot p}{\varepsilon \cdot p + (1-\varepsilon)\cdot(1-p)} \\
\mathbb{P}\{a_1 = C \mid \phi = s\} &= 1 - \mathbb{P}\{a_1 = S \mid \phi = s\} = \frac{\varepsilon\cdot(1-p)}{(1-\varepsilon)\cdot p + \varepsilon\cdot(1-p)} \\
\mathbb{P}\{a_1 = C \mid \phi = c\} &= 1 - \mathbb{P}\{a_1 = S \mid \phi = c\} = \frac{(1-\varepsilon)\cdot(1-p)}{\varepsilon\cdot p + (1-\varepsilon)\cdot(1-p)}
\end{aligned}$$

For Player 2 to optimally play S when $\phi = s$, we need that $u_2(S \mid s) \geq u_2(C \mid s)$, where

$$\begin{aligned}
u_2(S \mid s) &= \mathbb{P}\{a_1 = S \mid s\}\cdot 2 + \mathbb{P}\{a_1 = C \mid s\}\cdot 3 \\
&= \frac{2\cdot(1-\varepsilon)\cdot p + 3\cdot\varepsilon\cdot(1-p)}{(1-\varepsilon)\cdot p + \varepsilon\cdot(1-p)} \\
u_2(C \mid s) &= \mathbb{P}\{a_1 = S \mid s\}\cdot 1 + \mathbb{P}\{a_1 = C \mid s\}\cdot 4 \\
&= \frac{1\cdot(1-\varepsilon)\cdot p + 4\cdot\varepsilon\cdot(1-p)}{(1-\varepsilon)\cdot p + \varepsilon\cdot(1-p)}
\end{aligned}$$

So we have that

$$u_2(S \mid s) \geq u_2(C \mid s) \iff (1-\varepsilon)p \geq \varepsilon(1-p) \iff p \geq \varepsilon$$

meaning that Player 1 must mix with a greater probability than ε for Player 2 to be willing to play S when the signal is s (note that this subsumes Player 2's response in

[2] Note that we assumed in the question that $\varepsilon \in (0,1)$. I'm fairly sure answers that said 'This is only a PBE if $\varepsilon \in \{0,1\}$' were accepted, but technically we are assuming that the signal is never perfectly informative.

part (d), as in that case $p = 1$). For Player 2 to optimally mix when the signal is c, we need that $u_2(S \mid c) = u_2(C \mid c)$, where

$$\begin{aligned} u_2(S \mid c) &= \mathbb{P}\{a_1 = S \mid c\} \cdot 2 + \mathbb{P}\{a_1 = C \mid c\} \cdot 3 \\ &= \frac{2 \cdot \varepsilon \cdot p + 3 \cdot (1-\varepsilon) \cdot (1-p)}{\varepsilon \cdot p + (1-\varepsilon) \cdot (1-p)} \\ u_2(C \mid c) &= \mathbb{P}\{a_1 = S \mid c\} \cdot 1 + \mathbb{P}\{a_1 = C \mid c\} \cdot 4 \\ &= \frac{1 \cdot \varepsilon \cdot p + 4 \cdot (1-\varepsilon) \cdot (1-p)}{\varepsilon \cdot p + (1-\varepsilon) \cdot (1-p)} \end{aligned}$$

So we have that

$$u_2(S \mid c) = u_2(C \mid c) \iff \varepsilon p = (1-\varepsilon)(1-p) \iff p = (1-\varepsilon)$$

This means that Player 1 must mix with probability $(1-\varepsilon)$ on S, and additionally (from above) that $1 - \varepsilon \geq \varepsilon$, a condition that is met by our assumption that ε is relatively small.

The above suffices to show that this strategy is optimal for Player 2. It remains to show that it is optimal for Player 1. Assume that when observing the signal c, Player 2 chooses S with probability $q \in [0, 1]$. For Player 1 to optimally mix with probability $(1-\varepsilon) \in (0, 1)$ on S, we need that $u_1(S) = u_1(C)$, taking Player 2's actions as given. We have that:

$$\begin{aligned} u_1(S) &= (1-\varepsilon) \cdot 5 + \varepsilon\,(q \cdot 5 + (1-q) \cdot 3) \\ &= 5 - 2 \cdot \varepsilon \cdot (1-q) \\ u_1(C) &= \varepsilon \cdot 6 + (1-\varepsilon)\,(q \cdot 6 + (1-q) \cdot 4) \\ &= 4 + 2 \cdot q + 2 \cdot \varepsilon \cdot (1-q) \end{aligned}$$

So we have that

$$u_1(S) = u_1(C) \iff 5 - 4 \cdot \varepsilon \cdot (1-q) = 4 + 2 \cdot q \iff q = \frac{1-4\varepsilon}{2-4\varepsilon}$$

and this is a well-defined mix strategy as long as $\varepsilon \in \left(0, \frac{1}{4}\right)$, which we have by assumption. Since the associated beliefs are constructed initially from Bayes' Rule, this suffices to show that this is a Perfect Bayesian Equilibrium. The strategies are:

$$a_1 = (1-\varepsilon)S + \varepsilon C \quad ; \quad a_2(s) = S \quad ; \quad a_2(c) = \frac{1-4\varepsilon}{2-4\varepsilon} S + \left(1 - \frac{1-4\varepsilon}{2-4\varepsilon}\right) C$$

(f) There is no Perfect Bayesian Equilibrium where Player 1 mixes and Player 2 chooses a pure strategy. Any truly mixed strategy would require that $u_1(S \mid a_2) = u_2(C \mid a_2)$. However, if $a_2(\phi) = S$, then

$$u_1(S \mid a_2) = 5 < 6 = u_1(C \mid a_2)$$

and if $a_2(\phi) = C$, then

$$u_1(S \mid a_2) = 3 < 4 = u_1(C \mid a_2)$$

Since regardless of Player 2's strategy, $C \succ_1 S$, Player 1 would never mix as long as Player 2 is playing a pure strategy.

Part V

Question

In an economy, one consumer good x is produced from two inputs y and z, which may also be consumed. The three goods trade at prices p, q, and r respectively. Consumers own the inputs, and a large group of firms produce x from y and z. You are given data on aggregate production before and after a forced reallocation of inputs. The data are summarized in the following table.

	p	x	q	y	r	z
dist 1	5	2	2	1	1	3
dist 2	2	5/2	1	2	1	1
Δ	3	$-1/2$	1	-1	0	2

Good z is the numeraire good, so $r = 1$ in both cases. You are asked to analyze these data, so first you have to check them. Could these data points come from changes solely in the distribution of factor ownership?

Solution

Remark. We don't know what the answer to this is. I'm going to give the one I think is most reasonable, we're fairly sure that the answer is 'No' based on grading distributions. This is extremely low confidence, however.

These data points could *not* have come from changes solely in the distribution of factor ownership. This follows from the Weak Axiom of Revealed Preference (WARP). Note that profit in the first distribution is

$$\pi_1 = 5 \cdot 2 - 2 \cdot 1 - 3 \cdot 1 = 5$$

and the profit in the second distribution is

$$\pi_2 = 2 \cdot \frac{5}{2} - 1 \cdot 2 - 1 \cdot 1 = 3$$

However, under the first distribution's prices, the production in the second distribution would have attained profit

$$\Delta\pi = 5 \cdot \frac{5}{2} - 2 \cdot 2 - 1 \cdot 1 = 7.5$$

Since the same technology could have created a strictly higher profit under the first distribution, this is a direct violation of WARP and could not have come from simply a change in distribution of factor ownership.