Microeconomics

Some Game Theory Questions

Gabe Sekeres

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Question I

A seller is selling a single indivisible good using an auction mechanism. There is a group of potential buyers $\mathcal{N}$, where $|\mathcal{N}| = N$, and each buyer $i \in \mathcal{N}$ has a value for the object $v_i \in \mathbb{R}_+$. If buyer j receives the object and pays p, their utility is $u_j(p) = v_j - p$.

Assume the seller is using a first-price auction, where each buyer submits sealed bids (simultaneously), and the seller allocates the good to the buyer who submits the highest bid, at a price of their bid. If the bids are tied, the seller allocates the good randomly among the tied bids. For simplicity, restrict bids to be non-negative real numbers. Strategies in this game are functions from values to bids.

Initially, assume that $N = 2$ and that buyer values are common knowledge.

(1) Fix some b_2 as buyer 2's bid. Formally characterize buyer 1's best response correspondence. (**Hint:** Consider separately the cases where $b_2 < v_1$ and where $b_2 \geq v_1$).

(2) Fix some v_1 and v_2, and assume that $v_1 > v_2 > 0$. Find a Nash equilibrium in weakly undominated pure strategies, or prove that there are none.

(3) Find the unique pure-strategy Nash equilibrium in weakly undominated strategies when $v_1 = v_2$.

Maintain $N = 2$, but now assume that buyer values are private information. Formally, say that each buyer i knows their own private value v_i, but views v_j as a random variable drawn from a known distribution $F(\cdot)$, which has finite moments as needed and is continuous with full support over $\mathbb{R}_+$.

(4) Assume that buyer 2 plays a strategy where they always bid their value, so $b_2 = v_2$. Formally characterize buyer 1's best response correspondence. You do not need to solve the maximization problem. (**Hint:** Make use of the indicator function $\mathbb{1}\{v_2 < b_1\}$ when solving for the best response).

(5) How does each individual buyer's expected welfare in equilibrium in part (4) changed from parts (1) to (3)? Answer intuitively.

(6) Alternatively, now assume that there are n total buyers, and that $v_i \sim_{\text{i.i.d.}} \mathcal{U}[0, 1]$. Characterize a Nash equilibrium in weakly undominated strategies.

Solution

(1) Buyer 1's best response correspondence is

$$\text{BR}_1(b_2) = \begin{cases} \emptyset & b_2 < v_1 \\ [0, b_2] & b_2 = v_1 \\ [0, b_2) & b_2 > v_1 \end{cases}$$

The first case is the unintuitive one. It can most easily be found by showing that the utility maximization problem is over a set that is not closed below, and that the optimal bid is at the infimum of that set.

(2) There are no Nash equilibria. Observe that buyer 1's best response correspondence is nonempty if and only if $b_2 \geq v_1$. However, bidding b_2 would be higher than v_2, and bidding above one's value is weakly dominated by bidding one's value.

(3) The unique Nash equilibrium is $b_1 = b_2 = v_1 = v_2$. You can show this with intuition.

(4) Buyer 1's maximization problem is to choose b_1 to solve:

$$\max_{b_1 \in \mathbb{R}_+} \mathbb{1}\{v_2 < b_1\} \cdot (v_1 - b_1) + \mathbb{1}\{v_2 = b_1\} \cdot \frac{1}{2} \cdot (v_1 - b_1) + \mathbb{1}\{v_2 > b_1\} \cdot 0$$

subject to their expectations of v_2. Applying the known probabilities, we have that buyer 1 is solving

$$\max_{b_1 \in \mathbb{R}_+} F(b_1) \cdot (v_1 - b_1) + 0 + 0$$

so we have that

$$\text{BR}_1 \equiv \operatorname*{argmax}_{b_1 \in \mathbb{R}_+} F(b_1) \cdot (v_1 - b_1)$$

(5) It has increased, since any equilibrium in parts (1) to (3) would necessarily be in weakly dominated strategies, admitting expected utility of 0 or less, where we now have a maximization problem that admits greater than 0 expected utility.

(6) This is a classical problem. Note that the probability that a given bid is the highest bid is the probability that every other bid ($n-1$ of them) is below it. So buyer i's maximization problem given value v_i is

$$\max_{b_i \in [0,1]} (F(b_i))^{n-1} \cdot (v_i - b_i) \equiv \max_{b_i \in [0,1]} b_i^{n-1} \cdot (v_i - b_i) \equiv \max_{b_i \in [0,1]} b_i^{n-1} \cdot v_i - b_i^n$$

For finite n, this is strictly concave in b_i, so can be solved using first order conditions. We have that

$$(n-1) \cdot b_i^{n-2} \cdot v_i - n \cdot b_i^{n-1} = 0 \quad \Longrightarrow \quad b_i = \frac{n-1}{n} v_i$$

This is the unique symmetric Nash equilibrium in weakly undominated strategies.

Question II

Consider a society democratically deciding how to allocate a public good. There are a set $\mathcal{N}$ of agents, and each $i \in \mathcal{N}$ is exogenously assigned either income y_L or y_H. The measure (proportion) of agents assigned y_L is $\theta > \frac{1}{2}$. For simplicity, assume that there are always an odd number of voters, so that the low-income agents have a strict majority, and assume that $|\mathcal{N}| > 4$. Agents can either vote V or not vote N.

Voting incurs a cost of 1 for high-income agents and a cost of 2 for low-income agents.[1] If the high-income voters outnumber the low-income voters, each high-income agent attains 3 (less the cost of voting, if they voted) and each low-income agent attains 1 (again less the cost of voting if they voted). If the low-income voters outnumber the high-income voters, each high-income agent attains 1 (less the cost of voting, if they voted) and each low-income agent attains some $x \in \mathbb{R}_+$. In the case of a tie, nothing is provided and all agents attain zero payoff.

(1) Assume that all low-income agents and all high-income agents always do the same thing – either all agents of a type vote or all agents of a type do not vote. Represent this as a strategic game, with the low-income agents as the row and the high-income agents as the column.

(2) Still assuming all low-income agents and all high-income agents do the same thing, for which values of x will the low-income agents always vote? For which values of x will the low-income agents never vote? Find all of the Nash equilibria in those two cases.

For the remainder of this problem, assume x is sufficiently large that if all agents of a type were moving together, low-income agents would always vote. (This is a hint to part (2)!). However, now each agent acts individually.

(3) Show that the Nash equilibria you found in part (2) when x was sufficiently large no longer hold.

(4) Describe a pure-strategy Nash equilibrium, or prove that none exist.

(5) Describe a symmetric[2] pure-strategy Nash equilibrium, or prove that none exist.

Now assume that there are n total agents, who are all of the same income level. They do not know what income they have, but receive a signal $\phi \in \{h, \ell\}$ where $\mathbb{P}\{y_i = y_H \mid \phi = h\} = \mathbb{P}\{y_i = y_L \mid \phi = \ell\} = 1 - \varepsilon$ for some small $\varepsilon > 0$. They can choose between public good level H or level L. If all agents agree that they are high-income, they will get H. Otherwise, they will get L, the default. Their prior belief that they are high-income with probability π.

[1] Conceptually – think about how skipping an hour of work will affect someone with lower income versus someone with higher income. Game theory payoffs are truly utilities, not dollars. We often elide this point for simplicity, but it's important to fundamentally understand if you want to use game theory models in practice.

[2] Where all agents of a certain type pursue the same strategy.

They get a payoff of 0 if they provide the good that matches their type, $-z$ if they provide H when their type is L, and $-(1-z)$ if they provide L when their type is H.

(6) Identify a necessary and sufficient condition for an agent who receives signal ℓ to select L.

(7) What is this game really?

Solution

(1) The strategic-form game is:

y_L \ y_H	V	N
V	$(x-2,0)$	$(x-2,1)$
N	$(0,2)$	$(0,0)$

(2) When $x > 2$, voting strictly dominates not voting. When $x < 2$, not voting strictly dominates voting. Either all of the low-income agents will vote and the high-income agents will not vote, or the low-income agents will not vote and the high-income agents will vote. By IDSDS, when $x > 2$, there is a single Nash equilibrium of (V, N), and when $x < 2$, there is a single Nash equilibrium of (N, V).

(3) Assume that all low-income agents are voting and all high-income agents are not voting. Take an individual low-income agent i. They are currently attaining $x - 2$, but if they deviate to not voting there will still be a strict majority of low-income agents voting, so they will attain $x > x - 2$. This is a profitable deviation, so this strategy profile is no longer a Nash equilibrium.

(4) Consider the following: one low-income agent votes, and the rest do not vote, and no high-income agents vote. Obviously the low-income agents who are not voting would not prefer to deviate, the low-income agent who is voting prefers $x - 2$ to 0, and the high-income agents each prefer 1 to 0, so there are no profitable deviations.

(5) None exist. Consider the four possible symmetric pure strategy profiles. When all low-income agents are voting and no high-income agents are voting, we are in the case from (3). When all agents are voting, high-income agents would strictly improve by not voting. When no agents are voting, any individual agent would strictly improve by voting. When all high-income agents are voting and no low-income agents are voting, an individual high-income agent could strictly improve by not voting. Thus, no symmetric pure strategy profiles can be Nash equilibria.

(6) We need for, given a low signal, the expected utility of choosing L is higher than the expected utility of choosing H. Note that the two expected utilities are the same if one or more other agents are choosing L, so we will restrict attention to the case when

the agent in question is pivotal. The relevant posterior for agent i is

$$
\begin{aligned}
&\mathbb{P}\{H : \phi_i = \ell \text{ and } \phi_j = h \;\forall\; j \neq i\} = \\
&= \frac{\mathbb{P}\{\phi_i = \ell \text{ and } \phi_j = h \;\forall\; j \neq i \mid H\}\mathbb{P}\{H\}}{\mathbb{P}\{\phi_i = \ell \text{ and } \phi_j = h \;\forall\; j \neq i \mid H\}\mathbb{P}\{H\} + \mathbb{P}\{\phi_i = \ell \text{ and } \phi_j = h \;\forall\; j \neq i \mid L\}\mathbb{P}\{L\}} \\
&= \frac{\varepsilon \cdot (1-\varepsilon)^{n-1} \cdot \pi}{\varepsilon \cdot (1-\varepsilon)^{n-1} \cdot \pi + (1-\varepsilon) \cdot \varepsilon^{n-1} \cdot (1-\pi)} \\
&\coloneqq p
\end{aligned}
$$

The relevant expected utilities are

$$
\begin{aligned}
u(L) &= -(1-z) \cdot p + 0 \cdot (1-p) \\
&= -(1-z) \cdot \frac{\varepsilon \cdot (1-\varepsilon)^{n-1} \cdot \pi}{\varepsilon \cdot (1-\varepsilon)^{n-1} \cdot \pi + (1-\varepsilon) \cdot \varepsilon^{n-1} \cdot (1-\pi)} \\
u(H) &= 0 \cdot p - z \cdot (1-p) \\
&= -z \cdot \left(1 - \frac{\varepsilon \cdot (1-\varepsilon)^{n-1} \cdot \pi}{\varepsilon \cdot (1-\varepsilon)^{n-1} \cdot \pi + (1-\varepsilon) \cdot \varepsilon^{n-1} \cdot (1-\pi)}\right)
\end{aligned}
$$

So they will obey a signal of ℓ if and only if (after some annoying algebra)

$$
z \geq \frac{\varepsilon \cdot (1-\varepsilon)^{n-1} \cdot \pi}{\varepsilon \cdot (1-\varepsilon)^{n-1} \cdot \pi + (1-\varepsilon) \cdot \varepsilon^{n-1} \cdot (1-\pi)}
$$

(7) The juror game!