ECON 6090

Section 9

Gabe Sekeres

Fall 2025

Material

Definition. The *production function* $f : \mathbb{R}_+^{L-1} \to \mathbb{R}_+$ is defined by

$$f(x) = \max q \text{ s.t. } (q, -z) \in Y$$

Definition. The *input requirement set*

$$V(q) \coloneqq \{z \in \mathbb{R}_+^{L-1} : (q, -z) \in Y\}$$

gives all of the input vectors that can be used to produce an output q.
Definition. The *isoquant*

$$Q(q) \coloneqq \{z \in \mathbb{R}_+^{L-1} : z \in V(q) \text{ and } z \notin V(q') \text{ for any } q' > q\}$$

gives all the input vectors that can be used to produce at most q units of output.
Definition. The firm's *cost minimization problem* is

$$\min_{z \in \mathbb{R}_+^{L-1}} w \cdot z \text{ s.t. } f(z) \geq q$$

The associated value function is called the *cost function*

$$C(w, q) \coloneqq \min_{z \in \mathbb{R}_+^{L-1}} w \cdot z \text{ s.t. } f(z) \geq q$$

We also defined:

$$V^\star(q) = \{x \in \mathbb{R}_+^{L-1} : w \cdot x \geq C(w, q) \; \forall \; w \in \mathbb{R}_{++}^{L-1}\}$$

This leads to two different definitions of the cost function, which we showed in class are equal:

$$C(w, q) = \min_{z \in V(q)} w \cdot z \qquad \text{or} \qquad C^\star(w, q) = \min_{z \in V^\star(q)} w \cdot z$$

Proposition 1. (Properties of the Cost Function)

(i) C is homogeneous of degree 1 in w

(ii) C is concave in w

(iii) If we assume free disposal, C is nondecreasing in q

(iv) If f is homogeneous of degree k in z, C is homogeneous of degree $\frac{1}{k}$ in q

Proof.

(i) Increasing w by $\alpha > 0$ is a monotonic transformation and does not affect the choice of z, but it does increase $w \cdot z$ by a factor of α.

(ii) Fix $w, w' \in \mathbb{R}_+^{L-1}$, and suppose $C(w, q) = w \cdot z$ and $C(w', q) = w' \cdot z'$. Take $\alpha \in [0, 1]$ and let $w'' = \alpha w + (1-\alpha)w'$. Then for z'' a cost minimizer at w'', we have that

$$C(w'', q) = w'' \cdot z'' = \alpha w \cdot z'' + (1-\alpha)w' \cdot z''$$

We also know that $w \cdot z'' \geq C(w, q)$ and $w' \cdot z'' \geq C(w', q)$, so we have that $C(w'', q) \geq \alpha C(w, q) + (1-\alpha)C(w', q)$.

(iii) Suppose that $q' > q$. By free disposal, q can be produced using the same input vector used to produce q'.

(iv) Homogeneity of degree k of f implies that

$$f(z) = q \iff \frac{1}{q} f(z) = 1 \iff f\left(\frac{z}{q^{1/k}}\right) = 1$$

Thus, we get that

$$\begin{aligned} C(w, q) &= \min_z w \cdot z \text{ s.t. } f\left(\frac{z}{q^{1/k}}\right) = 1 \\ &= q^{1/k} \min_z w \cdot \frac{z}{q^{1/k}} \text{ s.t. } f\left(\frac{z}{q^{1/k}}\right) = 1 \\ &= q^{1/k} C(w, 1) \end{aligned}$$

□

Definition. The firm's *profit maximization problem* is

$$\max_y p \cdot y \text{ s.t. } y \in Y$$

The associated value function is called the *profit function*:

$$\pi(p) := \max_y p \cdot y \text{ s.t. } y \in Y$$

Proposition 2. (Properties of the Profit Function)

(i) Homogeneous of degree 1

(ii) Nondecreasing in p

(iii) Nonincreasing in w

(iv) Convex in (p, w)

(v) Continuous

Proof.

(i) $\max_z \alpha(pf(z) - w \cdot z) = \alpha \max_z pf(z) - w \cdot z$

(ii) $p' \geq p \Longrightarrow p'f(z) \geq pf(z) \; \forall \; z$

(iii) $w' \geq w \Longrightarrow w' \cdot z \geq w \cdot z$

(iv) Let $(p'', w'') := \alpha(p, w) + (1 - \alpha)(p', w')$ and let z, z', z'' be the solution to the profit maximization problem with the corresponding output prices and input price vectors. Then by definition

$$\pi(p, w) = pf(z) - w \cdot z \geq pf(z'') - w \cdot z''$$

$$\pi(p', w') = p'f(z) - w' \cdot z \geq p'f(z'') - w' \cdot z''$$

which implies that

$$\begin{aligned}
\alpha\pi(p, w) + (1 - \alpha)\pi(p', w') &\geq \alpha(pf(z'') - w \cdot z'') + (1 - \alpha)(p'f(z'') - w' \cdot z'') \\
&= (\alpha p + (1 - \alpha)p')f(z'') - (\alpha w + (1 - \alpha)w') \cdot z'' \\
&= \pi(p'', w'')
\end{aligned}$$

(v) Follows from Berge's Theorem

□

Definition. The *unconditional input demand function*

$$x(p, w) := \operatorname*{argmax}_{z \in \mathbb{R}_+^{L-1}} pf(z) - w \cdot z$$

is the solution to the profit maximization problem. The *output supply function*

$$q(w) := f(x(w))$$

is the output level where the profit is being maximized.

Proposition 3. (Hotelling's Lemma) *If π is differentiable, then for $(p, w) \in \mathbb{R}_{++}^L$,*

$$q(p, w) = \frac{\partial \pi(p, w)}{\partial p}$$

$$x_j(p, w) = -\frac{\partial \pi(p, w)}{\partial w_j}$$

Definition. The *conditional input demand function*

$$z(w,q) \coloneqq \operatorname*{argmin}_{z \in \mathbb{R}_+^{L-1}} w \cdot z \text{ s.t. } f(z) = q$$

is the solution to the cost minimization problem.

Proposition 4. (Shephard's Lemma) *If C is differentiable, then for $w \in \mathbb{R}_{++}^{L-1}$,*

$$z_i(w,q) = \frac{\partial C(w,q)}{\partial w_i}$$

Proposition 5. *Suppose the profit function is twice continuously differentiable. Then:*

(i) $\frac{\partial q(p,w)}{\partial p_i} \geq 0$

(ii) $\frac{\partial x_j(p,w)}{\partial w_j} \leq 0$

(iii) $\frac{\partial x_j(p,w)}{\partial w_i} = \frac{\partial x_i(p,w)}{\partial w_j}$

Proposition 6. *Suppose the cost function is twice continuously differentiable. Then:*

(i) $\frac{\partial z_i(w,q)}{\partial w_i} \leq 0$

(ii) $\frac{\partial z_j(w,q)}{\partial w_i} = \frac{\partial z_i(w,q)}{\partial w_j}$

(iii) $\frac{\partial z_i(w,q)}{\partial q} = \frac{\partial MC(w,q)}{\partial w_i} = \begin{cases} > 0 & \text{Normal Input} \\ < 0 & \text{Inferior Input} \end{cases}$

Practice Problems

1. **2025 June Q, Part I:** Consider the production possibilities set

$$Y = \left\{ (q, -z) \in \mathbb{R}^2_+ \times \mathbb{R}^2_- : z_1^{\alpha} z_2^{\beta} \geq [q_1^{\sigma} + q_2^{\sigma}]^{\frac{1}{\sigma}} \right\}$$

where $\alpha = \beta = \frac{1}{3}$ and $\sigma > 0$.

(a) Derive the conditional input demand function $z(w, q)$. (Note: Here we are considering a two-output technology, so $q \in \mathbb{R}^2_+$.)

(b) Derive the cost function $C(w, q)$.

(c) Now suppose $\sigma = \frac{3}{2}$. Derive the unconditional input demand function $x(p, w)$.

(d) Give an expression for the derivative of the profit function with respect to w_1 and explain how you arrived at this expression. You should **not** accomplish this by first finding an expression for the profit function and then differentiating.